




# *D*-finite multivariate series with arithmetic restrictions on their coefficients

Jason Bell  and Daniel Smertnig

*Abstract.* A multivariate, formal power series over a field  $K$  is a Bézivin series if all of its coefficients can be expressed as a sum of at most  $r$  elements from a finitely generated subgroup  $G \leq K^*$ ; it is a Pólya series if one can take  $r = 1$ . We give explicit structural descriptions of *D*-finite Bézivin series and *D*-finite Pólya series over fields of characteristic 0, thus extending classical results of Pólya and Bézivin to the multivariate setting.

## 1 Introduction

A fundamental result in the theory of linear recurrences due to Pólya [P621] asserts that if  $\{f(n)\}$  is a sequence satisfying a linear recurrence and taking all of its values in the set  $G \cup \{0\}$  for some finitely generated multiplicative subgroup  $G$  of  $\mathbb{Q}^*$ , then the generating series of  $f(n)$  is a finite sum of series of the form  $gx^i/(1 - g'x^n)$  with  $g, g' \in G$  and  $i, n$  natural numbers along with a finite set of monomials with coefficients in  $G$ ; moreover, these series can be chosen to have disjoint supports. In particular, Pólya's theorem gives a complete characterization of sequences  $\{f(n)\} \subseteq \mathbb{Q}^*$  that have the property that both  $\{f(n)\}$  and  $\{1/f(n)\}$  satisfy a nontrivial linear recurrence.

Pólya's result was later extended to number fields by Benzaghoul [Ben70] and then by Bézivin [Bé87] to all fields (even those of positive characteristic). A non-commutative multivariate version was recently proved by the authors [BS21]; the noncommutative variant can be interpreted as structural description of unambiguous weighted finite automata over a field.

The generating function of a sequence satisfying a linear recurrence is the power series in some variable  $x$  of a rational function about  $x = 0$ . When one adopts this point of view, it is natural to ask whether such results can be extended to *D*-finite (or differentially finite) power series. We recall that if  $K$  is a field of characteristic zero, then a univariate power series  $F(x) = \sum f(n)x^n \in K[[x]]$  is *D*-finite if  $F(x)$  satisfies a nontrivial homogeneous linear differential equation with polynomial coefficients:

$$\sum_{i=0}^n p_i(x)F^{(i)}(x) = 0,$$

with  $p_0(x), \dots, p_n(x) \in K[x]$ .

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Univariate  $D$ -finite series were introduced by Stanley [Sta80], and the class of  $D$ -finite power series is closed under many operations, such as taking  $K$ -linear combinations, products, sections, diagonals, Hadamard products, and derivatives, and it contains a multitude of classical generating functions arising from enumerative combinatorics [Sta99, Chapter 6]. In particular, algebraic series and their diagonals are  $D$ -finite, and a power series  $F$  is  $D$ -finite if and only if its sequence of coefficients satisfies certain recursions with polynomial coefficients [Lip89, Theorem 3.7].

Bézivin gave a sweeping extension of Pólya’s result, showing that a univariate  $D$ -finite Bézivin series over a field  $K$  of characteristic 0 with the property that there is a fixed  $r \geq 1$  and a fixed finitely generated subgroup  $G$  of  $K^*$  such that every coefficient of  $F(x)$  can be written as a sum of at most  $r$  elements of  $G$  is in fact a rational power series and has only simple poles [Bé86].

A natural, and thus far unexplored, direction in which to extend the results of Pólya and Bézivin is to consider multivariate analogues. A multivariate variant of  $D$ -finite series was given by Lipshitz [Lip89]. Here, one has a field  $K$  of characteristic zero and declares that a formal multivariate series

$$F = \sum_{\mathbf{n} \in \mathbb{N}^d} f(\mathbf{n}) \mathbf{x}^{\mathbf{n}} \in K[[\mathbf{x}]] = K[[x_1, \dots, x_d]]$$

is  $D$ -finite if all the partial derivatives  $(\partial/\partial x_1)^{e_1} \dots (\partial/\partial x_d)^{e_d} F$  for  $e_1, \dots, e_d \geq 0$  are contained in a finite-dimensional vector space over the rational function field  $K(\mathbf{x})$ . Equivalently, for each  $i \in [1, d]$ , the series  $F$  satisfies a linear partial differential equation of the form

$$P_{i,n} \cdot (\partial/\partial x_i)^n F + P_{i,n-1} \cdot (\partial/\partial x_i)^{n-1} F + \dots + P_{i,1} \cdot (\partial/\partial x_i) F + P_{i,0} \cdot F = 0,$$

with polynomials  $P_{i,0}, P_{i,1}, \dots, P_{i,n} \in K[\mathbf{x}]$ , at least one of which is nonzero.

In fact, many interesting classical Diophantine questions can be expressed in terms of questions about coefficients of multivariate rational power series and multivariate  $D$ -finite series. To give one example, Catalan’s conjecture (now a theorem due to Mihăilescu [Mih04]) states that the only solutions to the equation  $3^n = 2^m + 1$  are given by  $(n, m) = (1, 1)$  and  $(2, 3)$ . This is equivalent to the statement that the bivariate rational power series

$$1/((1 - 3x_1)(1 - x_2)) - 1/((1 - x_1)(1 - 2x_2)) - 1/((1 - x_1)(1 - x_2))$$

has nonzero coefficients except for the coefficients of  $x_1^2 x_2^3$  and  $x_1 x_2$ . On the other hand, it is typically much more difficult to obtain results about multivariate rational functions and multivariate  $D$ -finite series for several reasons. In the case of univariate rational series, the coefficients have a nice closed form and there is a strong structural description of the set of zero coefficients due to Skolem, Mahler, and Lech (see [EvdPSW03, Chapter 2.1]). Similarly, for  $D$ -finite series, there are many strong results concerning the asymptotics of their coefficients (see [FS09], in particular Chapters VII.9 and VIII.7) and one can often make use of these results when considering problems in the univariate case. Straightforward attempts at extending these approaches to higher dimensions typically fail or become too unwieldy. For this reason, new ideas are often important in obtaining multivariate extensions.

It is therefore of considerable interest to understand multivariate  $D$ -finite series, although doing so often presents additional technical difficulties. In this paper, we consider structural results for multivariate series that are implied by the additional restrictions on the coefficients of a  $D$ -finite series that were imposed by Bézivin and Pólya in the univariate case. For a field  $K$  of characteristic zero and a multiplicative subgroup  $G \leq K^*$ , we set  $G_0 := G \cup \{0\}$  and write

$$rG_0 := \{g_1 + \dots + g_r : g_1, \dots, g_r \in G_0\}$$

for the  $r$ -fold sumset of  $G_0$ .

In view of Bézivin’s and Pólya’s results, we give the following definitions.

**Definition 1.1** A power series  $F = \sum_{\mathbf{n} \in \mathbb{N}^d} f(\mathbf{n})\mathbf{x}^{\mathbf{n}} \in K[[\mathbf{x}]]$  is:

- a *Bézivin series* if there exists a finitely generated subgroup  $G \leq K^*$  and  $r \in \mathbb{N}$  such that  $f(\mathbf{n}) \in rG_0$  for all  $\mathbf{n} \in \mathbb{N}^d$ ;
- a *Pólya series* if there exists a finitely generated subgroup  $G \leq K^*$  such that  $f(\mathbf{n}) \in G_0$  for all  $\mathbf{n} \in \mathbb{N}^d$ .

The terminology of Pólya series is standard; we have chosen to name Bézivin series based on Bézivin’s results characterizing this type of series in the univariate case [Bé86, Bé87].

In this paper, we completely characterize multivariate  $D$ -finite Bézivin and Pólya series. As an immediate consequence, we obtain a characterization of  $D$ -finite series whose Hadamard (sub)inverse is also  $D$ -finite. The proofs make repeated use of unit equations and results from the theory of semilinear sets.

To state our main result, we recall the notion of semilinear subsets of  $\mathbb{N}^d$ . A subset  $\mathcal{S} \subseteq \mathbb{N}^d$  is *simple linear* if it is of the form  $\mathcal{S} = \mathbf{a}_0 + \mathbf{a}_1\mathbb{N} + \dots + \mathbf{a}_s\mathbb{N}$  with  $(\mathbb{Z}$ -)linearly independent  $\mathbf{a}_1, \dots, \mathbf{a}_s \in \mathbb{N}^d$ . The terminology comes from the theory of semilinear sets (see Section 2.1). The following is our main result.

**Theorem 1.2** Let  $K$  be a field of characteristic zero, let  $d \geq 0$ , and let

$$F = \sum_{\mathbf{n} \in \mathbb{N}^d} f(\mathbf{n})\mathbf{x}^{\mathbf{n}} \in K[[\mathbf{x}]] = K[[x_1, \dots, x_d]]$$

be a Bézivin series, with all coefficients of  $F$  contained in  $rG_0$  for some finitely generated subgroup  $G \leq K^*$  and  $r \in \mathbb{N}$ . Then the following statements are equivalent.

- (a)  $F$  is  $D$ -finite.
- (b)  $F$  is rational.
- (c)  $F$  is a (finite) sum of skew-geometric series with coefficients in  $G$ , that is, rational functions of the form

$$\frac{g_0 u_0}{(1 - g_1 u_1) \cdots (1 - g_l u_l)},$$

where  $u_0, u_1, \dots, u_l$  are monomials in  $x_1, \dots, x_d$  such that  $u_1, \dots, u_l$  are algebraically independent, and  $g_0, g_1, \dots, g_l \in G$ .

- (d) As in (c), but in addition, the sum may be taken in such a way that for any two summands, the support is either identical or disjoint. Moreover, every  $\mathbf{n} \in \mathbb{N}^d$  is contained in the support of at most  $r$  summands.

(e) *There exists a partition of  $\mathbb{N}^d$  into finitely many simple linear sets so that on each such set  $S = \mathbf{a}_0 + \mathbf{a}_1\mathbb{N} + \dots + \mathbf{a}_s\mathbb{N}$  with  $\mathbf{a}_1, \dots, \mathbf{a}_s$  linearly independent,*

$$f(\mathbf{a}_0 + m_1\mathbf{a}_1 + \dots + m_s\mathbf{a}_s) = \sum_{i=1}^l g_{i,0}g_{i,1}^{m_1}\dots g_{i,s}^{m_s} \quad \text{for } (m_1, \dots, m_s) \in \mathbb{N}^s,$$

where  $0 \leq l \leq r$  and  $g_{i,j} \in G$  for  $i \in [1, l]$  and  $j \in [0, s]$ .

The description of  $D$ -finite Pólya series, that is, the case  $r = 1$  of the previous theorem, deserves separate mention, although it follows readily from the more general result on Bézivin series. Following terminology from formal language theory, a sum of power series  $F_1, \dots, F_n$  is *unambiguous* if the support of  $F_i$  is disjoint from the support of  $F_j$  for  $i \neq j$  (see Definition 3.16).

**Theorem 1.3** *Let  $K$  be a field of characteristic zero, let  $d \geq 0$ , and let*

$$F = \sum_{\mathbf{n} \in \mathbb{N}^d} f(\mathbf{n})\mathbf{x}^{\mathbf{n}} \in K[[\mathbf{x}]] = K[[x_1, \dots, x_d]]$$

*be a Pólya series with coefficients contained in  $G_0$  for some finitely generated subgroup  $G \leq K^*$ . Then the following statements are equivalent.*

- (a)  *$F$  is  $D$ -finite.*
- (b)  *$F$  is rational.*
- (c)  *$F$  is a (finite) unambiguous sum of skew-geometric series with coefficients in  $G$ .*
- (d) *The support of  $F$  can be partitioned into finitely many simple linear sets so that on each such set  $S = \mathbf{a}_0 + \mathbf{a}_1\mathbb{N} + \dots + \mathbf{a}_s\mathbb{N}$  with  $\mathbf{a}_1, \dots, \mathbf{a}_s$  linearly independent,*

$$f(\mathbf{a}_0 + m_1\mathbf{a}_1 + \dots + m_s\mathbf{a}_s) = g_0g_1^{m_1}\dots g_s^{m_s} \quad \text{for } (m_1, \dots, m_s) \in \mathbb{N}^s,$$

*with  $g_j \in G$  for  $j \in [0, s]$ .*

For a power series  $F = \sum_{\mathbf{n} \in \mathbb{N}^d} f(\mathbf{n})\mathbf{x}^{\mathbf{n}} \in K[[\mathbf{x}]]$ , let

$$F^\dagger = \sum_{\mathbf{n} \in \mathbb{N}^d} f(\mathbf{n})^\dagger \mathbf{x}^{\mathbf{n}},$$

where  $f(\mathbf{n})^\dagger = f(\mathbf{n})^{-1}$  if  $f(\mathbf{n})$  is nonzero and  $f(\mathbf{n})^\dagger$  is zero otherwise. The series  $F^\dagger$  is the *Hadamard subinverse* of  $F$ .

We call a power series *finitary* if its coefficients are contained in a finitely generated  $\mathbb{Z}$ -subalgebra of  $K$ . The set of finitary power series is trivially closed under  $K$ -linear combinations, products, sections, diagonals, Hadamard products, and derivatives. Therefore, the set of finitary  $D$ -finite series is closed under the same operations. Algebraic series as well as their diagonals and sections are finitary  $D$ -finite (see Lemma 7.1).

**Corollary 1.4** *Let  $F \in K[[\mathbf{x}]]$  be finitary  $D$ -finite. Then  $F^\dagger$  is finitary if and only if  $F$  satisfies the equivalent conditions of Theorem 1.3 for some finitely generated subgroup  $G \leq K^*$ . In particular, if  $F$  and  $F^\dagger$  are both finitary  $D$ -finite, then they are in fact unambiguous sums of skew-geometric series.*

### 1.1 Notation

Throughout the paper, we fix a field  $K$  of characteristic 0. When considering a Bézivin series  $F \in K[[\mathbf{x}]]$ , we will always tacitly assume that  $G \leq K^*$  denotes a finitely generated

subgroup, and  $r \geq 1$  denotes a positive integer, such that every coefficient of  $F$  is contained in  $rG_0$ .

## 2 Preliminaries

For  $a, b \in \mathbb{Z}$ , let  $[a, b] := \{x \in \mathbb{Z} : a \leq x \leq b\}$ . Let  $\mathbb{N} = \{0, 1, 2, \dots\}$  and  $\mathbb{N}_{\geq k} = \{x \in \mathbb{N} : x \geq k\}$  for  $k \in \mathbb{N}$ .

### 2.1 Semilinear sets

We summarize a few results from the theory of semilinear sets, and refer to [Sak09, Chapter II.7.3] and [DIV12] for more details.

**Definition 2.1** Let  $d \geq 1$ . A subset  $\mathcal{A} \subseteq \mathbb{N}^d$  is:

- *linear* if there exist  $\mathbf{a}, \mathbf{b}_1, \dots, \mathbf{b}_l \in \mathbb{N}^d$  such that  $\mathcal{A} = \mathbf{a} + \mathbf{b}_1\mathbb{N} + \dots + \mathbf{b}_l\mathbb{N}$ ;
- *semilinear* if  $\mathcal{A}$  is a finite union of linear sets;
- *simple linear* if there exist  $\mathbf{a}, \mathbf{b}_1, \dots, \mathbf{b}_l \in \mathbb{N}^d$  such that  $\mathcal{A} = \mathbf{a} + \mathbf{b}_1\mathbb{N} + \dots + \mathbf{b}_l\mathbb{N}$  and  $\mathbf{b}_1, \dots, \mathbf{b}_l$  are linearly independent over  $\mathbb{Z}$ .

Whenever we consider a representation of a simple linear set of the form  $\mathbf{a} + \mathbf{b}_1\mathbb{N} + \dots + \mathbf{b}_l\mathbb{N}$ , we shall tacitly assume that the vectors  $\mathbf{b}_1, \dots, \mathbf{b}_l$  are taken to be linearly independent.

We make some observations on the uniqueness of the presentation of a linear set  $\mathcal{A}$ . Suppose  $\mathcal{A} = \mathbf{a} + \mathbf{b}_1\mathbb{N} + \dots + \mathbf{b}_l\mathbb{N}$  with  $\mathbf{a}, \mathbf{b}_1, \dots, \mathbf{b}_l$  as above. The element  $\mathbf{a}$  is uniquely determined by  $\mathcal{A}$ , as it is the minimum of  $\mathcal{A}$  in the coordinatewise partial order on  $\mathbb{N}^d$ . Therefore, also the associated monoid  $\mathcal{A} - \mathbf{a} \subseteq \mathbb{N}^d$  is uniquely determined by  $\mathcal{A}$ . The set  $\{\mathbf{b}_1, \dots, \mathbf{b}_l\}$  must contain every atom of  $\mathcal{A} - \mathbf{a}$ , that is, every element that cannot be written as a sum of two nonzero elements of  $\mathcal{A} - \mathbf{a}$ . If  $l$  is taken minimal, then  $\{\mathbf{b}_1, \dots, \mathbf{b}_l\}$  is equal to the set of atoms of  $\mathcal{A} - \mathbf{a}$ , and is therefore unique. In particular, if  $\mathcal{A}$  is simple linear and  $\mathbf{b}_1, \dots, \mathbf{b}_l$  are linearly independent, then the representation is unique (up to order of  $\mathbf{b}_1, \dots, \mathbf{b}_l$ ).

If  $\mathcal{A}$  and  $\mathcal{A}'$  are two linear sets with the same associated monoid, then there exist  $\mathbf{a}, \mathbf{a}', \mathbf{b}_1, \dots, \mathbf{b}_l \in \mathbb{N}^d$  with  $\mathcal{A} = \mathbf{a} + \mathbf{b}_1\mathbb{N} + \dots + \mathbf{b}_l\mathbb{N}$  and  $\mathcal{A}' = \mathbf{a}' + \mathbf{b}_1\mathbb{N} + \dots + \mathbf{b}_l\mathbb{N}$ . By choosing  $l$  minimal, the choice of  $\mathbf{b}_1, \dots, \mathbf{b}_l$  is again unique (up to order).

The semilinear subsets of  $\mathbb{N}^d$  are precisely the sets definable in the Presburger arithmetic of  $\mathbb{N}$ , by a theorem of Ginsburg and Spanier [GS66]. We shall make use of the following fundamental (but nontrivial) facts.

**Proposition 2.2** *The semilinear subsets of  $\mathbb{N}^d$  form a Boolean algebra under set-theoretic intersection and union. In particular, finite unions and finite intersections of semilinear sets, as well as complements of semilinear sets, are again semilinear.*

**Proof** By [Sak09, Proposition II.7.15], a subset of  $\mathbb{N}^d$  is semilinear if and only if it is rational. By [Sak09, Theorem II.7.3], the rational subsets of  $\mathbb{N}^d$  form a Boolean algebra. ■

One can show that every semilinear set is a finite union of simple linear sets. A stronger and much deeper result, which has been shown by Eilenberg and Schützenberger [ES69] and independently by Ito [Ito69], is the following.

**Proposition 2.3** *Every semilinear set is a finite disjoint union of simple linear sets.*

**Proof** For the proof of Ito, see [Ito69, Theorem 2]. Alternatively, one may apply the more general [ES69, Unambiguity Theorem] of Eilenberg and Schützenberger to the monoid  $\mathbb{N}^d$ . This second proof is also contained in the book of Sakarovitch, and one obtains the claim as follows: let  $S \subseteq \mathbb{N}^d$  be semilinear. Then  $S$  is a rational subset of  $\mathbb{N}^d$  (see [Sak09, Proposition II.7.5] and the discussion preceding it). By [Sak09, Theorem II.7.4] or [ES69, Unambiguity Theorem], every rational subset of  $\mathbb{N}^d$  is unambiguous. Again by [Sak09, Proposition II.7.15], every unambiguous rational subset of  $\mathbb{N}^d$  is a finite disjoint union of simple linear sets. ■

## 2.2 Unit equations

Unit equations play a central role in our proofs. We recall the fundamental finiteness result. For number fields, it was proved independently by Evertse [Eve84] and van der Poorten and Schlickewei [vdPS82]; the extension to arbitrary fields appears in [vdPS91]. We refer to [EG15, Chapter 6] or [BG06, Theorem 7.4.1] for more details.

Let  $G$  be a finitely generated subgroup of the multiplicative subgroup  $K^*$  of the field  $K$ . It is important here that  $\text{char } K = 0$ . Let  $m \geq 1$ . A solution  $(x_1, \dots, x_m) \in K^m$  to an equation of the form

$$(2.1) \quad X_1 + \dots + X_m = 0$$

is *nondegenerate* if  $\sum_{i \in I} x_i \neq 0$  for every  $\emptyset \subsetneq I \subsetneq [1, m]$ . In particular, if  $m \geq 2$ , then all  $x_i$  of a nondegenerate solution are nonzero.

**Proposition 2.4** (Evertse; van der Poorten and Schlickewei) *There exist only finitely many projective points  $(x_1 : \dots : x_m)$  with coordinates  $x_1, \dots, x_m \in G$  such that  $(x_1, \dots, x_m)$  is a nondegenerate solution to the unit equation (2.1).*

It is easily seen that there can be infinitely many degenerate solutions (even when considered as projective points), but one can recursively apply the theorem to subequations. In particular, we will commonly use an argument of the following type.

Let  $\Omega$  be some index set, and let  $g_1, \dots, g_m: \Omega \rightarrow G_0$  be maps such that

$$g_1(\omega) + \dots + g_m(\omega) = 0 \quad \text{for } \omega \in \Omega.$$

For every partition  $\mathcal{P} = \{I_1, \dots, I_t\}$  of the set  $[1, m]$ , let  $\Omega_{\mathcal{P}} \subseteq \Omega$  consist of all  $\omega$  such that for all  $j \in [1, t]$ , the tuple  $(g_i(\omega))_{i \in I_j}$  is a nondegenerate solution of the unit equation  $\sum_{i \in I_j} X_i = 0$ . Since every solution of a unit equation can be partitioned into nondegenerate solutions in at least one way, we obtain  $\Omega = \bigcup_{\mathcal{P}} \Omega_{\mathcal{P}}$ , where the union runs over all partitions of  $[1, m]$ .

Since there are only finitely many partitions of  $[1, m]$ , one can often deal with each  $\Omega_{\mathcal{P}}$  separately, or reduce to one  $\Omega_{\mathcal{P}}$  having some desirable property by an application of the pigeonhole principle. For example, if  $\Omega$  is infinite, then at least one  $\Omega_{\mathcal{P}}$  is infinite. Similarly, if  $\Omega = \mathbb{N}^d$ , then not all  $\Omega_{\mathcal{P}}$  can be contained in semilinear sets of rank at most  $d - 1$ , because  $\mathbb{N}^d$  cannot be covered by finitely many semilinear sets of rank  $\leq d - 1$ .

### 2.3 Hahn series

We recall that an abelian group  $G$  is totally ordered as a group if  $G$  is equipped with a total order  $\leq$  with the property that  $a + c \leq b + c$  whenever  $a, b, c \in G$  are such that  $a \leq b$ . For the group  $H = \mathbb{Q}^d$ , we will give  $H$  a total ordering that is compatible with the group structure by first picking positive linearly independent real numbers  $\varepsilon_1, \dots, \varepsilon_d$  and then declaring that  $(a_1, \dots, a_d) \leq (b_1, \dots, b_d)$  if and only if

$$\sum_{i=1}^d a_i \varepsilon_i \leq \sum_{i=1}^d b_i \varepsilon_i.$$

**Lemma 2.5** *The following hold for the above order:*

- $\mathbb{N}^d$  is a well-ordered subset of  $\mathbb{Q}^d$ .
- If  $(a_1, \dots, a_d) < (b_1, \dots, b_d)$  and if  $(c_1, \dots, c_d) \in \mathbb{Q}^d$ , then there is some  $N > 0$  such that  $n(b_1, \dots, b_d) > n(a_1, \dots, a_d) + (c_1, \dots, c_d)$  whenever  $n \geq N$ .

**Proof** Let  $S$  be a nonempty subset of  $\mathbb{N}^d$ . Pick  $(a_1, \dots, a_d) \in S$ , and let  $u := \sum_{i=1}^d a_i \varepsilon_i$ . Then, if  $(b_1, \dots, b_d) \in S$  is less than  $(a_1, \dots, a_d)$ , we must have  $b_i \leq u/\varepsilon_i$  for  $i \in [1, d]$ . Thus, there are only finitely many elements in  $S$  that are less than  $u$  and so  $S$  has a smallest element. Next, if  $(a_1, \dots, a_d) < (b_1, \dots, b_d)$ , then  $\theta := \sum_{i=1}^d (b_i - a_i) \varepsilon_i > 0$ . Then  $n\theta > \sum_{i=1}^d c_i \varepsilon_i$  for all  $n$  sufficiently large. ■

The second property is equivalent to  $\mathbb{Q}^d$  with the given order being archimedean. If  $G$  is a totally ordered abelian group, we can define the ring of *Hahn power series*

$$K((\mathbf{x}^G)) := \left\{ \sum_{g \in G} a_g \mathbf{x}^g : a_g \in K \text{ for } g \in G, \{g : a_g \neq 0\} \text{ is well ordered} \right\}.$$

Then  $K((\mathbf{x}^G))$ , together with the obvious operations, is a ring. For

$$F = \sum_{g \in G} a_g \mathbf{x}^g \in K((\mathbf{x}^G)),$$

one defines  $\text{supp}(F) := \{ \mathbf{x}^g : g \in G, a_g \neq 0 \}$  to be the support of  $F$ . We define  $[\mathbf{x}^g]F := a_g$ . Then there is a valuation  $v: K((\mathbf{x}^G)) \rightarrow G \cup \{\infty\}$ , defined as follows: one sets  $v(F) = g$  where  $\mathbf{x}^g$  is the minimal monomial in the support of  $F$  if  $F \neq 0$ , and  $v(0) = \infty$ .

For  $F_1 = \sum_{g \in G} a_g \mathbf{x}^g$  and  $F_2 = \sum_{g \in G} b_g \mathbf{x}^g \in K((\mathbf{x}^G))$ , the *Hadamard product* is defined as

$$F_1 \odot F_2 = \sum_{g \in G} a_g b_g \mathbf{x}^g.$$

If  $K$  is algebraically closed and  $G$  is divisible, then  $K((\mathbf{x}^G))$  is an algebraically closed field. In particular, if we use the order given above for  $H = \mathbb{Q}^d$ , we see that if  $K$  is algebraically closed, then  $K((\mathbf{x}^H))$  is algebraically closed. After making the identification  $x_i := \mathbf{x}^{e_i}$ , where  $e_i = (0, \dots, 1, \dots, 0)$  and where there is a 1 in the  $i$ th coordinate and zeros in all other coordinates, we have that the formal power series ring  $K[[x_1, \dots, x_d]]$  is a subalgebra of  $K((\mathbf{x}^H))$ .

We will find it convenient to write  $\mathbf{x}^{(a_1, \dots, a_d)} = x_1^{a_1} \dots x_d^{a_d}$  and write  $K((x_1^{\mathbb{Q}}, \dots, x_d^{\mathbb{Q}}))$  for  $K((\mathbf{x}^H))$ . In the other direction, we will find it convenient to abbreviate the power series ring as  $K[[\mathbf{x}]] = K[[x_1, \dots, x_d]]$ . These conventions are consistent with usual multi-index notation, and so for  $\mathbf{a} = (a_1, \dots, a_d)$  and  $\mathbf{b} = (b_1, \dots, b_d) \in \mathbb{Q}^d$ , we write  $\mathbf{a} + \mathbf{b} = (a_1 + b_1, \dots, a_d + b_d)$  and  $\mathbf{a}\mathbf{b} = (a_1 b_1, \dots, a_d b_d)$ . If  $\mathbf{a} \in \mathbb{Z}^d$  and  $\lambda = (\lambda_1, \dots, \lambda_d)$  with  $\lambda_i \in K((\mathbf{x}^H))$ , we also write  $\lambda^{\mathbf{a}} = \prod_{i=1}^d \lambda_i^{a_i}$ , although we will usually only use this for  $\lambda_i \in K^*$  or  $\lambda_i$  a monomial.

The set of [rational] Bézivin series is not closed under products or differentiation. However, it is closed under Hadamard products and it forms a  $K[\mathbf{x}]$ -submodule of  $K[[\mathbf{x}]]$ , as the following easy lemma shows.

**Lemma 2.6** *Let  $F \in K[[\mathbf{x}]]$  be a Bézivin series with coefficients in  $rG_0$ . If  $P \in K[\mathbf{x}]$  is a polynomial with  $s$  terms in its support, then there exists a set  $B \subseteq K$  of cardinality  $rs$  such that*

$$[\mathbf{x}^n]FP \in BG_0 = \sum_{b \in B} bG_0 \quad \text{for } \mathbf{n} \in \mathbb{N}^d.$$

In particular, the series  $PF$  is a Bézivin series with coefficients in  $rsG'_0$  for a suitable finitely generated subgroup  $G' \leq K^*$ .

We will need to understand the factorization of polynomials of the form  $1 - c\mathbf{x}^e$  with  $e \in \mathbb{N}^d$ .

**Lemma 2.7** *Let  $K$  be algebraically closed, and let  $e = (e_1, \dots, e_d) \in \mathbb{Z}^d \setminus \{0\}$ . In the factorial domain  $K[x_1^{\pm 1}] = K[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$ , the Laurent polynomial*

$$Q = 1 - c\mathbf{x}^e \quad \text{with } c \in K^*$$

*factors into irreducibles as*

$$Q = \prod_{j=1}^t (1 - \zeta^j b \mathbf{x}^{e/t}),$$

where  $t = \gcd(e_1, \dots, e_d)$ ,  $\zeta \in K^*$  is a primitive  $t$ -th root of unity, and  $b^t = c$ . In particular, the Laurent polynomial  $Q$  is irreducible if and only if  $\gcd(e_1, \dots, e_d) = 1$ .

**Proof** The proof follows an argument of Ostrowski [Ost76, Theorem IX]. Since  $(e_1/t, \dots, e_d/t)$  is a unimodular row, there exists a matrix  $A = (a_{i,j})_{i,j \in [1,d]} \in \text{GL}_d(\mathbb{Z})$  whose first row is  $(e_1/t, \dots, e_d/t)$ . This matrix  $A$  induces a ring automorphism  $\varphi$  of the Laurent polynomial ring  $K[x_1^{\pm 1}]$  with  $\varphi(x_i) = \prod_{j=1}^d x_j^{a_{i,j}}$ . Then  $\varphi^{-1}(Q) = 1 - cx_1^t$ . Since  $K[x_1^{\pm 1}]$  is divisor-closed in  $K[\mathbf{x}^{\pm 1}]$ , the problem reduces to that of factoring the univariate Laurent polynomial  $1 - cx_1^t$  in  $K[x_1^{\pm 1}]$ . Since  $K[x_1^{\pm 1}]$  is obtained from the factorial domain  $K[x_1]$  by inverting the prime element  $x_1$ , and clearly  $x_1$  is not a factor of  $1 - cx_1^t$  in  $K[x_1]$ , it suffices to consider the factorization of  $1 - cx_1^t$  in  $K[x_1]$ . However, here the result is clear. ■

### 3 Rational series with polynomial–exponential coefficients

In this section, we consider rational series whose denominator is a product of elements of the form  $1 - cu$  with  $c \in K^*$  and  $u \in K[\mathbf{x}]$  a nonconstant monomial. This will come



in handy in the later sections, as it will turn out that rational Bézivin series are always of such a form. For the class of rational series under investigation here, it is possible to give a fairly explicit structural description of their coefficient sequences, namely they are piecewise polynomial–exponential on simple linear subsets of  $\mathbb{N}^d$ .

**Definition 3.1** Let  $f: \mathbb{N}^d \rightarrow K$  be a sequence.

- The sequence  $f$  is *piecewise polynomial–exponential on simple linear subsets of  $\mathbb{N}^d$*  if there exists a partition of  $\mathbb{N}^d$  into simple linear sets  $\mathcal{S}_1, \dots, \mathcal{S}_m$  so that for each  $i \in [1, m]$ , there exist  $k_i \in \mathbb{N}$ , polynomials  $A_{i,1}, \dots, A_{i,k_i} \in K[x]$ , and  $\alpha_{i,1}, \dots, \alpha_{i,k_i} \in (K^*)^d$  such that

$$f(\mathbf{n}) = \sum_{j=1}^{k_i} A_{i,j}(\mathbf{n}) \alpha_{i,j}^{\mathbf{n}} \quad \text{for } \mathbf{n} \in \mathcal{S}_i.$$

- The sequence  $f$  is *piecewise polynomial on simple linear subsets of  $\mathbb{N}^d$*  if one can moreover take  $\alpha_{i,j} = (1, \dots, 1)$  for all  $i \in [1, m]$  and  $j \in [1, k_i]$ .
- The sequence  $f$  is *piecewise exponential on simple linear subsets of  $\mathbb{N}^d$*  if one can take the polynomials  $A_{i,j}$  to be constant for all  $i \in [1, m]$  and  $j \in [1, k_i]$ .

Note that in the piecewise exponential case, sums of exponentials ( $k_i > 1$ ) are allowed. There is no restriction on the ranks of the simple linear sets; the rank of  $\mathcal{S}_i$  may be smaller than  $d$ , and also need not be the same for  $\mathcal{S}_1, \dots, \mathcal{S}_m$ . Each of these three types of representation is trivially preserved under refinements of the partition. In particular, representations of the above types are not unique. It is not hard to see that every series  $F \in K[[\mathbf{x}]]$ , whose coefficient sequence is polynomial–exponential on simple subsets of  $\mathbb{N}^d$ , is rational (see Corollary 3.12). We give an easy example to illustrate the definition.

**Example 3.2** Consider the sequence  $f: \mathbb{N}^2 \rightarrow \mathbb{Q}$  defined by

$$\begin{aligned} \sum_{m,n=0}^{\infty} f(m,n) x^m y^n &= \frac{1}{1-2xy} + \frac{1}{1-3xy} + \frac{y}{(1-3xy)^2(1-5y)} + \frac{x}{(1-xy)(1-x)} \\ &= \sum_{k=0}^{\infty} (2^k + 3^k) x^k y^k + \sum_{k,l=0}^{\infty} (k+1) 3^k 5^l x^k y^{k+l+1} + \sum_{k,l=0}^{\infty} x^{k+l+1} y^l. \end{aligned}$$

Then

$$f(m,n) = \begin{cases} 2^m + 3^m, & \text{if } m = n, \\ \frac{1}{5}(m+1) \left(\frac{3}{5}\right)^m 5^n, & \text{if } m < n, \\ 1, & \text{if } m > n. \end{cases}$$

Since

$$\begin{aligned} \{ (n,n) \in \mathbb{N}^2 : n \in \mathbb{N} \} &= (1,1)\mathbb{N}, \\ \{ (m,n) \in \mathbb{N}^2 : m < n \} &= (0,1) + (1,1)\mathbb{N} + (0,1)\mathbb{N}, \text{ and} \\ \{ (m,n) \in \mathbb{N}^2 : m > n \} &= (1,0) + (1,1)\mathbb{N} + (1,0)\mathbb{N} \end{aligned}$$

are simple linear sets, the sequence  $f$  is piecewise polynomial–exponential on simple linear subsets of  $\mathbb{N}^2$ . However,  $f$  is neither piecewise polynomial nor piecewise

exponential on simple linear subsets of  $\mathbb{N}^2$ . The coefficient sequence of  $\frac{1}{1-2xy} + \frac{1}{1-3xy}$  is piecewise exponential on simple linear subsets of  $\mathbb{N}^2$ , and the coefficient sequence of  $\frac{x}{(1-xy)(1-x)}$  is piecewise polynomial on simple linear subsets of  $\mathbb{N}^2$ .

The following basic properties hold.

**Lemma 3.3** *Let  $\mathcal{PE}, \mathcal{P}, \mathcal{E} \subseteq K[[\mathbf{x}]]$  denote, respectively, the power series whose coefficient sequences are piecewise polynomial–exponential [polynomial, exponential] on simple linear subsets of  $\mathbb{N}^d$ .*

- (1) *Each of  $\mathcal{PE}, \mathcal{P}$ , and  $\mathcal{E}$  is a  $K[\mathbf{x}]$ -submodule of  $K[[\mathbf{x}]]$  and is closed under Hadamard products.*
- (2) *The sets  $\mathcal{PE}$  and  $\mathcal{P}$  are also closed under products and partial derivatives. In particular,  $\mathcal{PE}$  and  $\mathcal{P}$  form subalgebras of  $K[[\mathbf{x}]]$ .*

**Proof** (1) Let  $F$  and  $G$  be series that are [polynomial–exponential, polynomial, exponential] on simple linear subsets. It is clear that for every monomial  $u$  and every scalar  $\lambda \in K$ , also  $\lambda uF$  is of the same form. Thus, it suffices to show that the same is true for  $F + G$ . If  $\mathcal{S}$  and  $\mathcal{T} \subseteq \mathbb{N}^d$  are simple linear subsets, then the intersection  $\mathcal{S} \cap \mathcal{T}$  is semilinear. By Proposition 2.3, the intersection  $\mathcal{S} \cap \mathcal{T}$  can be represented as a finite disjoint union of simple linear sets. Thus, in representations of the coefficient sequences of  $F$  and  $G$  as in Definition 3.1, we may assume that the simple linear sets coincide for the two series, by passing to a common refinement. Then the claim about  $F + G$  is immediate.

(2) This can again be computed on each simple linear set. Alternatively, it will follow from Theorem 3.10 and Corollary 3.13. ■

The set  $\mathcal{E}$  is not closed under products or derivatives, since  $(1 - x)^{-1} \in \mathcal{E}$ , but  $(1 - x)^{-e} \in K[[x]]$  for  $e \geq 2$  is polynomial–exponential but not exponential on simple linear subsets of  $\mathbb{N}$ . We recall some easy facts about the algebraic independence of monomials.

**Lemma 3.4** *Let  $\mathbf{e}_1, \dots, \mathbf{e}_n \in \mathbb{N}^d$ , let  $c_1, \dots, c_n \in K^*$ , and let  $m_1, \dots, m_n \in \mathbb{N}_{\geq 1}$ . The following statements are equivalent.*

- (a) *The vectors  $\mathbf{e}_1, \dots, \mathbf{e}_n$  are linearly independent over  $\mathbb{Z}$ .*
- (b) *The monomials  $\mathbf{x}^{\mathbf{e}_1}, \dots, \mathbf{x}^{\mathbf{e}_n}$  generate a free abelian subgroup of the unit group of  $K[\mathbf{x}^{\pm 1}]$ .*
- (c) *The monomials  $\mathbf{x}^{\mathbf{e}_1}, \dots, \mathbf{x}^{\mathbf{e}_n}$  are algebraically independent over  $K$ .*
- (d) *The polynomials*

$$(1 - c_1 \mathbf{x}^{\mathbf{e}_1})^{m_1}, \dots, (1 - c_n \mathbf{x}^{\mathbf{e}_n})^{m_n}$$

*are algebraically independent over  $K$ .*

**Proof** (a)  $\Leftrightarrow$  (b) Clear.

(a)  $\Leftrightarrow$  (c) Consider the family  $(\mathbf{x}^{\mathbf{e}_1 c_1 + \dots + \mathbf{e}_n c_n})_{c_1, \dots, c_n \in \mathbb{N}}$  of monomials. The monomials in this family are pairwise distinct, and hence linearly independent over  $K$ , if and only if  $\mathbf{e}_1, \dots, \mathbf{e}_n$  are linearly independent. However, the linear independence of  $(\mathbf{x}^{\mathbf{e}_1 c_1 + \dots + \mathbf{e}_n c_n})_{c_1, \dots, c_n \in \mathbb{N}}$  is equivalent to the algebraic independence of  $\mathbf{x}^{\mathbf{e}_1}, \dots, \mathbf{x}^{\mathbf{e}_n}$ .

(c)  $\Leftrightarrow$  (d) If  $0 \neq P \in K[y_1, \dots, y_n]$ , then  $P$  vanishes on  $(1 - c_1 \mathbf{x}^{\mathbf{e}_1}, \dots, 1 - c_n \mathbf{x}^{\mathbf{e}_n})$  if and only if  $P(1 - c_1 y_1, \dots, 1 - c_n y_n)$  vanishes on  $(\mathbf{x}^{\mathbf{e}_1}, \dots, \mathbf{x}^{\mathbf{e}_n})$ . Hence,  $\mathbf{x}^{\mathbf{e}_1}, \dots, \mathbf{x}^{\mathbf{e}_n}$

are algebraically independent if and only if the polynomials  $1 - c_1 \mathbf{x}^{e_1}, \dots, 1 - c_n \mathbf{x}^{e_n}$  are algebraically independent.

Now, for any choice of polynomials  $f_1, \dots, f_n$ , the field  $K(f_1, \dots, f_n)$  is an algebraic extension of  $K(f_1^{m_1}, \dots, f_n^{m_n})$ , and therefore the two fields have the same transcendence degree over  $K$ . Thus,  $1 - c_1 \mathbf{x}^{e_1}, \dots, 1 - c_n \mathbf{x}^{e_n}$  is algebraically independent if and only if  $(1 - c_1 \mathbf{x}^{e_1})^{m_1}, \dots, (1 - c_n \mathbf{x}^{e_n})^{m_n}$  is algebraically independent. ■

Looking once more at Definition 3.1, in a sequence with piecewise polynomial-exponential coefficients on simple linear sets, for each simple linear set  $\mathcal{S}_i$ , we have  $f(\mathbf{n}) = \sum_{j=1}^{k_i} A_{i,j}(\mathbf{n}) \alpha_{i,j}^{\mathbf{n}}$  for  $\mathbf{n} \in \mathcal{S}_i$ . We can represent  $\mathcal{S}_i$  as  $\mathcal{S}_i = \mathbf{a} + \mathbf{b}_1 \mathbb{N} + \dots + \mathbf{b}_s \mathbb{N}$  with suitable  $\mathbf{a}, \mathbf{b}_1, \dots, \mathbf{b}_s \in \mathbb{N}^d$ , where  $\mathbf{b}_1, \dots, \mathbf{b}_s$  are linearly independent. On  $\mathcal{S}_i$ , we can therefore also consider representations of the form  $f(\mathbf{n}) = \sum_{j=1}^l B_{i,j}(\mathbf{m}) \beta_{i,j}^{\mathbf{m}}$  where  $\mathbf{m} = (m_1, \dots, m_s)$  with  $\mathbf{n} = \mathbf{a} + m_1 \mathbf{b}_1 + \dots + m_s \mathbf{b}_s$ , and  $B_{i,1}, \dots, B_{i,l} \in K[y_1, \dots, y_s]$ ,  $\beta_{i,1}, \dots, \beta_{i,l} \in (K^*)^s$ . One easily sees how the two notions relate.

**Lemma 3.5** *Let  $F = \sum_{\mathbf{n} \in \mathbb{N}^d} f(\mathbf{n}) \mathbf{x}^{\mathbf{n}} \in K[[\mathbf{x}]]$ , and let  $\mathcal{S} = \mathbf{a} + \mathbf{b}_1 \mathbb{N} + \dots + \mathbf{b}_s \mathbb{N}$  be simple linear. Consider the following statements.*

(a) *There exist polynomials  $A_1, \dots, A_l \in K[\mathbf{x}]$  and  $\alpha_1, \dots, \alpha_l \in (K^*)^d$  such that*

$$f(\mathbf{n}) = \sum_{j=1}^l A_j(\mathbf{n}) \alpha_j^{\mathbf{n}} \quad \text{for } \mathbf{n} \in \mathcal{S}.$$

(b) *There exist polynomials  $B_1, \dots, B_l \in K[y_1, \dots, y_s]$  and  $\beta_1, \dots, \beta_l \in (K^*)^s$  such that*

$$f(\mathbf{a} + \mathbf{b}_1 m_1 + \dots + \mathbf{b}_s m_s) = \sum_{j=1}^l B_j(\mathbf{m}) \beta_j^{\mathbf{m}} \quad \text{for } \mathbf{m} = (m_1, \dots, m_s) \in \mathbb{N}^s.$$

*Then (a)  $\Rightarrow$  (b). If  $K$  is algebraically closed, then also (b)  $\Rightarrow$  (a).*

**Proof** (a)  $\Rightarrow$  (b). By straightforward substitution.

(b)  $\Rightarrow$  (a). Let  $\mathbf{n} = (n_1, \dots, n_d) \in \mathcal{S}$ . Since  $\mathbf{b}_1, \dots, \mathbf{b}_s$  are linearly independent, there exist unique  $m_1, \dots, m_s \in \mathbb{N}^d$  with  $\mathbf{n} = \mathbf{a} + \mathbf{b}_1 m_1 + \dots + \mathbf{b}_s m_s$ . Solving this linear system over  $\mathbb{Q}$ , there exists  $N \in \mathbb{N}_{\geq 1}$  and, for every  $i \in [1, s]$  and  $j \in [1, d]$  integers  $p_i, q_{i,j}$  such that

$$m_i = p_i/N + \sum_{j=1}^d n_j q_{i,j}/N$$

for all  $\mathbf{n} = (n_1, \dots, n_d) \in \mathcal{S}$  and  $\mathbf{m} = (m_1, \dots, m_s) \in \mathbb{N}^s$  with  $\mathbf{n} = \mathbf{a} + \mathbf{b}_1 m_1 + \dots + \mathbf{b}_s m_s$ .

Let  $v \in [1, l]$ . Suppose  $\beta_v = (\beta_{v,1}, \dots, \beta_{v,s})$  and pick  $\gamma_{v,i} \in K^*$  with  $\gamma_{v,i}^N = \beta_{v,i}$  for  $i \in [1, s]$ . Set  $\alpha_v = (\alpha_{v,1}, \dots, \alpha_{v,d})$  with

$$\alpha_{v,j} := \prod_{i=1}^s \gamma_{v,i}^{q_{i,j}} \quad \text{for } j \in [1, d]$$

and

$$A_v(x_1, \dots, x_d) := B_v \left( p_1/N + \sum_{j=1}^d x_j q_{1,j}/N, \dots, p_s/N + \sum_{j=1}^d x_j q_{s,j}/N \right) \prod_{i=1}^s \gamma_{v,i}^{p_i}$$

If  $\mathbf{n} = \mathbf{a} + \mathbf{b}_1 m_1 + \dots + \mathbf{b}_s m_s$ , then

$$\begin{aligned} A_v(\mathbf{n}) \alpha_v^n &= B_v(\mathbf{m}) \prod_{i=1}^s \gamma_{v,i}^{p_i} \prod_{j=1}^d \alpha_{v,j}^{n_j} = B_v(\mathbf{m}) \prod_{i=1}^s \gamma_{v,i}^{p_i} \cdot \prod_{j=1}^d \prod_{i=1}^s \gamma_{v,i}^{q_{i,j} n_j} \\ &= B_v(\mathbf{m}) \prod_{i=1}^s \left( \gamma_{v,i}^{p_i} \prod_{j=1}^d \gamma_{v,i}^{q_{i,j} n_j} \right) = B_v(\mathbf{m}) \prod_{i=1}^s \gamma_{v,i}^{m_i N} \\ &= B_v(\mathbf{m}) \prod_{i=1}^s \beta_{v,i}^{m_i} = B_v(\mathbf{m}) \boldsymbol{\beta}_v^{\mathbf{m}}. \end{aligned}$$

The algebraic closure of  $K$  was only used to guarantee the existence of  $N$ th roots of the  $\beta_{v,i}$  in the proof of (b)  $\Rightarrow$  (a). Hence, the condition can clearly be weakened to the existence of suitable roots. For instance, if  $K = \mathbb{R}$  and all  $\beta_{v,j}$  are positive, the implication (b)  $\Rightarrow$  (a) holds. However, the condition cannot be entirely removed, as the following example shows.

**Example 3.6** Let

$$F = \frac{1}{1 - 2x^2} = \sum_{n=0}^{\infty} 2^n x^{2n} \in \mathbb{Q}[[x]].$$

Then  $f(1 + 2m) = 0$  and  $f(2m) = 2^m$  for all  $m \in \mathbb{N}$ . On  $2\mathbb{N}$ , we thus have a representation of the form (b).

However, suppose that for all  $n \in 2\mathbb{N}$ , there is a representation  $\sqrt{2}^n = \sum_{i=1}^l A_i(n) \alpha_i^n$  with polynomials  $A_1, \dots, A_l \in \mathbb{Q}[x]$  and pairwise distinct  $\alpha_1, \dots, \alpha_l \in \mathbb{Q}$ . Then  $2^m = \sum_{i=1}^l A_i(2m) \alpha_i^{2m}$  for all  $m \in \mathbb{N}$ . Since such a representation by an exponential polynomial is unique (see, e.g., [BR11, Corollary 6.2.2]), we must have  $\alpha_i^2 = 2$ , and hence without restriction  $l = 2$  and  $\alpha_1 = \sqrt{2}, \alpha_2 = -\sqrt{2}$ . Therefore,  $\alpha_i \notin \mathbb{Q}$ . Of course, over  $\mathbb{Q}(\sqrt{2})$ , one has  $f(n) = \frac{1}{2} \sqrt{2}^n + \frac{1}{2} (-\sqrt{2})^n$  for all  $n \in \mathbb{N}$ .

In a representation as in (a) of Lemma 3.5, it is easily seen that the polynomials are not uniquely determined: for instance, to represent the coefficient sequence  $f$  of  $\frac{1}{1-xy}$  on  $S = (1,1)\mathbb{N}$  as  $f(n, n) = A(n, n)$ , we can take any polynomial  $A \in 1 + (x - y)K[x, y]$ . However, we now show that the situation is better in a representation as in (b). The proof is essentially the same as the standard one in the univariate case (see, e.g., [BR11, Chapter 6.2]).

**Proposition 3.7** Let  $F = \sum_{\mathbf{n} \in \mathbb{N}^d} f(\mathbf{n}) \mathbf{x}^{\mathbf{n}} \in K[[\mathbf{x}]]$ , and let  $S = \mathbf{a} + \mathbf{b}_1 \mathbb{N} + \dots + \mathbf{b}_s \mathbb{N}$  be a simple linear set. Let  $B_1, \dots, B_l \in K[y_1, \dots, y_s] \setminus \{0\}$  and  $\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_l \in (K^*)^s$  where the vectors  $\boldsymbol{\beta}_j$  are pairwise distinct such that

$$f(\mathbf{a} + \mathbf{b}_1 m_1 + \dots + \mathbf{b}_s m_s) = \sum_{j=1}^l B_j(\mathbf{m}) \boldsymbol{\beta}_j^{\mathbf{m}} \quad \text{for } \mathbf{m} = (m_1, \dots, m_s) \in \mathbb{N}^s.$$

Then the polynomials  $B_j$  and the coefficient vectors  $\beta_j$  are uniquely determined (up to order).

**Proof** It suffices to consider  $S = \mathbb{N}^d$ . Let  $\Lambda \subseteq K^*$  be a finite set and  $e \in \mathbb{N}$ . Let  $A_e \subseteq K[[\mathbf{x}]]$  be the  $K$ -vector space of rational series spanned by rational functions of the form

$$\frac{1}{(1 - \lambda_1 x_1)^{k_1} \dots (1 - \lambda_d x_d)^{k_d}}$$

with  $\lambda_1, \dots, \lambda_d \in \Lambda$  and  $k_1, \dots, k_d \in [1, e]$ . Then  $\dim A_e = |\Lambda|^d e^d$ .

Now, in the group algebra  $K[\mathbf{x}][\langle \Lambda \rangle^d]$ , let  $V_e$  be the  $K$ -vector space consisting of elements of the form

$$\sum_{j=1}^l C_j(\mathbf{x}) \mathbf{y}_j,$$

with  $\mathbf{y}_j \in \Lambda^d$  and polynomials  $C_j \in K[\mathbf{x}]$  with  $\deg_{x_i}(C_j) \leq e - 1$  for all  $i \in [1, d]$  and  $j \in [1, l]$ . Then  $\dim V_e \leq |\Lambda^d| e^d$ .

For  $i \in [1, d]$  let  $\lambda_i \in \Lambda$  and  $k_i \in [1, e]$ . Since

$$\prod_{i=1}^d (1 - \lambda_i x_i)^{-k_i} = \sum_{n_1, \dots, n_d=0}^{\infty} \binom{k_1 + n_1 - 1}{n_1} \dots \binom{k_d + n_d - 1}{n_d} \lambda_1^{n_1} \dots \lambda_d^{n_d} x_1^{n_1} \dots x_d^{n_d},$$

the vector space  $V_e$  maps surjectively onto  $A_e$ . Since  $\dim V_e \leq \dim A_e$ , this is a bijection. This implies the claim. ■

We also record the following consequence.

**Corollary 3.8** Let  $u_0, u_1, \dots, u_s \in K[\mathbf{x}]$  be monomials with  $\mathbf{u} := (u_1, \dots, u_s)$  algebraically independent, and let  $C \subseteq K^*$  be a finite set. For  $\alpha = (\alpha_1, \dots, \alpha_s) \in C^s$  and  $\mathbf{e} = (e_1, \dots, e_s) \in \mathbb{N}_{\geq 1}^s$ , let

$$Q_{\alpha, \mathbf{e}} := (1 - \alpha_1 u_1)^{e_1} \dots (1 - \alpha_s u_s)^{e_s} \in K[\mathbf{u}].$$

Then, for  $F \in K[[\mathbf{x}]]$ , the following statements are equivalent.

- (1) For each  $\alpha \in C^s$  and  $\mathbf{e} = (e_1, \dots, e_s) \in \mathbb{N}_{\geq 1}^s$ , there exist  $\lambda_{\alpha, \mathbf{e}} \in K$ , all but finitely many of which are zero, such that

$$F = u_0 \sum_{\alpha \in C^s} \sum_{\mathbf{e} \in \mathbb{N}_{\geq 1}^s} \frac{\lambda_{\alpha, \mathbf{e}}}{Q_{\alpha, \mathbf{e}}}.$$

- (2) For each  $\alpha \in C^s$ , there exists a polynomial  $A_{\alpha} \in K[\mathbf{u}]$  such that

$$F = u_0 \sum_{\mathbf{n} \in \mathbb{N}^s} \sum_{\alpha \in C^s} A_{\alpha}(\mathbf{n}) \alpha^{\mathbf{n}} \mathbf{u}^{\mathbf{n}}.$$

Moreover, the polynomials  $A_{\alpha}$  are uniquely determined by  $F$ . If  $\alpha \in C^s$  and

$$m_i = \max\{e_i : \mathbf{e} = (e_1, \dots, e_i, \dots, e_s) \in \mathbb{N}^s \text{ with } \lambda_{\alpha, \mathbf{e}} \neq 0\}$$

(taking  $m_i = -\infty$  if the set is empty), then  $\deg_{u_i}(A_{\alpha}) = m_i - 1$ . In particular,  $A_{\alpha}$  is constant if and only if  $\lambda_{\alpha, \mathbf{e}} = 0$  whenever  $e_i > 1$  for some  $i \in [1, s]$ .

For the next step, we use a multivariate partial fraction decomposition. The result is due to Leinartas [Leĩ78], an easy to read exposition of the (short) proof in English is given by Raichev in [Rail2].

**Proposition 3.9** (Leinartas’s decomposition) *Let  $F = P/Q \in K(\mathbf{x})$  with  $P, Q \in K[\mathbf{x}]$  and  $Q \neq 0$ , and let  $Q = Q_1^{e_1} \cdots Q_t^{e_t}$  with  $Q_1, \dots, Q_t \in K[\mathbf{x}]$  irreducible and pairwise nonassociated and  $e_1, \dots, e_t \geq 1$ . Then*

$$F = \sum_{\mathbf{b}=(b_1,\dots,b_t) \in S} \frac{P_{\mathbf{b}}}{Q_1^{b_1} \cdots Q_t^{b_t}},$$

for a finite set  $S \subseteq \mathbb{N}^d$  and polynomials  $P_{\mathbf{b}} \in K[\mathbf{x}]$ , and for every  $\mathbf{b} = (b_1, \dots, b_t) \in S$ , the polynomials  $\{Q_i : b_i > 0\}$  appearing in the denominator are algebraically independent and have a common root in  $\overline{K}$ .

Unlike the univariate case, in this representation, unfortunately the exponents  $b_i$  can exceed  $e_i$  (see Example 3.11).

We can now characterize power series whose coefficient sequences are polynomial-exponential on simple linear subsets of  $\mathbb{N}^d$ .

**Theorem 3.10** *Let  $K$  be algebraically closed. The following statements are equivalent for a power series  $F = \sum_{\mathbf{n} \in \mathbb{N}^d} f(\mathbf{n})\mathbf{x}^{\mathbf{n}} \in K[[\mathbf{x}]]$ .*

(a) *One has*

$$F = \frac{P}{(1 - c_1 u_1) \cdots (1 - c_l u_l)}$$

with  $c_1, \dots, c_l \in K^*$ , with nonconstant monomials  $u_1, \dots, u_l \in K[\mathbf{x}]$ , and  $P \in K[\mathbf{x}]$ .

(b) *The sequence  $f$  is piecewise polynomial–exponential on simple linear subsets of  $\mathbb{N}^d$ .*

**Proof** (a)  $\Rightarrow$  (b) Lemma 2.7 allows us to assume that the polynomials  $1 - c_i u_i$  for  $i \in [1, l]$  are all irreducible. Gathering associated polynomials, and resetting the notation, we may write

$$F = \frac{P}{(1 - c_1 u_1)^{e_1} \cdots (1 - c_l u_l)^{e_l}},$$

with irreducible polynomials  $1 - c_i u_i$  that are pairwise coprime and  $e_1, \dots, e_l \geq 1$ .

By Proposition 3.9, we may now express

$$F = \sum_{I \subseteq [1, l]} \frac{P_I}{Q_I},$$

with  $P_I \in K[\mathbf{x}]$  and  $Q_I = \prod_{i \in I} (1 - c_i u_i)^{b_{I,i}}$  where  $b_{I,i} \geq 1$  for all  $I \subseteq [1, l]$  and  $i \in I$ , and the different irreducible factors of  $Q_I$  are algebraically independent. By Lemma 3.4, the monomials  $\{u_i : i \in I\}$  are algebraically independent for  $I \subseteq [1, l]$ . We can therefore apply Corollary 3.8 followed by (b)  $\Rightarrow$  (a) of Lemma 3.5, to deduce that every  $1/Q_I$  is piecewise polynomial–exponential on simple linear subsets of  $\mathbb{N}^d$ . By (1) of Lemma 3.3, the same is true for  $F$ .

(b)  $\Rightarrow$  (a) By assumption, there exists a partition of  $\mathbb{N}^d$  into simple linear sets such that on each simple linear set  $S$ , the coefficients of  $F$  can be represented as in (a) of

Lemma 3.5. Applying (a)  $\Rightarrow$  (b) of Lemma 3.5, followed by Corollary 3.8, we see that  $F$  is a  $K$ -linear combination of rational functions of the form

$$\frac{u_0}{(1 - c_1 u_1) \cdots (1 - c_s u_s)}$$

with  $c_1, \dots, c_s \in K^*$  and monomials  $u_0, u_1, \dots, u_s \in K[\mathbf{x}]$ . ■

**Example 3.11** The exponents in the multivariate partial fraction decomposition can increase. For instance,

$$F = \frac{1}{(1 - x)(1 - y)(1 - xy)} = \frac{x}{(1 - x)(1 - xy)^2} + \frac{1}{(1 - y)(1 - xy)^2}.$$

So, despite the fact that the denominator of  $F$  has no repeated factors, the coefficient polynomials in a polynomial–exponential decomposition may be nonconstant. In the example, we get  $[x^n y^n]F = (n + 1)$  for  $n \in \mathbb{N}$ .

The following is a corollary of the trivial direction of Theorem 3.10 (and hence could have been observed before).

**Corollary 3.12** *If the coefficient sequence of  $F \in K[[\mathbf{x}]]$  is piecewise polynomial–exponential on simple linear subsets of  $\mathbb{N}^d$ , then  $F$  is rational.*

Looking at the proof of Theorem 3.10 again, we see that the special case in which every constant in the denominator is a root of unity corresponds to the coefficient sequence being piecewise polynomial on simple linear subsets of  $\mathbb{N}^d$ . Hence, we recover the following (well-known) result.

**Corollary 3.13** *Let  $K$  be algebraically closed. The following statements are equivalent for a power series  $F = \sum_{\mathbf{n} \in \mathbb{N}^d} f(\mathbf{n})\mathbf{x}^{\mathbf{n}} \in K[[\mathbf{x}]]$ .*

(a) *One has*

$$F = \frac{P}{(1 - \omega_1 u_1) \cdots (1 - \omega_l u_l)}$$

*with  $\omega_1, \dots, \omega_l \in K^*$  roots of unity and with nonconstant monomials  $u_1, \dots, u_l \in K[\mathbf{x}]$ , and  $P \in K[\mathbf{x}]$ .*

(b) *The sequence  $f$  is piecewise polynomial on simple linear subsets of  $\mathbb{N}^d$ .*

**Remark 3.14** Generating series of the form

$$\frac{1}{(1 - u_1) \cdots (1 - u_l)}$$

with monomials  $u_1, \dots, u_l$  play a role in several areas of mathematics. They count the number of solutions to a Diophantine linear system over the natural numbers [DM88], and appear in combinatorics [Sta12, Chapter 4.6], representation theory [Hec82], and commutative algebra (as Hilbert series) [MS05, Sta96].

The coefficient sequences of such series are called *vector partition functions*. After some earlier work by Blakley [Bla64] and Dahmen and Micchelli [DM88], Sturmfels [Stu95] gave a description of their structure in terms of the chamber complex of the matrix whose columns are the exponents of the monomials  $u_1, \dots, u_l$ .

From a structural point of view, the set  $\mathbb{N}^d$  is partitioned into finitely many polyhedral cones with apex at the origin, and on each such cone the vector partition function  $f$  is given by a multivariate quasi-polynomial (however, the results in [Stu95] are much more specific than this description). This type of decomposition is equivalent to  $f$  being polynomial on simple linear subsets of  $\mathbb{N}^d$ —a good overview of the connection with the theory of semilinear sets is given by D’Alessandro, Intrigila, and Varricchio [DIV12],<sup>1</sup> and the specific result can be found in [DIV12, Proposition 2 and Corollary 1].

Proofs of this decomposition (e.g., in [Stu95]) typically involve results on counting lattice points in polytopes. While we only derived the structural form of the decomposition here, we only needed purely algebraic methods. Neither the generalization to the polynomial-exponential case nor the use of purely algebraic methods appears to be entirely new: positive real coefficients were also permitted, for instance, by Brion and Vergne [BV97]. A purely algebraic derivation of the polynomial case is given by Fields in [Fie00, Fie02], which we have discovered through O’Neill [O’N17, Theorem 2.10]. The result on the partial fraction decomposition is not used by Fields, but similar reductions are used for the denominators. The partial fraction decomposition in connection with multivariate rational generating series is used by Mishna in [Mis20, Chapter 92]. However, no structural decomposition result is derived there.

Since we could not locate a reference that derives the structural result for the polynomial-exponential case over an arbitrary field, we have chosen to include a proof, although we assume that this is at least “essentially known.”

### 3.1 The exponential case

We now look at the other extremal case, where the coefficients are piecewise exponential on simple linear subsets of  $\mathbb{N}^d$ .

**Definition 3.15** Let  $F \in K[[\mathbf{x}]]$ .

- (1) The series  $F$  is *skew-geometric* if there exist  $c_0 \in K, c_1, \dots, c_l \in K^*$  and monomials  $u_0, u_1, \dots, u_l$  such that  $u_1, \dots, u_l$  are algebraically independent and

$$F = \frac{c_0 u_0}{(1 - c_1 u_1) \cdots (1 - c_l u_l)}.$$

- (2) The series  $F$  is *geometric* if moreover  $\{u_1, \dots, u_l\} \subseteq \{x_1, \dots, x_d\}$ .

For a skew-geometric series  $F$  as above, the exponents of the monomials in the support of  $F$  form a simple linear set. The coefficient of  $u_0 u_1^{e_1} \cdots u_l^{e_l}$  in  $F$  is  $c_0 c_1^{e_1} \cdots c_l^{e_l}$ . Every skew-geometric series is rational by definition. Moreover, its coefficient series is exponential on simple linear subsets of  $\mathbb{N}^d$ .

**Definition 3.16** Let  $F_1, \dots, F_n \in K[[\mathbf{x}]]$ .

- (1) The sum  $F_1 + \dots + F_n$  is *unambiguous* if the sets  $\text{supp}(F_1), \dots, \text{supp}(F_n)$  are pairwise disjoint.

---

<sup>1</sup>The definition of a simple linear set in the paper contains a typo; only  $b_1, \dots, b_n$  are supposed to be linearly independent.



(2) The sum  $F_1 + \dots + F_n$  is *trivially ambiguous* if for all  $i, j \in [1, n]$ ,

$$\text{supp}(F_i) \cap \text{supp}(F_j) = \emptyset \quad \text{or} \quad \text{supp}(F_i) = \text{supp}(F_j).$$

**Example 3.17** For a set  $\mathcal{S} \subseteq \mathbb{N}^d$ , let

$$\mathbf{1}_{\mathcal{S}} := \sum_{\mathbf{n} \in \mathcal{S}} \mathbf{x}^{\mathbf{n}} \in K[[\mathbf{x}]].$$

If  $\mathcal{S} = \mathbf{a} + \mathbf{b}_1\mathbb{N} + \dots + \mathbf{b}_l\mathbb{N}$  is a simple linear set, then

$$\mathbf{1}_{\mathcal{S}} = \frac{\mathbf{x}^{\mathbf{a}}}{(1 - \mathbf{x}^{\mathbf{b}_1}) \dots (1 - \mathbf{x}^{\mathbf{b}_l})}$$

is skew-geometric. If  $\mathcal{S}$  is semilinear, then it is a finite disjoint union of simple linear sets, and therefore  $\mathbf{1}_{\mathcal{S}}$  is an unambiguous sum of skew-geometric series.

**Lemma 3.18** (1) If  $F \in K[[\mathbf{x}]]$  is skew-geometric, and  $\mathcal{S} \subseteq \mathbb{N}^d$  is a simple linear set with  $\{\mathbf{x}^{\mathbf{n}} : \mathbf{n} \in \mathcal{S}\} \subseteq \text{supp}(F)$ , then  $F \odot \mathbf{1}_{\mathcal{S}}$  is skew-geometric.

(2) If a series  $F \in K[[\mathbf{x}]]$  has a coefficient sequence that is piecewise exponential on simple linear subsets of  $\mathbb{N}^d$ , then  $F$  is a sum of skew-geometric series. If  $K$  is algebraically closed, the converse holds.

(3) Every sum of skew-geometric series can be expressed as a trivially ambiguous sum of skew-geometric series.

**Proof** (1) Suppose

$$F = \frac{c_0 \mathbf{x}^{\mathbf{a}}}{(1 - c_1 \mathbf{x}^{\mathbf{b}_1}) \dots (1 - c_l \mathbf{x}^{\mathbf{b}_l})},$$

with  $(\mathbf{b}_1, \dots, \mathbf{b}_l) \in \mathbb{N}^d$  linearly independent,  $\mathbf{a} \in \mathbb{N}^d$ , and  $c_0, c_1, \dots, c_l \in K^*$  (if  $c_0 = 0$  there is nothing to show). Let  $\mathcal{S}_0 = \mathbf{a} + \mathbf{b}_1\mathbb{N} + \dots + \mathbf{b}_l\mathbb{N}$  and suppose  $\mathcal{S} = \mathbf{p} + \mathbf{q}_1\mathbb{N} + \dots + \mathbf{q}_s\mathbb{N}$  with  $\mathcal{S} \subseteq \mathcal{S}_0$ .

Let  $\mathbf{p} = \mathbf{a} + \mu_1 \mathbf{b}_1 + \dots + \mu_l \mathbf{b}_l$  with  $\mu_1, \dots, \mu_l \in \mathbb{N}$ . Since  $\mathbf{p} + \mathbf{q}_i \in \mathcal{S}_0$  for all  $i \in [1, s]$ , we have  $\mathbf{q}_i = t_{i,1} \mathbf{b}_1 + \dots + t_{i,l} \mathbf{b}_l$  with  $t_{i,j} \in \mathbb{Z}$ . Since also  $\mathbf{p} + n\mathbf{q}_i \in \mathcal{S}_0$  for all  $n \in \mathbb{N}$ , we must have  $t_{i,j} \geq 0$  for all  $i \in [1, s]$  and  $j \in [1, l]$ . Now, if  $\mathbf{n} = \mathbf{p} + m_1 \mathbf{q}_1 + \dots + m_s \mathbf{q}_s \in \mathcal{S}$ , then

$$[\mathbf{x}^{\mathbf{n}}]F = c_0 \prod_{j=1}^l c_j^{\mu_j + \sum_{i=1}^s m_i t_{i,j}} = c_0 \prod_{j=1}^l c_j^{\mu_j} \cdot \prod_{i=1}^s \left( \prod_{j=1}^l c_j^{t_{i,j}} \right)^{m_i},$$

and so

$$F \odot \mathbf{1}_{\mathcal{S}} = \frac{d_0 \mathbf{x}^{\mathbf{p}}}{(1 - d_1 \mathbf{x}^{\mathbf{q}_1}) \dots (1 - d_s \mathbf{x}^{\mathbf{q}_s})},$$

for suitable  $d_0, d_1, \dots, d_s \in K^*$ . (The crucial observation in this straightforward proof was  $\mathbf{q}_i \in \mathbf{b}_1\mathbb{N} + \dots + \mathbf{b}_l\mathbb{N}$ .)

(2) Suppose  $F$  has a coefficient sequence that is piecewise exponential on simple linear subsets. We apply the direction (a)  $\Rightarrow$  (b) of Lemma 3.5, and observe that since the polynomials  $A_j$  are constant, so are the  $B_j$ . Then it is immediate that  $F$  is a sum of skew-geometric series. For the converse direction, we analogously use (b)  $\Rightarrow$  (a) of Lemma 3.5.

(3) The semilinear sets form a Boolean algebra, and every semilinear set is a finite disjoint union of simple linear sets. Given any sum of skew-geometric series, we can use (1) to refine their support in such a way that the sum can be represented as a trivially ambiguous one. ■

### 4 Rationality of $D$ -finite Bézivin series

In this section, we prove the following theorem.

**Theorem 4.1** *If  $F \in K[[x]]$  is  $D$ -finite and Bézivin, then  $F$  is rational.*

To do so, we first recall the notion of a  $P$ -recursive sequence, introduced by Lipshitz [Lip89], and prove a lemma that encapsulates, and extends to a multivariate setting, the crucial part of Bézivin’s argument. This lemma and its consequences will prove useful on several occasions.

Let  $f: \mathbb{N}^d \rightarrow K$  be a  $d$ -dimensional sequence. A  $k$ -section of  $f$  is a sequence (of dimension  $< d$ ) obtained by fixing some of the coordinates of  $f$  at values  $\leq k$ , where  $k \in \mathbb{N}$ . Formally, a  $k$ -section is a sequence  $g: \mathbb{N}^J \rightarrow K$  with  $J \not\subseteq [1, d]$  and  $c_i \in [0, k - 1]$  for every  $i \in [1, d] \setminus J$  such that

$$g((n_j)_{j \in J}) = f(n_1, \dots, n_d) \quad \text{with} \quad n_i = c_i \quad \text{for} \quad i \in [1, d] \setminus J.$$

**Definition 4.2** A sequence  $f: \mathbb{N}^d \rightarrow K$  is  $P$ -recursive of size  $k \geq 0$  if the following two conditions are satisfied.

- (i) For every  $j \in [1, d]$  and every  $\mathbf{a} \in [0, k]^d$ , there exist polynomials  $Q_{j,\mathbf{a}} \in K[y]$  such that

$$(4.1) \quad \sum_{\mathbf{a} \in [0, k]^d} Q_{j,\mathbf{a}}(n_j) f(\mathbf{n} - \mathbf{a}) = 0 \quad \text{for all} \quad \mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}_{\geq k}^d,$$

and for every  $j \in [1, d]$ , there exists at least one  $\mathbf{a} \in [0, k]^d$  with  $Q_{j,\mathbf{a}} \neq 0$ .

- (ii) If  $d > 1$ , then all  $k$ -sections of  $f(\mathbf{n})$  are  $P$ -recursive.

The sequence  $f$  is  $P$ -recursive if it is  $P$ -recursive of size  $k$  for some  $k \geq 0$ .

We remark that the sum in the recursion runs over a  $d$ -dimensional hypercube, but the coefficient polynomials are univariate.

By a theorem of Lipshitz, a power series is  $D$ -finite if and only if its sequence of coefficients is  $P$ -recursive [Lip89, Theorem 3.7]. While this is not completely obvious, it is possible to use these recursions and finitely many initial values to compute arbitrary values of  $f$  [Lip89, Section 3.10]. In particular, if  $f$  is  $P$ -recursive, then there exists a finitely generated subfield  $K_0 \subseteq K$  such that  $f(\mathbf{n}) \in K_0$  for all  $\mathbf{n} \in \mathbb{N}^d$ .

Every section of a  $P$ -recursive sequence, that is, a subsequence obtained by fixing some of the arguments, is again  $P$ -recursive [Lip89, Theorem 3.8(iv)].

We need the following easy observation.

**Lemma 4.3** *Let  $f: \mathbb{N}^d \rightarrow K$  be  $P$ -recursive of size  $k$ , let  $j \in [1, d]$ , and let  $Q_{j,\mathbf{a}} \in K[y]$  be polynomials, not all zero, such that*

$$\sum_{\mathbf{a} \in [0, k]^d} Q_{j,\mathbf{a}}(n_j) f(\mathbf{n} - \mathbf{a}) = 0 \quad \text{for all} \quad \mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}_{\geq k}^d.$$

Furthermore, let  $c \in \mathbb{N}$  be sufficiently large so that  $Q_{j,a}(c') \neq 0$  whenever  $c' \geq c$  and  $Q_{j,a} \neq 0$ . If there exist  $l_1, \dots, l_d \geq c$  such that  $f(\mathbf{n})$  vanishes on

$$\bigcup_{i=1}^d \mathbb{N}_{\geq c}^{i-1} \times [l_i, l_i + k - 1] \times \mathbb{N}_{\geq c}^{d-i},$$

then  $f(\mathbf{n})$  vanishes on  $\mathbb{N}_{\geq l_1} \times \dots \times \mathbb{N}_{\geq l_d}$ .

**Proof** Fix a well-order on  $\mathbb{N}^d$  for which  $(\mathbb{N}^d, +)$  is an ordered semigroup (for instance, a lexicographical order). We proceed by contradiction and assume that  $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{N}_{\geq l_1} \times \dots \times \mathbb{N}_{\geq l_d}$  is minimal with  $f(\mathbf{m}) \neq 0$ . Then  $m_i \geq l_i + k$  for all  $i \in [1, d]$ . Let  $\mathbf{b} \in [0, k]^d$  be the minimum, with respect to the well-order, with  $Q_{j,b} \neq 0$ . Taking  $\mathbf{n} = \mathbf{m} + \mathbf{b}$ , we see that equation (4.1) allows us to express  $f(\mathbf{m})$  as a linear combination of certain  $f(\mathbf{m}')$  with  $\mathbf{m}' < \mathbf{m}$  and  $\mathbf{m}' \in \mathbb{N}_{\geq l_1} \times \dots \times \mathbb{N}_{\geq l_d}$ , showing  $f(\mathbf{m}) = 0$ , a contradiction. ■

The following proof is a straightforward, if somewhat technical, adaption of Bézivin’s argument [Bé86] to cover multidimensional sequences. The restriction to a finitely generated field ensures the applicability of the following lemma.

**Lemma 4.4** *If  $K$  is a finitely generated as a field, then*

$$\sqrt{G} := \{x \in K^* : x^n \in G \text{ for some } n \geq 1\}$$

*is a finitely generated group.*

**Proof** By assumption,  $G$  is a finitely generated group. Therefore, there exists a finitely generated  $\mathbb{Z}$ -subalgebra  $A \subseteq K$  containing  $G$ . By a theorem of Nagata, the integral closure  $\bar{A}$  of  $A$  is a finitely generated  $A$ -module (see [Mat80, Theorem 31.H in Chapter 12]). Therefore,  $\bar{A}$  is a finitely generated  $\mathbb{Z}$ -algebra as well. A theorem of Roquette (see [Lan83, Corollary 7.5 of Chapter 2]) implies that the group of units  $(\bar{A})^\times$  is finitely generated. Since  $\sqrt{G} \subseteq (\bar{A})^\times$ , the claim follows. ■

A subgroup  $G \leq K^*$  is *root-closed* (in  $K^*$ ) if  $\sqrt{G} = G$ . Note that always  $\sqrt{G} \not\subseteq K^*$ . We can even find  $l \in (\mathbb{N}_{\geq 1} \setminus \sqrt{G}) \subseteq K^*$ , say, by choosing for  $l$  a suitable prime number.

**Lemma 4.5** *Suppose that the field  $K$  is finitely generated. Let  $G \leq K^*$  be a finitely generated subgroup and  $G_0 := G \cup \{0\}$ .*

*Let  $\Omega$  be a set, let  $n \geq 0$ , and let  $f_1, \dots, f_n: \Omega \rightarrow G_0$  and  $\pi: \Omega \rightarrow K$  be maps such that there exist polynomials  $Q_1, \dots, Q_n \in K[y]$  with*

$$\sum_{j=1}^n Q_j(\pi(\omega)) f_j(\omega) = 0 \quad \text{for all } \omega \in \Omega.$$

*Then, for all finite subsets  $B \subseteq K$ , there exists a finitely generated root-closed group  $G' \supseteq G$ , such that the following holds: for all  $l \in K^* \setminus G'$ , there exists  $e_0 \in \mathbb{N}$  such that*

$$(4.2) \quad \sum_{j=1}^n Q_j(0) f_j(\omega) = 0$$

*for all  $\omega \in \Omega$  with  $\pi(\omega) \in C$ , where*

$$C = \{l^e + b : e \in \mathbb{N}_{\geq e_0} \text{ and } b \in B\} \subseteq K.$$

**Proof** Let  $G'$  be a finitely generated group containing  $G$  and all coefficients of the polynomials  $Q_j(y + b)$  where  $b \in B$ . We may moreover assume  $\sqrt{G'} = G'$ . Enlarging  $G$  if necessary, we even assume  $G' = G = \sqrt{G}$  for notational simplicity.

If  $l \in K^* \setminus G$ , then in particular  $l$  is not a root of unity, and thus for every  $b \in K$ , the set  $\{l^e + b : e \in \mathbb{N}\}$  is infinite. Moreover, if  $l$  and  $b$  are fixed, then the exponent  $e$  is uniquely determined by  $l^e + b$ .

We proceed by induction on  $|B|$ . For  $B = \emptyset$ , there is nothing to show because then  $C = \emptyset$ . So fix  $b \in B$  such that  $B = B' \sqcup \{b\}$  with  $|B'| < |B|$ . Applying the induction hypothesis, there exists  $e'_0 \in \mathbb{N}$  such that (4.2) holds for all  $\omega \in \Omega$  with  $\pi(\omega) \in C'$ , where

$$C' := \{l^e + b' : e \in \mathbb{N}_{\geq e'_0}, b' \in B'\}.$$

Let

$$\Omega_b := \{\omega \in \Omega : \pi(\omega) = l^e + b \text{ with } e \in \mathbb{N}\}.$$

For  $\omega \in \Omega_b$  with  $\pi(\omega) = l^e + b$ , define  $\varepsilon(\omega) := e$ . For  $j \in [1, n]$ , let

$$Q_j(y + b) = \sum_{s=0}^m q_{j,s} y^s \quad \text{with} \quad q_{j,s} \in K.$$

Then

$$0 = \sum_{j=1}^n Q_j(\pi(\omega)) f_j(\omega) = \sum_{j=1}^n \sum_{s=0}^m q_{j,s} l^{\varepsilon(\omega)s} f_j(\omega)$$

may be considered as a solution to the unit equation  $X_1 + \dots + X_{n(m+1)} = 0$  over the group  $\langle G, l \rangle$ . Let  $I = [1, n] \times [0, m]$ . For every partition  $\mathcal{P} = \{I_1, \dots, I_p\}$  of  $I$ , let  $\Omega_{\mathcal{P}} \subseteq \Omega_b$  be the set satisfying: for all  $v \in [1, p]$ ,

- $\sum_{(j,s) \in I_v} q_{j,s} l^{\varepsilon(\omega)s} f_j(\omega) = 0$ , and
- $\sum_{(j,s) \in I} q_{j,s} l^{\varepsilon(\omega)s} f_j(\omega) \neq 0$  for all  $\emptyset \neq I \not\subseteq I_v$ .

Since the sets  $\Omega_{\mathcal{P}}$ , with  $\mathcal{P}$  ranging over all partitions, cover  $\Omega_b$ , it is sufficient to establish the claim of the lemma for each  $\Omega_{\mathcal{P}}$  separately. So fix  $\mathcal{P}$ . If  $\varepsilon(\Omega_{\mathcal{P}})$  is finite, the claim is trivially true by choosing  $e_0 \geq e'_0$  sufficiently large. We may therefore assume that  $\varepsilon(\Omega_{\mathcal{P}})$  is infinite.

The crucial step lies in showing that in the subsum indexed by  $I_v$ , all the monomials that occur must be equal. Thus, explicitly, we show: if  $(j_1, s_1), (j_2, s_2) \in I_v$  for some  $v \in [1, p]$ , then  $s_1 = s_2$ . Clearly, we only have to consider the case where  $|I_v| \geq 2$ . Then  $q_{j_1, s_1} l^{\varepsilon(\omega)s_1} f_{j_1}(\omega) \neq 0$  and  $q_{j_2, s_2} l^{\varepsilon(\omega)s_2} f_{j_2}(\omega) \neq 0$  for all  $\omega \in \Omega_{\mathcal{P}}$ . By construction,

$$\sum_{(j,s) \in I_v} q_{j,s} l^{\varepsilon(\omega)s} f_j(\omega) = 0$$

is a nondegenerate solution to the unit equation  $X_1 + \dots + X_{|I_v|} = 0$  over  $\langle G, l \rangle$ . Thus, there is a finite set  $Y$  with

$$l^{\varepsilon(\omega)(s_1 - s_2)} \underbrace{q_{j_1, s_1} q_{j_2, s_2}^{-1} f_{j_1}(\omega) f_{j_2}(\omega)^{-1}}_{\in G} \in Y.$$

for all  $\omega \in \Omega_{\mathcal{P}}$ . Then  $l^{\varepsilon(\omega)(s_1 - s_2)} \in \bigcup_{y \in Y} yG$  for all  $\omega \in \Omega_{\mathcal{P}}$ .

Since  $\varepsilon(\omega)$  takes infinitely many values, there exist  $y \in Y$  and  $\omega, \omega' \in \Omega_{\mathcal{P}}$  with  $\varepsilon(\omega) \neq \varepsilon(\omega')$ , and

$$l^{\varepsilon(\omega)(s_1-s_2)}, l^{\varepsilon(\omega')(s_1-s_2)} \in yG.$$

But then,  $l^{(\varepsilon(\omega)-\varepsilon(\omega'))(s_1-s_2)} \in G$ . By choice of  $l$ , this implies  $s_1 = s_2$ .

Taking the union over all  $I_v$  containing a fixed  $s \in [0, m]$  in the second coordinate, we conclude

$$l^{\varepsilon(\omega)s} \sum_{j=1}^n q_{j,s} f_j(\omega) = 0,$$

and therefore  $\sum_{j=1}^n q_{j,s} f_j(\omega) = 0$  for all  $s \in [0, m]$ . But then,

$$0 = \sum_{j=1}^n \sum_{s=0}^m q_{j,s} y^s f_j(\omega) = \sum_{j=1}^n Q_j(y+b) f_j(\omega)$$

for all  $\omega \in \Omega_{\mathcal{P}}$ . Substituting  $y = -b$  into this polynomial, we obtain

$$\sum_{j=1}^n Q_j(0) f_j(\omega) = 0 \quad \text{for all } \omega \in \Omega_{\mathcal{P}}. \quad \blacksquare$$

**Remark 4.6** Since  $K$  has characteristic 0, the group  $K^*$  is never finitely generated. In particular, one can always find  $l$  as required in the previous theorem.

As a first easy consequence, we obtain the following.

**Lemma 4.7** Let  $P \in K[y]$  be a polynomial, and suppose there exists  $r \in \mathbb{N}$  such that  $P(n) \in rG_0$  for all sufficiently large  $n \in \mathbb{N}$ . Then  $P$  is constant.

**Proof** Working in a subfield, we may assume that  $K$  is a finitely generated field. Let  $P = \sum_{i=0}^t c_i y^i$  with  $t \in \mathbb{N}$  and  $c_0, \dots, c_t \in K$ . Fix  $n_0 \in \mathbb{N}$  such that  $P(n) \in rG_0$  for  $n \geq n_0$ , and let  $g_1, \dots, g_r: \mathbb{N} \rightarrow G_0$  be such that

$$P(n) - g_1(n) - \dots - g_r(n) = 0 \quad \text{for } n \geq n_0.$$

We will apply a simple special case of Lemma 4.5 to this equation. To do so, we choose  $\Omega = \mathbb{N}$  with  $\pi: \mathbb{N} \rightarrow K$  given by  $\pi(n) = n$ ,  $Q_1 = P$ ,  $f_1 = 1$ , and  $f_j = g_{j-1}$ ,  $Q_j = -1$  for  $j \in [2, r+1]$ . For the set  $B$ , we choose  $B = \{0\}$ . By Lemma 4.5, there exists  $l \in \mathbb{N}_{\geq 1} \setminus \sqrt{G}$  and  $e_0 \in \mathbb{N}$  such that, for all  $e \geq e_0$ ,

$$P(0) = g_1(l^e) + \dots + g_r(l^e).$$

This implies  $P(0) = P(l^e)$  for  $e \geq e_0$ . Since a nonconstant polynomial can take a fixed value at most  $\deg(P) - 1$  times, it follows that  $P$  is constant.  $\blacksquare$

This immediately applies to series with polynomial-exponential coefficients as follows.

**Lemma 4.8** Let  $Q_1, \dots, Q_l \in K[x]$ , let  $\lambda_1, \dots, \lambda_l \in (K^*)^d$  be pairwise distinct, and let

$$F = \sum_{n \in \mathbb{N}^d} \left( \sum_{j=1}^l Q_j(n) \lambda_j^n \right) x^n.$$

If  $F$  is a Bézivin series, then  $Q_1, \dots, Q_l$  are constant.

**Proof** By Corollary 3.8, we have  $F = P/Q$  with  $Q = (1 - \alpha_1 y_1) \cdots (1 - \alpha_s y_s)$  where  $\alpha_1, \dots, \alpha_d \in K^*$  and  $y_1, \dots, y_s \in \{x_1, \dots, x_d\}$  (repetition is allowed). We proceed by contradiction. Suppose that one of the polynomials  $Q_j$  is not constant. Then a factor in  $Q$  occurs with multiplicity  $e \geq 2$ , again by Corollary 3.8. Multiplying  $F$  by some polynomial, the Bézivin property is preserved, and hence we may assume  $F = P/(1 - \alpha x_1)^2$  where  $P$  is not divisible by  $1 - \alpha x_1$  (reindexing the variables if necessary).

Through successive polynomial division, write  $P = B_2(1 - \alpha x_1)^2 + B_1(1 - \alpha x_1) + B_0$  with  $B_2 \in K[x]$  and  $B_1, B_0 \in K[x_2, \dots, x_d]$ . Since  $(1 - \alpha x_1)$  does not divide  $P$ , we have  $B_0 \neq 0$ . Then

$$\frac{P}{Q} = B_2 + \frac{B_1}{(1 - \alpha x_1)} + \frac{B_0}{(1 - \alpha x_1)^2} = B_2 + \sum_{n=0}^{\infty} B_1 \alpha^n x_1^n + (n + 1)B_0 \alpha^n x_1^n.$$

Let  $\mathbf{u} \in K[x_2, \dots, x_d]$  be a monomial in the support of  $B_0$ . For all sufficiently large  $n$ ,

$$\alpha^{-n} [x_1^n \mathbf{u}]P = (n + 1)[\mathbf{u}]B_0 + [\mathbf{u}]B_1.$$

Let  $A := (y + 1)[\mathbf{u}]B_0 + [\mathbf{u}]B_1 \in K[y]$ . By choice of  $\mathbf{u}$ , the polynomial  $A$  has degree 1. However, enlarging  $G$  to ensure  $\alpha \in G$ , there exists  $r \geq 0$  such that  $A(n) \in rG_0$  for all sufficiently large  $n$ . This contradicts Lemma 4.7. ■

We are now ready to prove the main theorem of this section.

**Proof of Theorem 4.1** We may assume that  $K$  is finitely generated. Let  $F = \sum_{\mathbf{n} \in \mathbb{N}^d} f(\mathbf{n})\mathbf{x}^{\mathbf{n}}$  be a  $D$ -finite power series all of whose coefficients are in  $rG_0$  for some  $r \geq 0$  and a finitely generated subgroup  $G \leq K^*$ . Since the sequence  $f: \mathbb{N}^d \rightarrow K$  is  $P$ -recursive, there exist  $Q_{j,a}$  as in equation (4.1). For every  $j \in [1, d]$ , there exists  $\mathbf{a} \in [0, k]^d$  with  $Q_{j,a}(0) \neq 0$ . Let

$$H := \sum_{\mathbf{a}_1, \dots, \mathbf{a}_d \in [0, k]^d} Q_{1, \mathbf{a}_1}(0) \cdots Q_{d, \mathbf{a}_d}(0) \mathbf{x}^{\mathbf{a}_1 + \cdots + \mathbf{a}_d} F.$$

Then  $H = PF$  with a nonzero polynomial  $P$ , and it suffices to show that  $H$  is rational. Since  $H$  is  $D$ -finite as well, the sequence of coefficients of  $H$ , denoted by  $h$ , is  $P$ -recursive of some size  $k'$ ; without restriction,  $k' \geq k$ .

For each  $j \in [1, d]$ , we will now apply Lemma 4.5 in the following way. We choose  $\Omega = \mathbb{N}^d$  and  $\pi: \mathbb{N}^d \rightarrow \mathbb{N} \subseteq K$  to be the projection on the  $j$ th coordinate. We set  $B = [0, k'd]$  and consider the equation

$$0 = \sum_{\mathbf{a} \in [0, k]^d} Q_{j, \mathbf{a}}(n_j) f(\mathbf{n} - \mathbf{a}) = \sum_{\mathbf{a} \in [0, k]^d} Q_{j, \mathbf{a}}(\pi(\mathbf{n})) f_{\mathbf{a}}(\mathbf{n}),$$

where  $f_{\mathbf{a}}: \mathbb{N}^d \rightarrow rG_0$  is defined by  $f_{\mathbf{a}}(\mathbf{n}) = f(\mathbf{n} - \mathbf{a})$ . By assumption on  $f$ , this equation holds for  $\mathbf{n} \in \mathbb{N}_{\geq k'}^d$ . Enlarging  $G = G'$  if necessary, Lemma 4.5 (after expanding  $f_{\mathbf{a}}(\vec{n}) = g_{\mathbf{a},1}(\vec{n}) + \cdots + g_{\mathbf{a},r}(\vec{n})$ , analogously to the proof of Lemma 4.7) implies that, for any choice of  $l \in \mathbb{N}_{\geq 1} \setminus G$ , there exists  $e_0 \in \mathbb{N}$  such that

$$\sum_{\mathbf{a} \in [0, k]^d} Q_{j, \mathbf{a}}(0) f(\mathbf{n} - \mathbf{a}) = \sum_{\mathbf{a} \in [0, k]^d} Q_{j, \mathbf{a}}(0) f_{\mathbf{a}}(\mathbf{n}) = 0 \quad \text{for} \\ \mathbf{n} \in \mathbb{N}_{\geq k'}^{j-1} \times (l^{e_0 + \mathbb{N}} + B) \times \mathbb{N}_{\geq k'}^{d-j}.$$

Further enlarging  $G = G'$  if necessary, we may assume that the same finitely generated group  $G$  works for all  $j \in [1, d]$ . Then we can also take the same  $l \in \mathbb{N}_{\geq 1} \setminus G$  for all  $j \in [1, d]$ , and finally, taking  $e_0$  large enough, we can also assume the same  $e_0$  works for all  $j \in [1, d]$ . As a further convenience, we may enlarge  $e_0$  to ensure  $l^{e_0} \geq k'd$ .

For  $j \in [1, d]$ , define  $C_j := l^{e_0 + \mathbb{N}} + [k'(d - 1), k'd]$ . Then, for all  $j \in [1, d]$  and  $\mathbf{n} \in \mathbb{N}_{\geq k'd}^{j-1} \times C_j \times \mathbb{N}_{\geq k'd}^{d-j}$ ,

$$\begin{aligned} h(\mathbf{n}) &:= \sum_{\mathbf{a}_1, \dots, \mathbf{a}_d \in [0, k]^d} \left( \prod_{i=1}^d Q_{i, \mathbf{a}_i}(0) \right) f(\mathbf{n} - \mathbf{a}_1 - \dots - \mathbf{a}_d) \\ &= \sum_{\mathbf{a}_1, \dots, \widehat{\mathbf{a}}_j, \dots, \mathbf{a}_d \in [0, k]^d} \left( \prod_{\substack{i=1 \\ i \neq j}}^d Q_{i, \mathbf{a}_i}(0) \right) \sum_{\mathbf{a}_j \in [0, k]^d} Q_{j, \mathbf{a}_j}(0) f\left(\mathbf{n} - \sum_{\substack{i=1 \\ i \neq j}}^d \mathbf{a}_i - \mathbf{a}_j\right) = 0. \end{aligned}$$

Because  $h$  is  $P$ -recursive of size  $k'$ , Lemma 4.3 implies that  $h$  vanishes on  $\mathbb{N}_{\geq m}^d$  with  $m = l^{e_0} + k'(d - 1)$ .

For a subset  $\emptyset \neq I \subseteq [1, d]$  and a tuple  $(j_i)_{i \in I} \in [0, m]^I$ , let

$$u_{I, (j_i)_{i \in I}} := \prod_{i \in I} x_i^{j_i} \prod_{i \in [1, d] \setminus I} x_i^{m+1}.$$

We may write

$$H = \sum_{\emptyset \neq I \subseteq [1, d]} \sum_{\substack{(j_i)_{i \in I} \\ j_i \in [0, m]}} u_{I, (j_i)_{i \in I}} H_{I, (j_i)_{i \in I}}((x_\mu)_{\mu \in [1, d] \setminus I}),$$

with suitable power series  $H_{I, (j_i)_{i \in I}}$  in  $d - |I|$  variables. Note that in each summand of the form

$$u_{I, (j_i)_{i \in I}} H_{I, (j_i)_{i \in I}}((x_\mu)_{\mu \in [1, d] \setminus I}),$$

the exponents of  $x_i$ , for  $i \in I$ , are fixed at some  $j_i \in [0, m]$ , whereas the exponents of  $x_i$  are greater than  $m$  for  $i \in [1, d] \setminus I$ . It follows that these summands have pairwise disjoint support, and hence the  $H_{I, (j_i)_{i \in I}}$  have their coefficients in  $rG_0$ . These coefficient sequences being shifted sections of  $h$ , moreover the  $H_{I, (j_i)_{i \in I}}$  are  $D$ -finite. Because  $I \neq \emptyset$ , and by induction on  $d$ , we may assume that the  $H_{I, (j_i)_{i \in I}}$  are rational, and therefore so is  $H$ . ■

## 5 The denominator of rational Bézivin series

Having shown that  $D$ -finite Bézivin series are rational, we now show that the denominator takes a particularly simple form. In this section, we work with the Hahn series ring  $K((x^H))$  and its additive valuation  $v: K((x^H)) \rightarrow H \cup \{\infty\}$  (see Section 2.3). At first, the group  $H$  will be an arbitrary totally ordered abelian group, but we restrict to  $H = \mathbb{Q}^d$  after Proposition 5.2.

We make use of the following easy fact.

**Lemma 5.1** *The polynomials*

$$\left\{ \binom{x}{l} = \frac{x(x-1)\cdots(x-l+1)}{l!} : l \geq 0 \right\}$$

form a basis of the polynomial ring  $K[x]$ .

The following is a crucial reduction step, used later in describing the denominators of rational Bézivin series.

**Proposition 5.2** *Let  $H$  be a totally ordered abelian group, and let  $L = K((x^H))$  be its Hahn power series ring. Let  $s \geq 1$ , and, for  $i \in [1, s]$ , let  $\alpha_i, \beta_i \in L^*$  with the  $\alpha_i$  pairwise distinct. For  $n \geq 0$ , let*

$$F_n = \sum_{i=1}^s \beta_i \alpha_i^n \in L.$$

*If there exist a finitely generated subgroup  $G \leq K^*$  and  $r \geq 0$  such that for all  $n \geq 0$  every coefficient of  $F_n$  is an element of  $rG_0$ , then there exists an  $i \in [1, s]$  such that the support of  $\alpha_i$  is a monomial, that is,*

$$\alpha_i = c_i x^{h_i} \quad \text{with} \quad c_i \in K^* \text{ and } h_i \in H.$$

**Proof** Let  $m = \min\{v(\alpha_1), \dots, v(\alpha_s)\}$  and  $m' = \min\{v(\beta_1), \dots, v(\beta_s)\}$ . We may without restriction replace  $F_n$  by  $x^{-m'-nm} F_n$ , and may thus assume  $m = m' = 0$ . After reindexing,

$$0 = v(\alpha_1) = \dots = v(\alpha_k) < v(\alpha_{k+1}) \leq \dots \leq v(\alpha_s) \quad \text{for some } k \in [1, s].$$

For  $i \in [1, k]$ , we have  $\alpha_i = c_i + \theta_i$  with  $c_i \in K^*$  and  $\theta_i \in L$  where  $v(\theta_i) > 0$ . We proceed with a proof by contradiction, and may therefore assume  $\theta_i \neq 0$  for all  $i \in [1, k]$ .

Reindexing the  $\alpha_1, \dots, \alpha_k$  again if necessary, we may partition

$$[1, k] = [k_1, k_2 - 1] \uplus [k_2, k_3 - 1] \uplus \dots \uplus [k_p, k_{p+1} - 1],$$

with  $1 = k_1 < k_2 < \dots < k_p < k_{p+1} = k + 1$ , such that  $c_i = c_j$  if and only if  $i, j \in [k_v, k_{v+1} - 1]$  for some  $v$ .

Let

$$T_n := \sum_{i=1}^k \beta_i (\alpha_i^n - c_i^n) = \sum_{v=1}^p \left[ \sum_{j=k_v}^{k_{v+1}-1} \beta_j \alpha_j^n - \left( \sum_{j=k_v}^{k_{v+1}-1} \beta_j \right) c_{k_v}^n \right].$$

Then  $v(T_n) > 0$  for all  $n$ , because the constant terms cancel.

**Step 1.** *There exists  $m \in H$  with  $m > 0$  such that  $v(T_n) \leq m$  for infinitely many  $n \geq 0$ .*

Let

$$A_n = \begin{pmatrix} \alpha_1^n & \dots & \alpha_k^n & c_{k_1}^n & \dots & c_{k_p}^n \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{n+k-1+p} & \dots & \alpha_k^{n+k-1+p} & c_{k_1}^{n+k-1+p} & \dots & c_{k_p}^{n+k-1+p} \end{pmatrix}.$$



Extracting scalars, these are Vandermonde matrices with

$$\det(A_n) = \alpha_1^n \cdots \alpha_k^n c_{k_1}^n \cdots c_{k_p}^n \prod_{1 \leq i < j \leq k} (\alpha_j - \alpha_i) \prod_{1 \leq i < j \leq p} (c_{k_j} - c_{k_i}) \prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq p}} (c_{k_j} - \alpha_i).$$

By construction,  $\det(A_n) \neq 0$ . Since  $v(\alpha_i) = 0$  for  $i \in [1, k]$ , moreover  $v(\det(A_n))$  is in fact independent of  $n$ . Let  $m_1 := v(\det(A_n))$  and  $\mathbf{w} = (w_1, \dots, w_{k+p})^T$  with

$$w_i = \begin{cases} \beta_i, & \text{if } 1 \leq i \leq k, \\ -\sum_{j=k_v}^{k_{v+1}-1} \beta_j, & \text{if } i = k + v \text{ with } 1 \leq v \leq p. \end{cases}$$

Since  $\beta_1 \neq 0$ , we have  $w_1 \neq 0$ . Let  $m_2 := v(w_1)$ . The  $i$ th entry of  $A_n \mathbf{w}$  is  $T_{n+i-1}$ . Hence, using the subscript 1 to denote the first coordinate of a vector,

$$\det(A_n)w_1 = (\text{adj}(A_n)A_n \mathbf{w})_1 = (\text{adj}(A_n)(T_n, \dots, T_{n+k+p-1})^T)_1 = \sum_{j=0}^{k+p-1} \gamma_j T_{n+j}$$

for some  $\gamma_j$  with  $v(\gamma_j) \geq 0$ . Since  $v(\det(A_n)w_1) = m_1 + m_2$ , we must have  $v(T_{n+\delta(n)}) \leq m_1 + m_2$  for some  $\delta(n) \in [0, k + p - 1]$ .

**Step 2.** There exist  $N_0 \geq 0$  such that for all  $n \geq N_0$ ,

$$v\left(T_n - \sum_{i=1}^k \sum_{l=1}^{N_0} \binom{n}{l} \beta_i c_i^{n-l} \theta_i^l\right) > m.$$

Substituting  $\alpha_i = c_i + \theta_i$  into the definition of  $T_n$  and expanding, for  $0 \leq N_0 \leq n$ ,

$$T_n = \sum_{i=1}^k \sum_{l=1}^{N_0} \binom{n}{l} \beta_i c_i^{n-l} \theta_i^l + \sum_{i=1}^k \sum_{l=N_0+1}^n \binom{n}{l} \beta_i c_i^{n-l} \theta_i^l.$$

There exists  $N_0$  such that  $v(\beta_i \theta_i^l) = lv(\theta_i) + v(\beta_i) > m$  for all  $l \geq N_0$ , and choosing such  $N_0$  establishes the claim of **Step 2**.

Enlarging  $N_0$  if necessary, we may also assume  $v(\alpha_i^{N_0}) > m$  for  $i \in [k + 1, s]$  by Lemma 2.5. Let

$$P_n := \sum_{i=1}^k \sum_{l=1}^{N_0} \binom{n}{l} \beta_i c_i^{n-l} \theta_i^l.$$

Fix  $h \in H$  with  $0 < h \leq m$  such that  $[x^h]P_n \neq 0$  for some  $n \geq N_0$ . The—crucial—existence of such an  $h$  is guaranteed by Steps 1 and 2. Substituting Hahn series expressions for  $\beta_i$  and  $\theta_i$ , we can express  $[x^h]P_n$  as

$$[x^h]P_n = \sum_{i \in I} p_{h,i}(n) c_i^n,$$

where  $I \subseteq [1, k]$  is such that the  $c_i$  are pairwise distinct, and  $p_{h,i} \in K[x]$  are polynomials. Since  $l$  is always at least 1 in the expression for  $P_n$ , the polynomials  $p_{h,i}$  are either zero or nonconstant as a consequence of Lemma 5.1. By choice of  $h$ , there must be at least one  $i$  with  $p_{h,i} \neq 0$ .

For  $n \geq N_0$ ,

$$[x^h]P_n = [x^h]T_n = [x^h]F_n - \sum_{i=1}^k [x^h]\beta_i c_i^n - \underbrace{[x^h] \sum_{i=k+1}^s \beta_i \alpha_i^n}_{=0}.$$

Enlarging  $G$ , we may assume  $c_i, -[x^h]\beta_i \in G$ , and hence  $[x^h]P_n \in (r+k)G_0$ . Now, the univariate case of Lemma 4.8 implies that the  $p_{h,i}$  are constant, and by our construction therefore  $p_{h,i} = 0$  for all  $i \in I$ . This is a contradiction to  $[x^h]P_n \neq 0$  for some  $n \geq N_0$ . ■

**Lemma 5.3** *Let  $K$  be algebraically closed, and let*

$$P = (1 - c_1 \mathbf{x}^{a_1}) \cdots (1 - c_s \mathbf{x}^{a_s}) \in K((x_1^{\mathbb{Q}}, \dots, x_d^{\mathbb{Q}})),$$

with  $c_i \in K^*$  and  $\mathbf{a}_1, \dots, \mathbf{a}_s \in \mathbb{Q}^d \setminus \{\mathbf{0}\}$ .

• If  $P \in K[\mathbf{x}^{\pm 1}]$ , then there exist  $\mathbf{b}_1, \dots, \mathbf{b}_t \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$  and  $c'_1, \dots, c'_t \in K^*$  such that

$$P = (1 - c'_1 \mathbf{x}^{b_1}) \cdots (1 - c'_t \mathbf{x}^{b_t}).$$

• If  $P \in K[\mathbf{x}]$  with  $P(\mathbf{0}) = 1$ , one can even take  $\mathbf{b}_1, \dots, \mathbf{b}_t \in \mathbb{N}^d \setminus \{\mathbf{0}\}$ .

**Proof** Let  $\mathbf{a}_i = (a_{i,1}, \dots, a_{i,d})$  for  $i \in [1, s]$ . Let  $N \in \mathbb{N}$  be such that  $Na_{i,j} \in \mathbb{Z}$  for all  $i \in [1, s]$  and  $j \in [1, d]$ . Since

$$1 - c_i^N \mathbf{x}^{Na_i} = (1 - c_i \mathbf{x}^{a_i}) \sum_{j=0}^{N-1} c_i^j \mathbf{x}^{ja_i},$$

there exists  $F \in K((x_1^{\mathbb{Q}}, \dots, x_d^{\mathbb{Q}}))$ , having finite support, such that  $PF = Q$  with  $Q = (1 - c_1^N \mathbf{x}^{Na_1}) \cdots (1 - c_s^N \mathbf{x}^{Na_s})$ . Since  $P$  and  $Q$  are Laurent polynomials, the quotient  $F = Q/P$  is a rational function in  $\mathbf{x}$ . Therefore,  $Q/P$  has a Laurent series expansion at the origin, which coincides with the Hahn series  $F$ . Thus,  $F$  is in fact a Laurent series in  $\mathbf{x}$ . Since  $F$  has finite support, it is even a Laurent polynomial. We conclude that  $P$  divides  $Q$  in the Laurent polynomial ring  $K[\mathbf{x}^{\pm 1}]$ .

The Laurent polynomial ring is a factorial domain, arising from  $K[\mathbf{x}]$  by inverting the prime elements  $x_1, \dots, x_d$ . By Lemma 2.7, we know that all factors of  $Q$  in  $K[\mathbf{x}^{\pm 1}]$  are—up to units of  $K[\mathbf{x}^{\pm 1}]$ —of the form  $1 - c' \mathbf{x}^{\mathbf{b}}$  with  $c' \in K^*$  and  $\mathbf{b} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ . Therefore,

$$P = (1 - c'_1 \mathbf{x}^{b_1}) \cdots (1 - c'_t \mathbf{x}^{b_t})$$

with  $\mathbf{b}_1, \dots, \mathbf{b}_t \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ , and this factorization is unique up to order and associativity in  $K[\mathbf{x}^{\pm 1}]$ .

Suppose now that  $P$  is a polynomial and  $P(\mathbf{0}) = 1$ . Then  $P = Q_1 \cdots Q_r$  with irreducible polynomials  $Q_i \in K[\mathbf{x}]$ . Since  $P(\mathbf{0}) = 1$ , we have  $Q_1(\mathbf{0}) \cdots Q_r(\mathbf{0}) = 1$ . Replacing each  $Q_i$  by  $Q_i/Q_i(\mathbf{0})$ , we can therefore assume  $Q_i(\mathbf{0}) = 1$  for each  $i \in [1, r]$ . In particular, no  $Q_i$  is a monomial. Therefore,  $Q_1, \dots, Q_r$  remain prime elements in the localization  $K[\mathbf{x}^{\pm 1}]$ . Since  $K[\mathbf{x}^{\pm 1}]$  is factorial, this implies  $r = t$  and that, after reindexing the polynomials  $Q_i$  if necessary, we may assume that  $Q_i$  is associated with  $1 - c'_i \mathbf{x}^{b_i}$  for each  $i \in [1, r]$ . Explicitly, this means  $Q_i = q_i \mathbf{x}^{e_i} (1 - c'_i \mathbf{x}^{b_i})$  with  $e_i \in \mathbb{Z}^d$

and  $q_i \in K^*$ . Then  $Q_i \in K[\mathbf{x}]$  forces  $\mathbf{e}_i \in \mathbb{N}^d$ , and so there are two possibilities: either  $\mathbf{e}_i = \mathbf{0}$  and  $q_i = 1$ , in which case necessarily  $\mathbf{b}_i \in \mathbb{N}^d$ , or  $q_i c_i' \mathbf{x}^{\mathbf{e}_i + \mathbf{b}_i} = 1$ . In either case,  $Q_i$  is of the desired form. ■

**Lemma 5.4** *Let  $Q \in K[x_1, \dots, x_{d-1}, x_d]$  be irreducible with  $\deg_{x_d}(Q) \geq 1$ , and let  $L := K((x_1^{\mathbb{Q}}, \dots, x_{d-1}^{\mathbb{Q}}))$ . Let*

$$Q = \mu \prod_{i=1}^s (1 - \lambda_i x_d)$$

with  $\mu \in L^*$  and  $\lambda_1, \dots, \lambda_s \in L$ . If some  $\lambda_i$  is a monomial in  $L$ , then all of them are.

**Proof** Let  $R := K[x_1, \dots, x_{d-1}]$  and  $L_0 := K(x_1, \dots, x_{d-1})$ . Since  $Q$  is irreducible in  $K[x_1, \dots, x_d]$  and is not contained in  $R$ , it is also irreducible as a univariate polynomial in  $R[x_d]$ . By Gauss’s lemma, then  $Q$  is irreducible in  $L_0[x_d]$ . Since  $L$  is algebraically closed,  $Q$  can be expressed as a product of linear factors as above.

Suppose now without restriction that  $\lambda_1^{-1} = a\mathbf{x}^{\mathbf{e}}$  with  $a \in K^*$  and  $\mathbf{e} \in \mathbb{Q}^{d-1}$ . Let  $N \in \mathbb{Z}_{\geq 1}$  with  $N\mathbf{e} \in \mathbb{Z}^{d-1}$ . Then  $\lambda_1^{-1}$  is a root of  $P = x_d^N - a^N \mathbf{x}^{\mathbf{e}N} \in L_0[x_d]$ . However,  $P = \prod_{j=0}^{N-1} (x_d - a\zeta^j \mathbf{x}^{\mathbf{e}})$  in  $L[\mathbf{x}]$ , where  $\zeta \in L^*$  is a primitive  $N$ th root of unity. Since all roots of  $P$  are monomials, and  $Q$  divides  $P$  by irreducibility, all roots of  $Q \in L[x_d]$  are monomials. ■

**Theorem 5.5** *Let  $K$  be algebraically closed, and let  $F \in K[\mathbf{x}]$  be rational, with  $F = P/Q$ , where  $P, Q \in K[\mathbf{x}]$  are coprime polynomials and  $Q(\mathbf{0}) = 1$ . If  $F$  is a Bézivin series, then there exist  $s \geq 0$ , nonconstant monomials  $u_1, \dots, u_s \in K[\mathbf{x}]$ , and  $c_1, \dots, c_s \in K^*$  such that*

$$Q = (1 - c_1 u_1) \cdots (1 - c_s u_s).$$

**Proof** We proceed by induction on  $d$ . If  $d = 0$ , then the claim holds trivially with  $s = 0$ . Suppose  $d \geq 1$  and that the claim holds for  $d - 1$ .

Let  $Q = Q_1 \cdots Q_n$  where  $Q_1, \dots, Q_n \in K[\mathbf{x}]$  are irreducible ( $n \geq 0$ ). For each  $j \in [1, n]$ , the series  $P/Q_j = FQ_1 \cdots Q_{j-1} Q_{j+1} \cdots Q_n$  is also a rational Bézivin series. It therefore suffices to show: if  $F = P/Q$  with an irreducible polynomial  $Q \in K[\mathbf{x}]$ , not dividing  $P$ , and  $Q(\mathbf{0}) = 1$  and  $F$  is a Bézivin series, then  $Q$  is of the form  $1 - cu$  with  $c \in K^*$  and a monomial  $u \in K[\mathbf{x}]$ .

By reindexing, we may assume that  $Q$  has degree at least one in the variable  $x_d$ . Then, since  $Q$  is irreducible and does not divide  $P$ , there are polynomials  $A$  and  $B$  in  $K[\mathbf{x}]$  such that  $C := AP + BQ \in K[x_1, \dots, x_{d-1}] \setminus \{0\}$ . Since  $FA$  and  $B$  are both rational Bézivin series, the same is true for  $C/Q = FA + B$ . Now, since  $Q$  has degree at least one in  $x_d$  and since it has distinct roots as a polynomial in  $x_d$ , we may use the fact that  $L := K((x_1^{\mathbb{Q}}, \dots, x_{d-1}^{\mathbb{Q}}))$  is algebraically closed to factor  $Q$  and write

$$C/Q = CQ_0^{-1} (1 - \lambda_1 x_d)^{-1} \cdots (1 - \lambda_s x_d)^{-1},$$

where  $Q_0 \in K[x_1, \dots, x_{d-1}]$  and  $\lambda_1, \dots, \lambda_s \in L^*$  are pairwise distinct. Using partial fractions, we may write this as

$$C/Q = \alpha + \sum_{i=1}^s \frac{\beta_i}{1 - \lambda_i x_d},$$

where  $\alpha, \beta_i \in L$  and the  $\beta_i$  are nonzero. Then, if we expand, we have that for  $n \geq 1$ , the coefficient of  $x_d^n$  in  $C/Q$  is

$$F_n := \sum_{i=1}^s \beta_i \lambda_i^n \in L.$$

Applying Proposition 5.2 to this family of Hahn series, one of the  $\lambda_i$  is of the form  $\lambda_i = c_i u_i$  with  $c_i \in K^*$  and  $u_i \in L$  a monomial, that is, of the form  $u_i = x_1^{e_{i,1}} \cdots x_{d-1}^{e_{i,d-1}}$  with  $e_{i,j} \in \mathbb{Q}$ . By Lemma 5.4, then all  $\lambda_i$  are of the form  $\lambda_i = c_i u_i$  with  $c_i \in K^*$  and  $u_i \in L$  a monomial.

Since  $CQ_0^{-1} = CQ^{-1}(1 - c_1 u_1 x_d) \cdots (1 - c_s u_s x_d)$ , it follows that the rational series  $CQ_0^{-1}$  is a Bézivin series. The induction hypothesis implies  $Q_0 = (1 - c_{s+1} u_{s+1}) \cdots (1 - c_t u_t)$  with  $c_j \in K^*$  and monomials  $u_{s+1}, \dots, u_t \in K[x_1, \dots, x_{d-1}]$ . Replacing  $u_i$  by  $u_i x_d$  for  $i \in [1, s]$ , we have

$$Q = (1 - c_1 u_1) \cdots (1 - c_t u_t) \in K[\mathbf{x}].$$

By Lemma 5.3, we can rewrite this product in such a way that the Hahn monomials  $u_j$  are monomials with nonnegative integer exponents, that is, monomials in the polynomial ring  $K[\mathbf{x}]$ . Then  $t = 1$  by irreducibility of  $Q$  and the claim is shown. ■

### 6 Structural decomposition of rational Bézivin series

*Proposition 6.1* Let  $K$  be algebraically closed. Every rational Bézivin series over  $K$  is expressible as a (trivially ambiguous) sum of skew-geometric rational series.

**Proof** By Theorems 3.10 and 5.5, the coefficient sequence of  $F$  is piecewise polynomial-exponential on simple linear subsets of  $\mathbb{N}^d$ . Let  $\mathcal{S} = \mathbf{a} + \mathbf{b}_1 \mathbb{N} + \cdots + \mathbf{b}_s \mathbb{N}$  be such a simple linear set on which  $F$  is polynomial-exponential and set  $u_i := \mathbf{x}^{\mathbf{b}_i}$  for  $i \in [1, s]$  and  $\mathbf{u} = (u_1, \dots, u_s)$ . By Lemma 3.5, there exist  $Q_1, \dots, Q_l \in K[y_1, \dots, y_s]$  and  $\lambda_1, \dots, \lambda_s \in (K^*)^s$  such that

$$F \odot \mathbf{1}_{\mathcal{S}} = \mathbf{x}^{\mathbf{a}} \sum_{\mathbf{m} \in \mathbb{N}^s} \sum_{j=1}^l Q_j(\mathbf{m}) \lambda_j^{\mathbf{m}} \mathbf{u}^{\mathbf{m}}.$$

We consider  $H(\mathbf{u}) := \mathbf{x}^{-\mathbf{a}}(F \odot \mathbf{1}_{\mathcal{S}})$  as a power series in  $\mathbf{u}$ . Clearly,  $H$  is a Bézivin series. By Lemma 4.8, the series  $H(\mathbf{u})$  is a sum of geometric series in  $\mathbf{u}$ , and thus  $F \odot \mathbf{1}_{\mathcal{S}}$  is a sum of skew-geometric series in  $\mathbf{x}$ . Since we can partition  $\mathbb{N}^d$  into finitely many disjoint such simple linear sets  $\mathcal{S}$ , the series  $F$  is also a sum of skew-geometric series. ■

*Remark 6.2* Instead of using Theorem 3.10, the main reduction in the proof of the previous theorem can also be derived from the fact that the coefficient sequence of a generating series

$$\frac{1}{(1 - v_1) \cdots (1 - v_l)}$$

with monomials  $v_1, \dots, v_l$  is piecewise polynomial on simple linear subsets of  $\mathbb{N}^d$ . Therefore, one may substitute results on vector partitions [Stu95] for Theorem 3.10.

(In contrast to Theorem 3.10, here the coefficients in front of the monomials are all equal to 1.)

We sketch the main part of the argument. Consider an expression of the form  $1/Q$  with

$$Q := (1 - c_1 u_1) \cdots (1 - c_l u_l),$$

where  $c_1, \dots, c_l \in K^*$  and  $u_1, \dots, u_l$  are nonconstant monomials. First, we may assume that the monomials  $u_1, \dots, u_l$  have a common zero  $(\alpha_1, \dots, \alpha_d)$  in the algebraic closure  $\overline{K}$ —otherwise, we may use Hilbert’s Nullstellensatz to reduce the expression  $1/Q$  into a sum of expressions with fewer factors in the denominator. (This is actually also the first step of the partial fraction decomposition [Rai12] used in the proof of Theorem 3.10.)

Now, let  $v$  be a monomial. The coefficient of  $v$  in  $1/Q$  is

$$\begin{aligned} \sum_{\substack{e_1, \dots, e_l \geq 0 \\ u_1^{e_1} \cdots u_l^{e_l} = v}} \prod_{j=1}^l c_j^{e_j} &= \sum_{\substack{e_1, \dots, e_l \geq 0 \\ u_1^{e_1} \cdots u_l^{e_l} = v}} \prod_{j=1}^l u_j(\alpha_1, \dots, \alpha_d)^{-e_j} \\ &= \sum_{\substack{e_1, \dots, e_l \geq 0 \\ u_1^{e_1} \cdots u_l^{e_l} = v}} v(\alpha_1, \dots, \alpha_d)^{-1} = \mu(v) v(\alpha_1, \dots, \alpha_d)^{-1}. \end{aligned}$$

Here,  $\mu(v) \in \mathbb{N}$  is the number of ways of writing  $v$  as a product of  $u_1, \dots, u_l$ . Thus,  $1/Q$  decomposes as a Hadamard product

$$1/Q = \frac{1}{(1 - u_1) \cdots (1 - u_l)} \odot \frac{1}{(1 - \alpha_1^{-1} x_1) \cdots (1 - \alpha_d^{-1} x_d)}.$$

The coefficients of the left factor are therefore polynomial on simple linear subsets of  $\mathbb{N}^d$ , and one proceeds from there.

### 6.1 The constants can be taken in the group

Let  $\alpha = (\alpha_1, \dots, \alpha_s), \beta = (\beta_1, \dots, \beta_s) \in (K^*)^s$  with  $s \in \mathbb{N}$ . We say that  $\alpha$  and  $\beta$  are *relatively nontorsion* if none of the quotients  $\alpha_i/\beta_i$  for  $i \in [1, s]$  is a nontrivial root of unity. Let  $u_1, \dots, u_s \in K[\mathbf{x}]$  be algebraically independent monomials. Consider an expression of the form

$$\frac{P}{(1 - \alpha_1 u_1) \cdots (1 - \alpha_s u_s)} + \frac{Q}{(1 - \beta_1 u_1) \cdots (1 - \beta_s u_s)}$$

with polynomials  $P, Q \in K[\mathbf{x}]$ . If, say,  $\alpha_1/\beta_1$  is a root of unity of order  $n$ , then we may use the identity

$$(1 - \alpha_1^n u_1^n) = (1 - \alpha_1 u_1) \left( \sum_{j=0}^{n-1} \alpha_1^j u_1^j \right).$$

to replace  $1 - \alpha_1 u_1$  in the denominator by  $1 - \alpha_1^n u_1^n$  and analogously for  $1 - \beta_1 u_1$ . We may thus assume that in any representation, the coefficient vectors in the denominator are relatively nontorsion (also for more than two summands). In regard to the

coefficient sequence, and its description in terms of simple linear sets, this amounts to a refinement

$$\mathbf{a} + \mathbf{b}_1\mathbb{N} + \dots + \mathbf{b}_s\mathbb{N} = \bigcup_{\mathbf{j} \in [0, n-1]^d} (\mathbf{a} + \mathbf{j}\mathbf{b}_1) + n\mathbf{b}_1\mathbb{N} + \mathbf{b}_2\mathbb{N} \dots + \mathbf{b}_s\mathbb{N}.$$

**Lemma 6.3** *Let  $\lambda = (\lambda_1, \dots, \lambda_d) \in (K^*)^d$ .*

- (1) *If some  $\lambda_i$  is not a root of unity, then  $\{\lambda^n : \mathbf{n} \in \mathbb{N}^d\} \subseteq K^*$  is infinite.*
- (2) *For every  $c \in K^*$ , the set  $\mathcal{L} := \{\mathbf{n} \in \mathbb{N}^d : \lambda^n = c\}$  is a semilinear set. If moreover  $\{\lambda^n : \mathbf{n} \in \mathbb{N}^d\}$  is infinite, then  $\mathcal{L}$  has rank at most  $d - 1$ .*

**Proof** (1) Clear.

(2) We may assume  $\mathcal{L} \neq \emptyset$ . First, observe that  $U := \{\mathbf{n} \in \mathbb{Z}^d : \lambda^n = 1\}$  is a subgroup of  $\mathbb{Z}^d$  and therefore free abelian. If the set  $\{\lambda^n : \mathbf{n} \in \mathbb{N}^d\}$  is infinite, then  $\mathbb{Z}^d/U$  must be infinite, and therefore  $U$  has rank at most  $d - 1$ . Now,  $\mathcal{L}_0 := U \cap \mathbb{N}^d$  is a linear set.

By Dickson’s lemma (see [Sak09, Lemma II.7.1]), the set  $\mathcal{L}$  has finitely many minimal elements  $\mathbf{a}_1, \dots, \mathbf{a}_k$  with respect to the partial order on  $\mathbb{N}^d$ , and it follows that  $\mathcal{L} = \bigcup_{i=1}^k \mathbf{a}_i + \mathcal{L}_0$ . ■

**Proposition 6.4** *Let  $F$  be a trivially ambiguous sum of skew-geometric series that is Bézivin and, more specifically, has coefficients in  $rG_0$  for some  $r \geq 0$ . Then  $F$  can be written as an unambiguous sum of series of the form*

$$F_S = \sum_{i=1}^l \frac{g_{i,0}u_0}{(1 - g_{i,1}u_1) \dots (1 - g_{i,s}u_s)},$$

where  $u_0, u_1, \dots, u_s \in K[\mathbf{x}]$  are monomials with  $u_1, \dots, u_s$  algebraically independent, and where  $g_{i,v} \in G$  for all  $i \in [1, l]$  and  $v \in [1, s]$ . Moreover, one can take  $l \leq r$ .

**Proof** We may partition  $\mathbb{N}^d$  into simple linear sets so that on each such simple linear set  $S$ ,

$$(6.1) \quad F \odot \mathbf{1}_S = F_S = \sum_{i=1}^l \frac{c_{i,0}u_0}{(1 - c_{i,1}u_1) \dots (1 - c_{i,s}u_s)},$$

with  $c_{i,v} \in K^*$ , with algebraically independent monomials  $u_1, \dots, u_s$ , and with an arbitrary monomial  $u_0$ . We first show that the coefficients in the denominators can be taken in  $G$ . After that, we will deal with the numerators and show that we can also achieve  $l \leq r$ .

First, by combining summands with the same denominator, we may assume that no denominator occurs twice in (6.1). Then the uniqueness statement of Corollary 3.8, applied over the polynomial ring  $K[u_1, \dots, u_s]$ , implies that the representation of  $F_S$  in this form is unique. In particular, each  $1 - c_{i,v}u_v$  indeed occurs as a factor of the reduced denominator of  $F_S$ , considered as a rational function in  $K(u_1, \dots, u_s)$ .

To deal with the denominators, it suffices to show that for every  $i \in [1, l]$  and  $v \in [1, s]$ , there exists  $N \geq 1$  such that  $c_{i,v}^N \in G$ . Then we can use the remarks preceding this proposition to replace the denominators in such a way that the coefficients are in  $G$ . By symmetry, it suffices to show this for  $i = v = 1$ . Multiplying  $u_0^{-1}F_S$  by a suitable

polynomial in  $K[u_1, \dots, u_s]$  to clear all denominators other than  $1 - c_{1,1}u_1$  and setting  $c := c_{1,1}$ , we obtain a rational series

$$H := \frac{P(u_1, \dots, u_s)}{1 - cu_1},$$

with  $1 - cu_1$  not dividing  $P(u_1, \dots, u_s) \in K[u_1, \dots, u_s]$  (note that the elements  $1 - c_{i,v}u_v$  are irreducible in  $K[u_1, \dots, u_s]$ ). By Lemma 2.6, there is a finite set  $B$  such that all coefficients of  $H$  are contained in  $BG_0$ .

Working now in the polynomial ring  $K[u_2, \dots, u_s][u_1]$ , we can use polynomial division to write

$$P = Q(1 - cu_1) + R$$

with  $Q \in K[u_1, u_2, \dots, u_s]$  and  $0 \neq R \in K[u_2, \dots, u_s]$ . Fix any monomial  $v \in \text{supp}(R)$ , and let its coefficient in  $R$  be  $a$ . Then, for large  $n$ ,

$$[vu_1^n]H = ac^n \in BG_0.$$

For  $b \in B$ , let  $g_b: \mathbb{N} \rightarrow G_0$  be such that

$$ac^n = \sum_{b \in B} bg_b(n) \quad \text{for large } n.$$

Fix a subset  $B' \subseteq B$  such that, for infinitely many  $n$ ,

$$ac^n - \sum_{b \in B'} bg_b(n) = 0$$

yields a nondegenerate solution to the unit equation  $X_0 + \sum_{b \in B'} X_b = 0$  over the group  $G'$  generated by  $G, B'$ , and  $-1$ . Let  $b \in B'$ . Then there exists a constant  $\beta \in K^*$  such that

$$\frac{ac^n}{bg_b(n)} = \beta,$$

for infinitely many  $n$ . Let  $n_1 < n_2$  be two such values. Then

$$\frac{ac^{n_1}}{bg_b(n_1)} = \frac{ac^{n_2}}{bg_b(n_2)}$$

implies  $c^{n_2-n_1} \in G$ . This finishes the claim about the denominators.

Going back to the representation of  $F_S$  in (6.1), and refining the set  $S$  if necessary, we can therefore assume  $c_{i,v} \in G$  for all  $i \in [1, l]$  and  $v \in [1, s]$ . Using the same type of reduction, we may assume that whenever  $i, j \in [1, l]$  and  $v \in [1, s]$  are such that  $c_{i,v} \neq c_{j,v}$ , then  $c_{i,v}/c_{j,v}$  is not a root of unity.

Now, we deal with the numerators  $c_{0,v}$  for  $v \in [1, s]$  and simultaneously with the number of summands. Let  $\mathbf{c}_i = (c_{i,1}, \dots, c_{i,s})$  for  $i \in [1, l]$ , and  $\mathbf{u} = (u_1, \dots, u_s)$ . Then

$$[u_0 \mathbf{u}^n]F = \sum_{i=1}^l c_{i,0} \mathbf{c}_i^n = g_1(\mathbf{n}) + \dots + g_r(\mathbf{n}),$$

for some functions  $g_1, \dots, g_r: \mathbb{N}^d \rightarrow G_0$ . Consider this as a unit equation over the group generated by  $G, -1$ , and  $c_{i,v}$  with  $i \in [1, l]$  and  $v \in [0, s]$ .

Consider a partition  $\mathcal{P} = \{(I_1, J_1), \dots, (I_t, J_t)\}$  of the disjoint union  $[1, l] \uplus [1, r]$ , that is,  $I_\tau \subseteq [1, l]$ ,  $J_\tau \subseteq [1, r]$ , at least one of  $I_\tau$  and  $J_\tau$  is nonempty, and  $I_\tau \cap I_{\tau'} = \emptyset = J_\tau \cap J_{\tau'}$  for  $\tau \neq \tau'$ . Let  $\Omega_{\mathcal{P}} \subseteq \mathbb{N}^s$  denote the set of all  $\mathbf{n} \in \mathbb{N}^s$  such that

$$\sum_{i \in I_\tau} c_{i,0} \mathbf{c}_i^n - \sum_{j \in J_\tau} g_j(\mathbf{n}) = 0$$

yields a nondegenerate solution to the unit equation  $\sum_{i \in I_\tau} X_i + \sum_{j \in J_\tau} X_j = 0$ . Then the sets  $\Omega_{\mathcal{P}}$  cover  $\mathbb{N}^s$  as  $\mathcal{P}$  ranges through all partitions.

We claim that there exists a partition  $\mathcal{P}$  such that for all  $i \neq j \in [1, l]$  the quotient  $(\mathbf{c}_i/\mathbf{c}_j)^{\mathbf{n}}$  takes infinitely many values as  $\mathbf{n}$  ranges through  $\Omega_{\mathcal{P}}$ . Suppose  $\beta \in K^*$ . The set of all  $\mathbf{n} \in \mathbb{N}^s$  such that  $(\mathbf{c}_i/\mathbf{c}_j)^{\mathbf{n}} = \beta$  is a semilinear set of rank at most  $s - 1$  by Lemma 6.3. Hence, if  $(\mathbf{c}_i/\mathbf{c}_j)^{\mathbf{n}}$  takes finitely many values on  $\Omega_{\mathcal{P}}$ , then  $\Omega_{\mathcal{P}}$  is contained in a semilinear set of rank at most  $s - 1$ . However, since the  $\Omega_{\mathcal{P}}$  cover  $\mathbb{N}^s$ , there must be at least one  $\mathcal{P}$  such that  $(\mathbf{c}_i/\mathbf{c}_j)^{\mathbf{n}}$  takes infinitely many values for  $\mathbf{n} \in \Omega_{\mathcal{P}}$ .

Now, fix such a partition  $\mathcal{P}$ . Then necessarily  $|I_\tau| \leq 1$  for all  $\tau \in [1, t]$  by the theorem on unit equations. If  $I_\tau = \{i\}$ , it follows that

$$c_{i,0} = \sum_{j \in J_\tau} g_j(\mathbf{n})/\mathbf{c}_i^n \quad \text{for } \mathbf{n} \in \Omega_{\mathcal{P}}.$$

Thus,  $c_{i,0}$  is a sum of at most  $|J_\tau|$  elements of  $G$ . Substituting into (6.1) and splitting the sums accordingly (now we allow the same denominator to appear multiple times), we achieve  $c_{i,0} \in G$  and  $l \leq |J_1| + \dots + |J_t| \leq r$  in this representation. ■

## 7 Proofs of main theorems

At this point, Theorems 1.2 and 1.3 can easily be proved as follows.

**Proof of Theorem 1.2** (a)  $\Rightarrow$  (b) By Theorem 4.1, every  $D$ -finite Bézivin series is rational.

(b)  $\Rightarrow$  (d) Let  $F \in K\langle\langle \mathbf{x} \rangle\rangle$  be a rational Bézivin series. By Proposition 6.1, the series is a trivially ambiguous sum of skew-geometric series. That the constants in the skew-geometric summands can be taken in  $G$ , and that the sum can be taken in such a way that no  $\mathbf{n} \in \mathbb{N}^d$  is contained in the support of more than  $r$  summands, follows from Proposition 6.4.

(d)  $\Rightarrow$  (c) Clear.

(c)  $\Rightarrow$  (a) Trivial, because every rational series is  $D$ -finite.

(d)  $\Leftrightarrow$  (e) This equivalence is immediate from the definition of a skew-geometric series, Definition 3.15, together with Lemma 3.4. ■

**Proof of Theorem 1.3** We apply Theorem 1.2 with  $r = 1$ . Then (a)  $\Leftrightarrow$  (b) of Theorem 1.3 follows from (a)  $\Leftrightarrow$  (b) of Theorem 1.2. The implication (b)  $\Rightarrow$  (c) of Theorem 1.3 follows from (b)  $\Rightarrow$  (d) of Theorem 1.2, taking into account  $r = 1$ .

To obtain (c)  $\Rightarrow$  (d) of Theorem 1.3, apply (d)  $\Rightarrow$  (e) of Theorem 1.2, noting again  $r = 1$ . This gives a partition of  $\mathbb{N}^d$  into simple linear sets, so that on each such set  $\mathcal{S} = \mathbf{a}_0 + \mathbf{a}_1\mathbb{N} + \dots + \mathbf{a}_s\mathbb{N}$ , one has  $f(\mathbf{a}_0 + m_1\mathbf{a}_1 + \dots + m_s\mathbf{a}_s) = 0$  or  $f(\mathbf{a}_0 + m_1\mathbf{a}_1 + \dots + m_s\mathbf{a}_s) = g_0g_1^{m_1}\dots g_s^{m_s} \neq 0$  with  $g_0, g_1, \dots, g_s \in G$ . Taking only these simple linear sets of the partition on which  $f$  does not vanish, we obtain a partition of the support of  $F$ , as claimed.



Finally, the implication (d)  $\Rightarrow$  (a) of Theorem 1.3 follows from the implication (e)  $\Rightarrow$  (a) of Theorem 1.2. ■

Before proving Corollary 1.4, we show that algebraic series and their diagonals and sections are finitary  $D$ -finite. Recall that the operator  $I_{1,2}: K[[\mathbf{x}]] \rightarrow K[[\mathbf{x}]]$  assigns to

$$F = \sum_{n_1, n_2, \dots, n_d \in \mathbb{N}} f(n_1, n_2, n_3, \dots, n_d) x_1^{n_1} x_2^{n_2} x_3^{n_3} \dots x_d^{n_d}$$

its primitive diagonal

$$I_{1,2}F = \sum_{n_1, n_3, \dots, n_d \in \mathbb{N}} f(n_1, n_1, n_3, \dots, n_d) x_1^{n_1} x_3^{n_3} \dots x_d^{n_d}.$$

For  $1 \leq i < j \leq d$ , the primitive diagonal operators  $I_{i,j}$  are defined analogously. A *diagonal* of  $F$  is any composition of diagonal operators applied to  $F$ . For instance, the complete diagonal of  $F$  is  $I_{1,2}I_{2,3} \dots I_{d-1,d}F = \sum_{n \in \mathbb{N}} f(n, n, \dots, n) x_1^n$ . A series  $F \in K[[\mathbf{x}]]$  is *algebraic* if it is algebraic over the field  $K(\mathbf{x})$ .

**Lemma 7.1** *If  $F \in K[[\mathbf{x}]]$  is:*

- algebraic, or
- a diagonal of an algebraic series, or
- a section of an algebraic series,

*then  $F$  is finitary  $D$ -finite.*

**Proof** Algebraic series are  $D$ -finite, and diagonals and sections of  $D$ -finite series are  $D$ -finite [Lip89, Proposition 2.5 and Theorem 2.7]. Moreover, diagonals and sections of finitary series are trivially finitary. It therefore suffices to show that algebraic series are finitary.

Let  $F \in K[[\mathbf{x}]]$  be algebraic. Replacing  $F$  by  $F - F(0)$ , we may assume  $F(0) = 0$ . Since  $F$  is algebraic, there exist  $p_2, \dots, p_m, q \in K[\mathbf{x}]$  with  $q \neq 0$  such that  $F = \frac{p_2}{q}F^2 + \dots + \frac{p_m}{q}F^m$ . Setting  $G = F/q$  and multiplying the previous equation by  $1/q$ , we obtain an equation

$$G = p_2G^2 + p_3qG^3 + \dots + p_mq^{m-2}G^m.$$

Keeping in mind  $G(0) = 0$ , this yields a recursion for the coefficients of  $G$ . It follows from this recursion that  $G$  is finitary. Since  $F = qG$ , also  $F$  is finitary. ■

**Lemma 7.2** *If  $F \in K[[\mathbf{x}]]$  is a finitary series and  $F^\dagger$  is also finitary, then  $F$  is a Pólya series.*

**Proof** Since  $F$  is finitary, there is a finitely generated  $\mathbb{Z}$ -algebra  $A \subseteq K$  containing all coefficients of  $K$ . Since  $F^\dagger$  is finitary as well, we may also assume that  $A$  contains  $f(\mathbf{n})^{-1}$  whenever  $f(\mathbf{n}) \neq 0$ . Therefore, all nonzero coefficients of  $F$  are contained in the unit group  $G$  of  $A$ . By a theorem of Roquette (see [Lan83, Corollary 7.5 of Chapter 2]), the group  $G$  is finitely generated. ■

**Proof of Corollary 1.4** Since  $F$  and  $F^\dagger$  are finitary,  $F$  is a Pólya series by Lemma 7.2. Being also  $D$ -finite,  $F$  satisfies condition (a) of Theorem 1.3. Conversely, if  $F$  satisfies condition (d) of Theorem 1.3, then clearly the same is true for  $F^\dagger$ . Thus, the series  $F^\dagger$  is  $D$ -finite.

The “in particular” statement of the corollary follows from Theorem 1.3 applied to  $F$ , respectively,  $F^\dagger$ . ■

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*Department of Pure Mathematics, University of Waterloo, Waterloo, ON N2L 3G1, Canada*  
*e-mail:* [jpbell@uwaterloo.ca](mailto:jpbell@uwaterloo.ca)

*Institute for Mathematics and Scientific Computing, University of Graz, NAWI Graz, Heinrichstrasse 36, 8010 Graz, Austria*  
*e-mail:* [daniel.smertnig@uni-graz.at](mailto:daniel.smertnig@uni-graz.at)