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THE POISSON INTEGRAL OF A SINGULAR MEASURE

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Let σ be a finite positive singular Borel measure defined on Euclidean space $\mathbf{R}^N$. For $w \in \mathbf{R}^N$ and $y > 0$, its Poisson integral is defined by the formula

$$\hat{\sigma}(w, y) = C_N \int_{\mathbf{R}^N} \frac{y d\sigma(x)}{(|x-w|^2+y^2)^{(N+1)/2}}$$

where C_N is chosen so that

$$C_N \int_{\mathbf{R}^N} \frac{dx}{(|x|^2+1)^{(N+1)/2}} = 1.$$

Since σ is singular, $\lim_{y\to 0}\hat{\sigma}(w, y) = 0$, almost everywhere with respect to Lebesgue measure on $\mathbf{R}^N$. On the other hand, $\lim_{y\to 0}\hat{\sigma}(w, y) = \infty$ almost everywhere $d\sigma$. It follows that for all sufficiently small y,

$$E_y = \{w \in \mathbf{R}^N : \hat{\sigma}(w, y) > 1\}$$

is a non-empty open subset of $\mathbf{R}^N$. If σ has compact support then $|E_y| \to 0$ as $y \to 0$, where $|E_y|$ denotes the Lebesgue measure of E_y. In this paper we give a lower bound on the rate at which $|E_y|$ may go to zero. The lower bound depends on the smoothness of the measure; the smoother the measure, the more slowly $|E_y|$ may approach 0. This may be somewhat unexpected in that if the measure σ is smooth then the maximum of $\hat{\sigma}(w, y)$ will go to ∞ more slowly, which should tend to make $|E_y|$ to to 0 more rapidly. The opposite is the case, however. The reason is that if the measure is smooth it must be well "spread out", and this spreading out of the measure has a greater influence on the size of E_y than does the slow growth of $\hat{\sigma}(w, y)$ to ∞. One consequence is that for the finite positive singular measure we have

$$\varliminf_{y\to 0} y^{-N/(N+1)} |E_y| > 0.$$

This result is sharp.

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In the second section we see the extent to which the results of the first section are sharp. We begin by getting bounds from above on $|E_\gamma|$ in case the support of σ is compact and has Lebesgue measure 0. Then we show that the bounds from above agree with the bounds from below if σ satisfies a certain homogeneity condition. If σ is not homogeneous then we must apply our results not to σ but to non-zero measures bounded by σ but smoother than σ.

In the third section we give an application to a problem about inner functions in the unit disc. In [**2**] it was shown that if φ is a singular inner function then there is an $\epsilon > 0$ such that

$$\int_0^\pi (1 - |\varphi(re^{i\theta})|)d\theta \geqq \epsilon\sqrt{1-r}.$$

Using the results of Section 1 we can improve this to show that if $0 < q < \infty$ there is an $\epsilon_q > 0$ such that

$$\int_0^{2\pi} (1 - |\varphi(re^{i\theta})|)^q d\theta \geqq \epsilon_q\sqrt{1-r}.$$

This result together with a generalization of a theorem in [**2**] can be used to show that if φ is an inner function and if

$$\int_0^1 (1-r)^\alpha \left(\int_0^{2\pi} |\varphi'(re^{i\theta})|^q d\theta\right)^{p/q} dr < \infty$$

for some p, q, such that

$$p\left(1 - \frac{1}{2q}\right) \geqq 1 + \alpha,$$

then φ is a Blaschke product. This was conjectured by M. Jevtić [**6**] when $p = q$.

1. For $x, w \in \mathbf{R}^N$ we let $d(x, w) = \max|x_i - w_i|$. Then we have

$$d(x, w) \leqq |x - w| \leqq \sqrt{N}d(x, w)$$

where

$$|x - w|^2 = \sum_{i=1}^N (x_i - w_i)^2.$$

We define the closed cube of side 2δ about x as

$$Q(x, \delta) = \{w \in \mathbf{R}^N : d(x, w) \leqq \delta\}.$$

Note that $|Q(x, \delta)| = (2\delta)^N$. The symbol σ will always denote a finite positive singular Borel measure on $\mathbf{R}^N$.

We will want to apply Egorov's theorem to the family of functions

$$f_\delta(x) = \sigma(Q(x, \delta))\delta^{-N}, \quad \delta > 0.$$

Since Egorov's theorem doesn't hold without restriction for families indexed by an uncountable set, (see [**9**]), we include the following.

LEMMA 1. *If $\epsilon > 0$ is given there exists a measurable set E such that $\sigma(\mathbf{R}^N \backslash E) < \epsilon$ and $\sigma(Q(x, \delta))\delta^{-N} \to \infty$ uniformly on E, as $\delta \to 0$.*

Proof. Let $\{r_j\}$ be an enumeration of the positive rational numbers, and let

$$S(n, k) = \bigcap_{r_j < n^{-1}} \{x : \sigma(Q(x, r_j))r_j^{-N} \geqq k\}.$$

Then $S(n, k)$ is measurable and $S(n + 1, k) \supseteq S(n, k)$. By the differentiation theorem for singular measures [**3**],

$$\sigma\left(\bigcup_n S(n, k)\right) = \sigma(\mathbf{R}^N),$$

and hence

$$\lim_{n\to\infty} \sigma(S(n, k)) = \sigma(\mathbf{R}^N), \quad \text{for each } k > 0.$$

Now choose n_k so that

$$\sigma(\mathbf{R}^N \backslash S(n_k, k)) < \epsilon\, 2^{-k}$$

and let

$$E = \bigcap_k S(n_k, k).$$

Then $\sigma(\mathbf{R}^N \backslash E) < \epsilon$. Let $k > 0$ be given, then for all $x \in E$ we have

$$x \in S(n_k, k) = \bigcap_{r_j < n_k^{-1}} \{x : \sigma(Q(x, r_j))r_j^{-N} \geqq k\}.$$

Now if $0 < \delta \leqq n_k^{-1}$ and $0 < r_j \leqq \delta$ then

$$\sigma(Q(x, \delta)) \geqq \sigma(Q(x, r_j)) \geqq k r_j^{-N}.$$

Since there are such r_j as close to δ as we please it follows that if $x \in E$ and if $\delta \leqq n_k^{-1}$ then

$$\sigma(Q(x, \delta))\, \delta^{-N} \geqq k.$$

LEMMA 2. *There exists an $\epsilon_N > 0$, depending only on N, such that if $w, z \in \mathbf{R}^N$, $y \in \mathbf{R}$, with $d(w, z) \leqq \delta$ and $0 < y \leqq 2\,\delta$. Then*

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$$\sigma(w, y) \geqq \epsilon_N y \sigma(Q(z, \delta))\delta^{-(N+1)}.$$

Proof. If $d(w, z) \leqq \delta$ and $x \in Q(z, \delta)$, then $d(x, \omega) \leqq 2\delta$, it follows that

$$\begin{aligned}
\sigma(w, y) &= C_N \int \frac{y d\sigma(x)}{(y^2 + |x - w|^2)^{(N+1)/2}} \\
&\geqq C_N \int_{Q(z,\delta)} \frac{y d\sigma(x)}{(y^2 + |x - w|^2)^{(N+1)/2}} \\
&\geqq C_N \int_{Q(z,\delta)} \frac{y d\delta(x)}{[4\delta^2 + Nd(x, w)^2]^{(N+1)/2}} \\
&\geqq \frac{C_N}{(N + 4N)^{(N+1)/2}} y\sigma(Q(z, \delta))\, \delta^{-(N+1)} \\
&= \epsilon_N y \sigma(Q(z, \delta))\, \delta^{-(N+1)}.
\end{aligned}$$

We define the modulus of continuity of σ by

$$\omega(\delta) = \sup_{x \in \mathbf{R}^N} \sigma(Q(x, \delta))$$

and we let

$$\delta(y) = \inf\{\delta : \omega(\delta)\delta^{-(N+1)} \leqq (\epsilon_N y)^{-1}\}.$$

We will need an elementary covering lemma; $\dot{Q}(x, \delta)$ denotes the interior of $Q(x, \delta)$. $\dot{Q}(x, \delta)$ is called an open cube.

LEMMA 3. *If A is a finite collection of open cubes, then there is a disjoint subcollection, B, such that if $\dot{Q} \in A$ then there exists $\dot{Q}(x, \delta) \in B$ with $\dot{Q} \subseteq \dot{Q}(x, 3\delta)$.*

THEOREM 1. *There is a constant τ_N, depending only on N, such that for any σ there is a $y_0 > 0$ with the property that if $0 < y \leqq y_0$,*

$$|E_y| = |\{w : \hat{\sigma}(w, y) > 1\}| \geqq \tau_N \|\sigma\|\, y/\delta(y).$$

(Here $\|\sigma\| = \int_{\mathbf{R}_N} d\sigma.$*)*

Proof. Let σ be given. By Egorov's theorem (Lemma 1) and the regularity of σ there is a compact set K such that

$$\sigma(K) \geqq \|\sigma\|/2 \quad \text{and} \quad \sigma(Q(x, \delta))\delta^{-N} \to \infty,$$

uniformly on K, as $\delta \to 0$. In particular, there is a $y_0 > 0$ such that if $0 < \delta \leqq y_0$ then

$$\sigma(Q(x, \delta))\delta^{-N} \geqq \epsilon_N{}^{-1}, \quad \text{for all } x \in K.$$

Now fix $0 < y \leqq y_0$ and $x \in K$ and define

$$\delta(x, y) = \inf\{\delta : \sigma(Q(x, \delta))\delta^{-(N+1)} \leqq (\epsilon_N y)^{-1}\}.$$

We list some properties.

i) $\sigma(Q(x, \delta(x, y)))\delta(x, y)^{-(N+1)} \leqq (\epsilon_N y)^{-1}$,

ii) $\sigma(Q(x, \delta))\delta^{-(N+1)} > (\epsilon_N y)^{-1}$, if $0 < \delta < \delta(x, y)$,

iii) $y \leqq \delta(x, y)$,

iv) $\delta(x, y) \leqq \delta(y)$,

v) If $w \in \dot{Q}(x, \delta(x, y))$, then $\hat{\sigma}(w, y) > 1$.

To prove i) note that there exists $\delta_j \searrow \delta(x, y)$ such that

$$\sigma(Q(x, \delta_j))\delta_j{}^{-(N+1)} \leqq (\epsilon_N y)^{-1}.$$

Now for each j,

$$\sigma(Q(x, \delta(x, y))\delta_j{}^{-(N+1)} \leqq \sigma(Q(x, \delta_j))\delta_j{}^{-(N+1)} \leqq (\epsilon_N y)^{-1}.$$

Now let $j \to \infty$. Number ii) follows from the definition of infimum. As for iii), if $0 < \delta < y$, then

$$\sigma(Q(x, \delta))\delta^{-(N+1)} = \sigma(Q(x, \delta))\delta^{-N}\delta^{-1} \geqq \epsilon_N{}^{-1}\delta^{-1} > (\epsilon_N y)^{-1}$$

and hence $\delta(x, y) \geqq y$. Number iv) follows immediately from the fact that

$$\sigma(Q(x, \delta)) \leqq \omega(\delta).$$

Finally, we want to use Lemma 2 to prove v). If $w \in \dot{Q}(x, \delta(x, y))$, then $d(x, w) < \delta(x, y)$. By iii), $y \leqq \delta(x, y)$ and hence there is a δ, $d(x, y) < \delta < \delta(x, y)$, and $y \leqq 2\delta$. It now follows from Lemma 2 that

$$\hat{\sigma}(w, y) \geqq \epsilon_N y\, \sigma(Q(x, \delta))\delta^{-(N+1)}.$$

This last expression is greater than 1 by ii). This establishes i)-v).

To proceed with the proof of Theorem 1, we take a fixed y, $0 < y \leqq y_0$. Now

$$K \subseteq \bigcup_{x \in K} \dot{Q}(x, \delta(x, y)/3),$$

and so by the compactness of K and the covering Lemma 3 there exists $x_1, \ldots, x_n \in K$ such that $\{\dot{Q}(x_j, \delta(x_j, y)/3\}$ is a disjoint collection and

$$K \subseteq \bigcup_{j=1}^{n} \dot{Q}\,(x_j, \delta\,(x_j, y)\,).$$

We have

$$\begin{aligned}
\frac{\|\sigma\|}{2} &\leqq \sigma(K) \leqq \sum_{j=1}^{n} \sigma(\,\dot{Q}(x_j, \delta(x_j, y)\,)\,) \\
&\leqq \sum_{j=1}^{n} \sigma(Q(x_j, \delta(x_j, y)\,)\,) \\
&\leqq (\epsilon_N y)^{-1} \sum_{j=1}^{n} \delta(x_j, y)^{N+1} \\
&\leqq (\epsilon_N y)^{-1}\, \delta(y) \sum_{j=1}^{n} \delta(x_j, y)^{N} && \text{(by iv)} \\
&= (3/2)^N(\epsilon_N y)^{-1}\delta(y) \sum_{j=1}^{n} \left[\frac{2}{3}\,\delta(x_j, y)\right]^N \\
&= (3/2)^N(\epsilon_N y)^{-1}\delta(y) \sum_{j=1}^{n} |\,\dot{Q}(x_j, \delta(x_j, y)/3| \\
&= (3/2)^N(\epsilon_N y)^{-1}\delta(y)|\bigcup_{j=1}^{n}\,\dot{Q}(x_j, \delta(x_j, y)/3)| \\
&\leqq (3/2)^N(\epsilon^N y)^{-1}\delta(y)|\bigcup_{j=1}^{n}\,\dot{Q}(x_j, \delta(x_j, y)\,)| \\
&\leqq (3/2)^N\,(\epsilon_N y)^{-1}\,\delta(y)|E_y|, && \text{(by v).}
\end{aligned}$$

It follows that

$$|E_y| \geqq \tau_N\|\sigma\|y/\delta(y), \quad \text{where } \tau_N = (2/3)^N\epsilon_N/2.$$

The proof is complete.

Before stating a corollary, note that if σ and μ are two such measures, and $\omega_\sigma \leqq \omega_\mu$, then $\delta_\sigma \leqq \delta_\mu$. This follows from the inequality

$$\omega_\sigma(\delta)\delta^{-(N+1)} \leqq \omega_\mu(\delta)\delta^{-(N+1)}.$$

COROLLARY. *There is an* $\eta_N > 0$, *depending only on N, such that for every* σ *there is an* $y_0 > 0$ *with the property that if* $0 < y \leqq y_0$ *then*

$$|E_y| \geqq \eta_N(\,\|\sigma\|\,y)^{N/(N+1)}.$$

Proof. Let μ be a point mass at $0 \in \mathbf{R}^N$ with total mass $\|\sigma\|$. Then

$$\omega_\mu(\delta) \equiv \|\sigma\|, \quad \delta > 0,$$

and hence

$$\omega_\delta \leqq \|\sigma\| \leqq \omega_\mu.$$

We conclude that $\delta_\sigma \leqq \delta_\mu$. We may calculate that

$$\delta_\mu(y) = (\|\sigma\| \, \epsilon_N y)^{1/(N+1)}$$

and hence that

$$|E_y| \geqq \tau_N \|\sigma\| \, y/\delta_\sigma(y) \geqq \tau_N \|\sigma\| y/(\|\sigma\| \, \epsilon_N y)^{1/(N+1)} = \eta_N(\|\sigma\| \, y)^{N/((N+1)}.$$

We shall see in the next section that if σ consists of a single point mass then $y^{N/(N+1)}$ is the exact rate of vanishing for $|E_y|$. Of course it is not difficult to check this by direct calculation.

2. In this section we investigate the extent to which Theorem 1 is sharp. To see what is happening we make the following observation: suppose that $\mu \leqq \sigma$, then of course $\omega_\mu \leqq \omega_\sigma$ and hence $\delta_\mu \leqq \delta_\sigma$. It follows that

$$|\{w:\hat{\sigma}(w, y) > 1\}| \geqq |\{w:\hat{\mu}(w, y) > 1\}| \geqq \tau_N \, \|\mu\| \, y/\delta_\mu(y).$$

Note that $y/\delta_\mu(y)$ goes to zero more slowly than $y/\delta_\sigma(y)$. In other words, we get the most information from Theorem 1 by applying it to the smoothest possible (non-zero) measure μ, with $\mu \leqq \sigma$. That is to say that we can't expect the inequality of Theorem 1 to be reversed unless σ has the property that if $0 \leqq \mu \leqq \sigma$ and $\mu \not\equiv 0$ then μ is not really smoother than σ. The point of this section is to show that for such "homogeneous" measures the inequality of Theorem 1 can be reversed.

Definition. The measure σ is said to be *homogeneous* if there exists $\epsilon > 0$ such that for all $x \in$ support σ, $\sigma(Q(x, \delta)) \geqq \epsilon\omega(\delta)$.

First we shall get a bound from above on $|E_y|$ for any σ whose support has Lebesgue measure 0. Then we will show that this bound agrees with the bound from below given by Theorem 1 when σ is homogeneous.

Definition. If K is compact and $|K| = 0$ define

$$\rho(\delta) = \rho_K(\delta) = |\bigcup_{x \epsilon K} \dot{Q}(x, \delta)|.$$

LEMMA 4. $\rho_K(2\delta) \leqq 6^N \rho_K(\delta)$.

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Proof. Fix $\delta > 0$ and let $\epsilon > 0$ be given. Take L to be a compact subset of $K_{2\delta} \ni |L| \geqq (1-\epsilon)\rho(2\delta)$. Now

$$L \subseteq \bigcup_{x\epsilon K} \dot{Q}(x, 2\delta),$$

so by compactness of L and Lemma 3, there are $x_1, \ldots, x_n \in K$ such that $\{\dot{Q}(x_j, 2\delta)\}$ is disjoint and

$$L \subseteq \bigcup_{j=1}^{n} \dot{Q}(x_j, 6\delta).$$

It follows that $|L| \leqq n(12\delta)^N$. On the other hand

$$\bigcup_{j=1}^{n} \dot{Q}(x_j, \delta) \subseteq K_\delta$$

and since these cubes are disjoint

$$n(2\delta)^N \leqq |K_\delta| = \rho_K(\delta).$$

So, we have

$$(1-\epsilon)\rho(2\delta) \leqq |L| \leqq n(12\delta)^N = 6^N n(2\delta)^N \leqq 6^N \rho(\delta).$$

Since $\epsilon > 0$ is arbitrary, the proof is complete.

THEOREM 2. *There is a constant D_N, depending only on N, such that if K = support σ is compact and* $|K| = 0$, *then*

$$|E_y| \leqq D_N \rho_K(\delta(y)).$$

Proof. Suppose that $\sigma(Q(w, \delta)) = 0$, then

$$\begin{aligned}
\hat{\sigma}(\omega, y) &= C_N \int_{d(x,w)>\delta} \frac{y d\sigma(x)}{[y^2 + |x-w|^2]^{(N+1)/2}} \\
&\leqq C_N \sum_{k=0}^{\infty} \int_{2^k\delta < d(x,w) \leqq 2^{k+1}\delta} \frac{y d\sigma(x)}{[y^2 + d(x, w)^2]^{(N+1)/2}} \\
&\leqq C_N \sum_{k=0}^{\infty} \frac{y}{(2^k\delta)^{N+1}} \sigma(Q(w, 2^{k+1}\delta)) \\
&\leqq C_N y \delta^{-(N+1)} \sum_{k=0}^{\infty} 2^{-(N+1)k} \omega(2^{k+1}\delta) \\
&\leqq C_N y \delta^{-(N+1)} \omega(\delta) \sum_{k=0}^{\infty} 2^{-(N+1)k} 2^{(k+1)N} \\
&= 2^{N+1} C_N y \omega(\delta) \delta^{-(N+1)}.
\end{aligned}$$

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(We have used the obvious inequality $\omega(2\delta) \leqq 2^N \omega(\delta)$.) Now fix m, depending only on N, so that

$$\epsilon_N^{-1}2^{N+1}C_N 2^{-m} \leqq 1$$

and suppose that $d(w, K) > 2^m \delta(y) = \delta$, then our calculation shows that

$$\begin{aligned}\hat{\sigma}(\omega, y) &\leqq 2^{N+1}C_N\omega(2^m\delta(y))(2^m\delta(y))^{-(N+1)}\\ &\leqq 2^{N+1}C_N 2^{mN}y\omega(\delta(y))2^{-(N+1)m}\delta(y)^{-(N+1)} \leqq 1,\end{aligned}$$

by the definitions of m and $\delta(y)$. In other words,

$$E_y = \{w:\hat{\sigma}(\omega, y) > 1\} \subseteq \{w:d(w, K) < 2^m\delta(y)\}$$

and so

$$|E_y| \leqq \rho_K(2^m\delta(y)) \leqq 6^{mN}\rho_K(\delta(y)) = D_N\rho_K(\delta(y)).$$

THEOREM 3. *Suppose that* σ *is homogeneous and that* $K =$ *support* σ *is compact and* $|K| = 0$. *Then there is a constant* A *such that*

$$|E_y| \leqq Ay\delta(y)^{-1}.$$

Proof. Because of Theorem 2 it is sufficient to show that

$$\rho(\delta(y)) \leqq Ay\delta(y)^{-1} \quad \text{for some constant } A.$$

For this it is sufficient to show that there is a constant A such that

$$\rho_K(\delta)\omega(\delta) \leqq A\delta^N, \quad \text{for all } \delta > 0.$$

For suppose this is established, then for $\delta < \delta(y)$ we have

$$\rho_K(\delta) \leqq A\delta^N\omega(\delta)^{-1} \leqq A\delta^{N+1}\omega(\delta)^{-1}\delta^{-1} < A\epsilon_N y\delta^{-1},$$

now let $\delta \to \delta(y)$ and we see that

$$\rho_K(\delta(y)) \leqq A\epsilon_N y\delta(y)^{-1}.$$

To show that $\rho_K(\delta)\omega(\delta) \leqq A\delta^N$, take $\delta > 0$ and let $\eta > 0$ be given. There is a compact set L, $K \subseteq L \subseteq K_\delta$ such that

$$|L| \geqq (1-\eta)\rho(\delta).$$

By compactness of L and Lemma 3 there are $x_1, \dots, x_n \in K$ such that $\{Q(x_j, \delta)\}$ is a disjoint collection and

$$L \subseteq \bigcup_{j=1}^{n} Q(x_j, 3\delta).$$

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Hence $|L| \leqq n(6\delta)^N$. Now since $\{\dot{Q}(x_j, \delta)\}$ are disjoint and σ is homogeneous

$$\|\sigma\| \geqq \sum_{j=1}^{n} \sigma(\dot{Q}(x_j, \delta)) \geqq n \in \omega(\delta)$$

$$\geqq \epsilon\omega(\delta)| L |(6\delta)^{-N} \geqq \epsilon(1 - \eta)6^{-N}\rho(\delta)\omega(\delta)\delta^{-N}.$$

The theorem follows.

We conclude this section by stating a theorem due essentially to Hausdorff, which guarantees the existence of homogeneous singular measures of any preassigned modulus of continuity. For the proof when $N = 1$, see [**1**], or [**7**].

THEOREM (Hausdorff). *Suppose that ω is a positive increasing function defined for $\delta > 0$ such that $\omega(2\delta) \leqq 2^N\omega(\delta)$ and*

$$\lim_{\delta \to 0} \omega(\delta)\delta^{-N} = \infty.$$

Then there is a homogeneous singular measure σ and positive constants A and B such that

$$A\omega_\sigma(\delta) \leqq \omega(\delta) \leqq B\omega_\sigma(\delta), \ \delta > 0.$$

3. In this section we give some applications of the previous sections to inner functions. We refer to [**8**] for details about the structure of inner functions. We shall use Theorem 1 in the unit disc rather than the upper half plane when $N = 1$. In the disc it takes the following form.

THEOREM 4. *There is a constant $\tau > 0$ such that for any positive singular Borel measure σ on the unit circle there is an r_0 with the property that*

$$| \{\theta:\hat{\sigma}(re^{i\theta}) > 1\} | \geqq \tau\|\sigma\| (1-r)\delta(r)^{-1} \quad \text{if } r_0 \leqq r < 1.$$

Of course

$$\hat{\sigma}(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-r^2}{|re^{i\theta} - e^{it}|^2} d\sigma(t) \quad \text{and}$$

$$\delta(r) = \inf \{\delta:\omega(\delta)\, \delta^{-2} \leqq (\epsilon_1(1-r))^{-1}\}.$$

In particular we have the following corollary.

COROLLARY. *For each σ there are constants $\epsilon > 0$ and $0 < r_0 < 1$ such that*

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$$|\{\theta:\hat{\sigma}(re(\theta) > 1\}| \geqq \epsilon\sqrt{1-r}, \quad \textit{if } r_0 \leqq r < 1.$$

THEOREM 5. *If σ is a measure on the unit circle and*

$$\varphi(z) = \exp\left\{-\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, d\sigma(t)\right\}$$

is the associated inner function, then for each $q < \infty$ there is a constant $\epsilon_q > 0$ such that

$$\int_0^{2\pi} (1-|\varphi(re^{i\theta})|)^q \frac{d\theta}{2\pi} \geqq \epsilon_q(1-r)\delta(r)^{-1}.$$

Proof. Let $E_r = \{\theta:\hat{\sigma}(re^{i\theta}) > 1\}$, by Theorem 4,

$$|E_r| \geqq \epsilon(1-r)\delta(r)^{-1}$$

for some $\epsilon > 0$ if r is close to 1. Now

$$\int_0^{2\pi} (1-|\varphi(re^{i\theta})|)^q d\theta \geqq \int_{E_r} (1-|\varphi(re^{i\theta})|)^q d\theta$$
$$\geqq (1-e^{-1})^q |E_r| = \epsilon_q (1-r)\delta(r)^{-1}.$$

COROLLARY 1. *If φ is a singular inner function then for each $q < \infty$ there is a constant $\epsilon_q > 0$ such that*

$$\int_0^{2\pi} (1-|\varphi(re^{i\theta})|)^q \frac{d\theta}{2\pi} > \epsilon_q\sqrt{1-r}.$$

Proof. This follows from the corollary to Theorem 4 in the same way Theorem 5 follows from Theorem 4.

COROLLARY 2. *If φ is a singular inner function, $p > 0$, $q > 0$, $\alpha > -1$, and $p(1-(2q)^{-1}) \geqq 1 + \alpha$, then*

$$\int_0^1 (1-r)^\alpha \left(\int_{-\pi}^{\pi} \left(\frac{1-|\varphi(re^{i\theta})|}{1-r}\right)^q \frac{d\theta}{2\pi}\right)^{p/q} dr = \infty.$$

The result we are aiming for in this section is the corollary to Theorem 6 below. Theorem 6 allows us to replace the quantity $1-|\varphi(re^{i\theta}|/1-r$ by $|\varphi'(re^{i\theta})|$ in Corollary 2 to Theorem 5. We need two preliminaries. The first is just a formal statement of a method used by Hardy and Littlewood.

LEMMA 5. *Suppose that $f:[0, 1) \to [0, \infty)$ and $F(r) = \sup_{0\leqq\rho\leqq r} f(\rho)$ and that $0 < p \leqq 1$, then there is a constant C_p such that*

$$\left(\int_0^1 f(r)dr\right)^p \leqq C_p \int_0^1 (1-r)^{p-1}F(r)^p dr.$$

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Proof. Let $r_n = 1 - 2^{-n}$, then

$$\left(\int_0^1 f(r)dr\right)^p = \left(\sum_{n=1}^{\infty} \int_{r_{n-1}}^{r_n} f(r)dr\right)^p$$

$$< \left(\sum_{n=1}^{\infty} F(r_n)2^{-n}\right)^p \leqq \sum_{n=1}^{\infty} F(r_n)^p 2^{-np}$$

$$= \frac{(1-2^{-p})}{p} \sum_{n=1}^{\infty} F(r_n)^p \int_{r_n}^{r_{n+1}} (1-r)^{p-1}\, dr$$

$$< C_p \sum_{n=1}^{\infty} \int_{r_n}^{r^{n+1}} F(r)^p(1-r)^{p-1}dr \leqq C_p \int_0^1 F(r)^p(1-r)^{p-1}dr.$$

We have used the fact that F is increasing.

We also need a general version of Hardy's inequality, see [**5**], page 245.

LEMMA 6. *If $f:(0, \infty) \to [0, \infty)$, $p > 1$, $\alpha > -1$, and $p > 1 + \alpha$, then*

$$\int_0^{\infty} x^{\alpha}\left(\frac{1}{x} \int_0^x f(t)dt\right)^p dx \leqq \left(\frac{p}{p-1-\alpha}\right)^p \int_0^{\infty} x^{\alpha} f(x)^p dx.$$

After a change of variables in Lemma 6 we get the following.

LEMMA 7. *If $f:(0, 1) \to [0, \infty)$, $p > 1$, $\alpha > -1$, and $p > 1 + \alpha$*

$$\int_0^1 (1-r)^{\alpha}\left(\frac{1}{1-r} \int_r^1 f(\rho)d\rho\right)^p dr \leqq \left(\frac{p}{p-1-\alpha}\right)^p \int_0^1 (1-r)^{\alpha} f(r)^p\, dr.$$

The case $p > 1$, $q = 1$ and $\alpha = 0$ of the next theorem was proved by Alan Gluchoff [**4**]. The case $p = q = 1$ was proved in [**2**].

THEOREM 6. *Suppose that $\alpha > -1$, $q > 0$, $p > 1 + \alpha$, and that φ is an inner function. Then*

$$\int_0^1 (1-r)^{\alpha}\left(\int_0^{2\pi} |\varphi'(re^{i\theta})|^q \frac{d\theta}{2\pi}\right)^{p/q} dr < \infty,$$

if and only if

$$\int_0^1 (1-r)^{\alpha}\left(\int_0^{2\pi} \left(\frac{1-|\varphi(re^{i\theta})|}{1-r}\right)^q \frac{d\theta}{2\pi}\right)^{p/q} dr < \infty.$$

Proof. For any function φ, holomorphic and bounded by 1 we have

$$|\varphi'(re^{i\theta})| \leqq \frac{1-|\varphi(re^{i\theta})|^2}{1-r^2}$$

and so one direction is clear. In the other direction we start with the inequality

3.1 $$1 - |\varphi(re^{i\theta})| \leqq \int_r^1 |\varphi'(\rho e^{i\theta})|\, d\rho,$$

which is valid for almost all θ because φ is an inner function. First we suppose that $q \geqq 1$ and apply the continuous form of Minkowski's inequality to 3.1 to obtain

3.2 $$\left(\int_0^{2\pi} (1-|\varphi(re^{i\theta})|)^q \frac{d\theta}{2\pi}\right)^{1/q} \leqq \int_r^1 M_q(\rho, \varphi')\, d\rho$$

where as usual,

$$M_q(\rho, \varphi^1) = \left(\frac{1}{2\pi}\int_0^{2\pi} |\varphi'(\rho e^{i\theta})|^q d\theta\right)^{1/q}.$$ Now raise both sides of 3.2 to the power p, multiply by $(1-r)^{\alpha-p}$ and integrate on r to get

3.3. $$\int_0^1 (1-r)^\alpha \left(\int_0^{2\pi}\left(\frac{1-|\varphi(re^{i\theta})|}{1-r}\right)^q \frac{d\theta}{2\pi}\right)^{p/q} dr$$
$$\leqq \int_0^1 (1-r)^{\alpha-p}\left(\int_r^1 M_q(\rho, \varphi')d\rho\right)^p dr.$$

At this point we distinguish 2 sub-cases; $p > 1$, and $p \leqq 1$. If $p \leqq 1$ we apply Lemma 5 and use the fact that $M_q(\rho, \varphi')$ is increasing to see that the right hand side of 3.3 is at most a constant times

3.4 $$\int_0^1 (1-r)^{\alpha-p}\int_r^1 M_q(\rho, \varphi')^p (1-\rho)^{p-1} d\rho dr$$
$$= \int_0^1 M_q(\rho, \varphi')(1-\rho)^{p-1}\int_0^\rho (1-r)^{\alpha-p} dr d\rho$$
$$\leqq \frac{1}{p-1-\alpha}\int_0^1 M_q(\rho, \varphi')^p (1-\rho)^\alpha d\rho,$$

because $p > 1 + \alpha$. This completes the proof if $q \geqq 1$ and $p \leqq 1$. If $q \geqq 1$ and $p > 1$, we return to 3.3 and write the right hand side as

3.5 $$\int_0^1 (1-r)^\alpha\left(\frac{1}{1-r}\int_r^1 M_q(\rho, \varphi')d\rho\right)^p dr.$$

By Lemma 7, this is bounded by a constant times

3.6 $$\int_0^1 (1-r)^\alpha M_q(r, \varphi')^p dr.$$

This completes the proof when $q \geqq 1$.

When $q < 1$ we return to 3.1 and raise both sides to the power q to obtain

3.7 $$(1-|\varphi(re^{i\theta})|)^q \leqq \left(\int_r^1 |\varphi'(\rho e^{i\theta})|\,d\rho\right)^q.$$

By Lemma 5 the right hand side of 3.7 is at most a constant times

3.8 $$\int_r^1 (1-\rho)^{q-1} \max_{0<t<\rho} |\varphi'(te^{i\theta})|^p d\rho.$$

If we now integrate on θ and use the Hardy-Littlewood complex maximal theorem we obtain

3.9 $$\int_0^{2\pi} (1-|\varphi(re^{i\theta})|)^q \frac{d\theta}{2\pi} \leqq C_q \int_r^1 (1-\rho)^{q-1} M_q(\rho, \varphi')^q\,d\rho.$$

Now raise both sides of 3.9 to the power p/q, multiply by $(1-r)^{\alpha-p}$ and integrate on r to arrive at

3.10 $$\int_0^1 (1-r)^\alpha \left(\int_0^{2\pi} \left(\frac{1-|\varphi(re^{i\theta})|}{1-r}\right)^q \frac{d\theta}{2\pi}\right)^{p/q} dr$$
$$\leqq C \int_0^1 (1-r)^{\alpha-p} \left(\int_r^1 (1-\rho)^{q-1} M_q(\rho, \varphi')^q\,d\rho\right)^{p/q} dr.$$

Again two sub-cases are considered. First suppose that $p/q \leqq 1$. We note that $(1-\rho)^{q-1} M_q(\rho, \varphi')^q$ is increasing since $q < 1$ and apply Lemma 5 to see that the right hand side of 3.10 is bounded by a constant times

3.11 $$\int_0^1 (1-r)^{\alpha-p} \int_r^1 (1-\rho)^{(q-1)p/q} M_q(\rho, \varphi')^p (1-\rho)^{\frac{p}{q}-1} ds dr$$
$$= \int_0^1 (1-r)^{\alpha-p} \int_r^1 M_q(\rho, \varphi')^p (1-\rho)^{p-1}\,d\rho dr$$
$$< C \int_0^1 M_q(\rho, \varphi')^p (1-\rho)^\alpha d\rho.$$

If $p/q > 1$, we return to the right hand side of 3.10 and apply Lemma 7 as at an earlier point in the proof. We omit the details. This completes the proof of Theorem 6.

COROLLARY. *If φ is a singular inner function, $p > 0$, $q > 0$, $\alpha > -1$, and* $p\left(1-\dfrac{1}{2q}\right) \geqq 1 + \alpha$, *then*

$$\int_0^1 (1-r)^\alpha M_q(r, \varphi')^p dr = \infty.$$

We point out that it follows from the inequalities for the atomic inner function proved by Jevtić, **[6]**, that if φ is the atomic function and

$p\left(1-\frac{1}{2q}\right) < 1 + \alpha$ then

$$\int_0^1 (1-r)^\alpha M_q(r, \varphi')^p dr < \infty.$$

As a final remark we say that Theorems 5 and 6 can be used to show how the smoothness of the measure σ determines whether or not

$$\int_0^1 (1-r)^\alpha M_q(r, \varphi')^p dr < \infty$$

as was done in [**1**] for the case $p = q = 1$.

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