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LETTERS TO THE EDITOR

AN ORDERING INEQUALITY FOR EXCHANGEABLE RANDOM VARIABLES

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Abstract

Let $X_1, \cdots, X_n$ be exchangeable random variables with finite variance and two sequences of constants satisfying $a_1 \leqq \cdots \leqq a_n$, $b_1 \leqq \cdots \leqq b_n$. Suppose that $a'_1, \cdots, a'_n$ is a rearrangement of $a_1, \cdots, a_n$ and that $g(x)$ is a non-decreasing function. Then

$$E \sum a'_i X_i g\left(\sum b_i X_i\right) \leqq E \sum a_i X_i g\left(\sum b_i X_i\right).$$

1. Introduction

Let $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ be two sequences with the same ordering, for example, both non-decreasing. Then a classic inequality (Hardy et al. (1934), Chapter 10) tells us that, if $a'_1, \cdots, a'_n$ is a rearrangement of $a_1, \cdots, a_n$, then $\sum a'_i b_i \leqq \sum a_i b_i$. Indeed this classic inequality shows that to establish

$$E \sum a'_i X_i g\left(\sum b_i X_i\right) \leqq E \sum a_i X_i g\left(\sum b_i X_i\right) \tag{1}$$

we have only to prove, under the conditions of the theorem, that

$$EX_1 g\left(\sum b_i X_i\right) \leqq EX_2 g\left(\sum b_i X_i\right) < \cdots \leqq EX_n g\left(\sum b_i X_i\right). \tag{2}$$

This will be done in Section 2.

In the course of proving (see Watson (1985)) a multivariate result concerned with rank-s orthogonal projectors Q uniformly distributed on the appropriate Grassmann manifold, it was observed that since the diagonal elements Q_{ii} of Q have an exchangeable distribution, the proof would go through if the inequality (1), with g replaced by exp, were true. This application is sketched in Section 3. The result may be useful outwith multivariate analysis.

2. Proof of (2)

We begin with a special case which is entirely analogous to the deterministic result.

Lemma 1. For exchangeable real random variables $X_1, \cdots, X_n$ constants $a_1 \leqq \cdots \leqq a_n$,

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$b_1 \leqq \cdots \leqq b_n$, and $a'_1, \cdots, a'_n$ any rearrangement of $a_1, \cdots, a_n$,

$$E(\sum a'_i X_i)(\sum b_i X_i) \leqq E(\sum a_i X_i)(\sum b_i X_i). \tag{3}$$

Proof. The left-hand side of (3) may be written

$$\sum \sum a'_i b_j E X_i X_j = \sum a'_i b_i (c-d) + d(\sum a'_i)(\sum b_i)$$

where, by exchangeability, $EX_i^2 = EX_j^2 = c$, $EX_iX_j = d$, $E(X_i - X_j)^2 = 2(c-d) \geqq 0$. But since $\sum a'_i = \sum a_i$ and $\sum a'_i b_i \leqq \sum a_i b_i$, (3) is proved.

To prove (2), there is no loss of generality in assuming that $P(X_1 > X_2) = P(X_1 < X_2) = \frac{1}{2}$, for the only alternative to this is $X_1 = \cdots = X_n$ when there is nothing to prove. Further, it suffices to prove that

$$E(X_2 - X_1)g(\sum b_i X_i) \geqq 0, \tag{4}$$

the first inequality in (2), since all others follow from the same argument. But the left-hand side of (4) is $P(X_2 > X_1)$ times the sum of two conditional expectations,

$$E\{(X_2 - X_1)g(\sum b_i X_i) \mid X_2 > X_1\} + E\{(X_2 - X_1)g(\sum b_i X_i) \mid X_2 < X_1\}. \tag{5}$$

By the exchangeability of X_1 and X_2, X_1 and X_2 may be interchanged in the second term of (5) which can then be rewritten as

$$E\{(X_2 - X_1)[g(b_1X_1 + b_2X_2 + \sum{}' b_iX_i) - g(b_1X_2 + b_2X_1 + \sum{}' b_iX_i)\} \mid X_2 > X_1\},$$

where $\sum' b_iX_i$ is $\sum b_iX_i$ excluding the first two terms. Clearly the first factor $X_2 - X_1$ is positive. The second factor is non-negative because $b_1 \leqq b_2$, $X_1 < X_2$ implies that $b_1X_1 + b_2X_2 \geqq b_1X_2 + b_2X_1$ and g is a non-decreasing function. Hence the theorem is proved.

3. Remarks

The seemingly trivial inequality (1), or its equivalent form (2), enables us to prove easily results which are otherwise rather baffling. For example, let M be a symmetric $q \times q$ matrix with eigenvalues $\lambda_1(M) \geqq \cdots \geqq \lambda_q(M)$, Q a symmetric $q \times q$ idempotent matrix of rank $s < q$ uniformly distributed on its Grassmann manifold, G, and define

$$N = \int_G Q \exp \operatorname{trace}(MQ)\, \Delta(dQ), \tag{6}$$

where $\Delta(dQ)$ is the invariant measure on G integrating to unity. It may be shown without too much trouble that N and M commute and so have the same eigensubspaces. It then follows that

$$\lambda_i(N) = \int_G Q_{ii} \exp \sum \lambda_i(M) Q_{ii}\, \Delta(dQ). \tag{7}$$

We may use the inequality of this paper, and the 'classic' inequality, to show that

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$\lambda_1(N) \geqq \cdots \geqq \lambda_q(N)$. For let $w_1, \cdots, w_q$ be a reordering of the $\lambda_i(M)$ and consider

$$\sum w_i \lambda_i(N) = \int_G \sum w_i Q_{ii} \exp \sum \lambda_i(M) Q_{ii} \, \Delta(dQ). \tag{8}$$

By symmetry, the Q_{ii} are exchangeable. The inequality (1) then tells us that the right-hand side of (6) is a maximum if $w_1 \geqq \cdots \geqq w_q$. Since this is true of the left-hand side of (7), the 'classic' inequality requires that $\lambda_1(N) \geqq \cdots \geqq \lambda_q(N)$.

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