A NOTE ON HYPERCONVEXITY IN RIEMANNIAN MANIFOLDS

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1. Summary. Let M denote a connected Riemannian manifold of class C^3, with positive definite C^2 metric. The curvature tensor then exists, and is continuous.

By a classical theorem of J. H. C. Whitehead (**1**), every point x of M has the property that all sufficiently small spherical neighbourhoods V of x are *convex*; that is, (i) to every $y, z \in V$ there is one and only one geodesic segment yz in M which is the shortest path joining them: $f:[0, 1] \to M, f(0) = y$, $f(1) = z$; and (ii) this segment yz lies entirely in $V: f([0, 1]) \subset V$; (iii) if f is parametrized proportional to arc length, then $f(t)$ is a C^2 function of y, t, and z.

Let V be a convex set in M; and let $y_1, y_2, z_1, z_2 \in V$. Let f_1, f_2: $[0, 1] \to V$ denote the geodesic segments y_1z_1, y_2z_2, each parametrized proportional to arc length. Then for each t the points $f_1(t), f_2(t)$ are called *corresponding points* of the geodesic segments y_1z_1, y_2z_2. In particular, y_1, y_2 are corresponding points; and so are z_1, z_2.

The distance between points x and y, denoted by $\rho(x, y)$, is the greatest lower bound of the lengths of rectifiable paths joining x and y. The diameter $D(A)$ of a set A is, as usual

$$D(A) = \sup_{x,y \in A} \rho(x, y).$$

Let V be a convex open set of M, and γ a positive number. Then V is called *γ-hyperconvex* if for every positive number ϵ and any geodesic segments y_1z_1, y_2z_2 in V, the inequalities $\rho(y_1, y_2) < \epsilon$, $\rho(z_1, z_2) < \gamma\epsilon$ imply that corresponding points of y_1z_1, y_2z_2 have distance less than ϵ. Clearly, γ has to satisfy $\gamma \leqslant 1$.

If V is γ-hyperconvex, and W is a convex subset of V, then also W is γ-hyperconvex. If V is γ-hyperconvex, and if $0 < \gamma' < \gamma$, then V is also γ'-hyperconvex.

THEOREM I. *Every point of a Riemannian manifold has a $\frac{1}{2}$-hyperconvex neighbourhood.*

Received June 2, 1958. Research supported by an Office of Naval Research Contract at the University of Washington.

https://doi.org/10.4153/CJM-1959-052-6 Published online by Cambridge University Press

This theorem* is a corollary of Theorem II, which makes use of the concept of maximal curvature, defined as follows.

If u and v are tangent vectors at some $x \in M$, with components u^λ, v^λ $(\kappa, \lambda, \mu, \nu, \ldots, = 1, \ldots, n)$ with respect to some local co-ordinates; and if $K_{\kappa\lambda\mu\nu}$ are the covariant components of the curvature tensor $K_{\kappa\lambda\mu}^{\cdots\nu}$, then $K(u, v)$ is defined as $- K_{\kappa\lambda\mu\nu} u^\kappa v^\lambda u^\mu v^\nu$. When u and v are perpendicular unit vectors, $K(u, v)$ is the *sectional curvature*, which clearly depends only on the 2-plane at x, spanned by u and v. If A is a subset of M, then $k(A)$, the *maximum curvature* of A, denotes the least upper bound of all numbers $K(u, v)$, with u, v being perpendicular unit vectors spanning all 2-planes at all points of A. $k(A)$ may be $+\infty$.

Our main result is the following theorem.

THEOREM II. *If the maximum curvature $k = k(V)$ of a convex open set V in M is non-positive: $k \leqslant 0$, then V is 1-hyperconvex. If $k > 0$, and if the diameter $D = D(V)$ is such that $kD^2 < \pi^2/4$, then V is γ-hyperconvex, where* $\gamma = \cos Dk^{1/2}$.

The case $k \leqslant 0$ is a much weakened formulation of results obtained by H. Busemann (**3**, Theorems (36.4) and (36.17)); the first of which states that the distance between corresponding points of geodesics in a G-space is a convex function of the linear parameter on the geodesics. Since, however, the case $k \leqslant 0$ is naturally included in our line of argument, the reader will find a new proof for this result.

Every sufficiently small spherical neighbourhood V of a point $x \in M$ is convex and has compact closure, which implies that $k(V)$ is finite. Since $k(V)$ is non-increasing when the radius of V tends to zero, it follows that $k(V)D(V)^2$ tends to zero as the radius of V approaches zero. This proves Theorem I, assuming Theorem II.

The proof of Theorem II is based on estimates of the solution of systems of linear differential equations whose prototype is $y'' = f(x)y$.

2. Estimates on certain systems of linear differential equations.

LEMMA 1. *Let $K(s)$ be a continuous family of linear transformations in E^n, with origin O, defined in the finite closed interval $0 \leqslant s \leqslant l$. Let $\mathbf{y}(s)$ be a solution of $\mathbf{y} + K(s)\mathbf{y} = 0$ in the interval $[0, l]$, and let S denote the unit sphere around O. Then there is a C^2 path $\mathbf{x}(s)$ on S such that for each $s \in [0, l]$, the points O, $\mathbf{x}(s)$, $\mathbf{y}(s)$ are collinear ($\mathbf{x}(s)$ is called a spherical image of $\mathbf{y}(s)$).*

*This statement was conjectured by E. A. Michael in a slightly weaker form obtained by replacing $\frac{1}{2}$-hyperconvexity of a convex set V by the following property: to every $\epsilon > 0$ there is a $\delta > 0$ such that if $y_1, y_2, z_1, z_2 \in V$, $\rho(y_1, y_2) < \epsilon$, $\rho(z_1, z_2) < \delta$, then the distance between corresponding points of $y_1, z_1, y_2 z_2$ is less than ϵ. This conjecture was submitted by Dr. Michael to a number of mathematicians, including H. E. Rauch, who vouched for it, and L. W. Green, who obtained a written proof (spring, 1957) of the weaker result which one gets by permitting δ to be equal to zero. Theorem I, in its present form, has been applied in (**2**) to the theory of continuous selections.

https://doi.org/10.4153/CJM-1959-052-6 Published online by Cambridge University Press

Proof. If $\mathbf{y} \equiv 0$ the lemma is trivial, because any C^2 path $\mathbf{x}(s)$ on S is a spherical image. If $\mathbf{y}(s) \neq 0$ for all $s \in [0, l]$ the proof is simple because $\mathbf{x}(s) = \mathbf{y}(s) \,.\, |\mathbf{y}(s)|^{-1}$ is a spherical image. Now assume that $\mathbf{y}(s)$ has the zeros $s_1, s_2, \ldots,$ but $\mathbf{y} \not\equiv 0$. The number N of these s_i must be finite, because at any accumulation point s' one has

$$\mathbf{y}(s') = \dot{\mathbf{y}}(s') = 0;$$

hence $\mathbf{y} \equiv 0$. Consider

$$\mathbf{z}(s) = \mathbf{y}(s)(s - s_1)^{-1}(s - s_2)^{-1} \ldots (s - s_N)^{-1}$$

for $s \neq s_1, \ldots, s_N$, and

$$\mathbf{z}(s_i) = \lim_{s \to s_i} \mathbf{z}(s) = \dot{\mathbf{y}}(s_i) \prod_{j \neq i} (s_i - s_j)^{-1}.$$

Then $\mathbf{z}(s) \neq 0$ for all $s \in [0, l]$ because

$$\dot{\mathbf{y}}(s_i) \neq 0.$$

Every spherical image of $\mathbf{z}(s)$ is a spherical image of $\mathbf{y}(s)$; and thus the problem has been reduced to a previous case provided $\mathbf{z}(s)$ is of class C^2 for $s = s_1, \ldots, s_N$. In verifying this, the following simple application of the mean value theorem is helpful: "*If a function f is continuous in $[a, b]$, differentiable at all points of $[a, b]$ except some $c \in [a, b]$, and if*

$$\lim_{s \to c} f'(s) = L$$

exists, then $f'(c)$ exists and equals L; whence f' is continuous at c". To show that $\mathbf{z}(s)$ is of class C^2 at s_i it suffices to show this for $\mathbf{z}_i(s) = \mathbf{y}(s)(s - s_i)^{-1}$. We use de l'Hopital's Rule:

$$\lim_{s \to s_i} \dot{\mathbf{z}}_i(s) = \lim \frac{\dot{\mathbf{y}}(s)(s - s_i) - \mathbf{y}(s)}{(s - s_i)^2} = \lim \frac{\ddot{\mathbf{y}}(s)(s - s_i)}{2(s - s_i)} = -\tfrac{1}{2}K(s_i)\mathbf{y}(s_i) = 0.$$

Hence, $\dot{\mathbf{z}}_i(s_i) = 0$, and $\dot{\mathbf{z}}_i$ is continuous in a neighbourhood of s_i in $[0, l]$. The procedure is repeated for $\ddot{\mathbf{z}}_i$, and one thus finds $\ddot{\mathbf{z}}_i(s_i) = -\tfrac{1}{3}K(s_i)\dot{\mathbf{y}}(s_i)$; and $\ddot{\mathbf{z}}_i(s)$ is continuous at s_i. Hence, $\ddot{\mathbf{z}}(s)$ exists and is continuous in $[0, l]$.

LEMMA 2. *If $K(s)$ is a continuous family of linear transformations in E^n; $0 \leqslant s \leqslant l$, and m is an upper bound for the inner product $(\mathbf{u}, K(s)\mathbf{u})$ for all $s \in [0, l]$ and all unit vectors $\mathbf{u}$; if $\mathbf{y}(s)$ is a solution of*

$$\ddot{\mathbf{y}} + K(s)\mathbf{y} = 0,$$

and if $\mathbf{x}(s)$ is a spherical image of $\mathbf{y}(s)$; with $\mathbf{y}(s) = \lambda(s)\mathbf{x}(s)$; then λ satisfies a differential equation $\ddot{\lambda}(s) + \phi(s)\lambda(s) = 0$, where $\phi(s)$ is continuous in s; $0 \leqslant s \leqslant l$; and $\phi(s) \leqslant m$.

Proof. Since $\mathbf{y}(s)$ and $\mathbf{x}(s)$ are of class C^2, and $\mathbf{x}(s) \neq 0$; $\lambda(s)$ is also of class C^2. Substituting $\mathbf{y} = \lambda\mathbf{x}$ into $\mathbf{y} + K(s)\mathbf{y} = 0$ one finds $\lambda\ddot{\mathbf{x}} + 2\dot{\lambda}\dot{\mathbf{x}} + \ddot{\lambda}\mathbf{x}$

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$+ \lambda K\mathbf{x} = 0$. Take the inner product with $\mathbf{x}$, remembering $(\mathbf{x}, \dot{\mathbf{x}}) = 0$ and $(\dot{\mathbf{x}}, \dot{\mathbf{x}}) + (\mathbf{x}, \ddot{\mathbf{x}}) = 0$. Thus follows $\ddot{\lambda} + \{(\mathbf{x}, K\mathbf{x}) - (\dot{\mathbf{x}}, \dot{\mathbf{x}})\}\lambda = 0$. Hence, $\phi(s) = (\mathbf{x}, K\mathbf{x}) - (\dot{\mathbf{x}}, \dot{\mathbf{x}})$ satisfies, and is continuous. In case $\mathbf{y} \not\equiv 0$, $\phi(s)$ is unique. Furthermore, $\phi \leqslant (\mathbf{x}, K\mathbf{x}) \leqslant m$.

LEMMA 3. *If $K(s)$ is as before; m as in Lemma 2, then $ml^2 < \pi^2$ implies that no non-trivial solution of $\ddot{\mathbf{y}} + K(s)\mathbf{y} = 0$, with $\mathbf{y}(0) = 0$ has a zero in the interval $0 < s \leqslant l$. Furthermore, for all $s \in [0, l]$, every solution with $\mathbf{y}(0) = 0$ satisfies*

$$|\mathbf{y}(s)| \leqslant \frac{\sin m^{\frac{1}{2}}s}{\sin m^{\frac{1}{2}}l}\,|\mathbf{y}(l)|, \qquad m > 0,$$

and

$$|\mathbf{y}(s)| \leqslant \frac{s}{l}\,|\mathbf{y}(l)|, \qquad m \leqslant 0.$$

Proof. There is no restriction in assuming $m \geqslant 0$ and $\mathbf{y} \not\equiv 0$. Every zero of $\mathbf{y}(s)$ is a zero of $|\mathbf{y}(s)| = \pm\lambda(s)$. λ satisfies $\ddot{\lambda} + \phi\lambda = 0$; $\lambda(0) = 0$, with $\phi \leqslant m$. The function $\psi(s) = \alpha \sin m^{\frac{1}{2}}s$ if $m > 0$, and $\psi(s) = \beta s$ if $m = 0$ (α and β any non-zero numbers) are solutions of $\ddot{\psi} + m\psi = 0$, with $\psi(0) = 0$. The smallest positive zero of the first function is $\pi m^{-\frac{1}{2}}$, which does not lie in $[0, l]$ because $\pi m^{-\frac{1}{2}} > (ml^2)^{\frac{1}{2}} m^{-\frac{1}{2}} = l$. Thus neither function ψ has a positive zero in the interval $[0, l]$. A well-known Sturmian theorem states that every positive zero of λ is preceded by one of ψ, which means, in this case, that λ has no positive zero in $[0, l]$. In order to prove this, *suppose* that λ has positive zeros in $[0, l]$; let s' be the smallest one. One may assume without loss of generality that $\lambda(s) \geqslant 0$ and $\psi(s) \geqslant 0$ for $s \in [0, s']$; then $\lambda(s') = 0$, $\dot{\lambda}(s') < 0$, $\psi(s') > 0$. Further, $m - \phi$ is strictly positive in some sub-interval of $[0, s']$; otherwise $m = \phi$ in $[0, s']$; and ψ is a multiple of λ; hence $\psi(s') = \lambda(s') = 0$, which is false. We have now

$$\dot{\lambda}\psi - \lambda\dot{\psi}\big|_0^{s'} = \int_0^{s'} (\ddot{\lambda}\psi - \lambda\ddot{\psi})\, ds = \int_0^{s'} \lambda\psi(m - \phi)\, ds > 0,$$

and also

$$\begin{aligned}\dot{\lambda}\psi - \lambda\dot{\psi}\big|_0^{s'} &= \dot{\lambda}(s')\psi(s') - \lambda(s')\dot{\psi}(s') \\ &\quad - \dot{\lambda}(0)\psi(0) + \lambda(0)\dot{\psi}(0) = \dot{\lambda}(s')\psi(s') < 0.\end{aligned}$$

Thus the assumption that λ has zeros in $[0, l]$ leads to a contradiction. This proves the first statement of the lemma; the second part uses the same formulas. The function λ/ψ is defined and of class C^1 for all $s \in [0, l]$; for $s = 0$ by limit procedure. Take $\lambda \geqslant 0$ in $[0, l]$, and

$$\psi(s) = \frac{\sin sm^{\frac{1}{2}}}{\sin lm^{\frac{1}{2}}}\,\lambda(l), \qquad\qquad \psi(s) = \frac{s}{l}\,\lambda(l)$$

respectively; then $\lambda(l) = \psi(l)$. For any s, with $0 < s \leqslant l$, we have

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$$\psi^2\left(\frac{\lambda}{\psi}\right)'\Bigg|_0^s = \dot{\lambda}\psi - \lambda\dot{\psi}\Bigg|_0^s = \int_0^s \lambda\psi(m - \phi) \geqslant 0.$$

Since $\psi(0) = 0$, we have $\psi^2(0)(\lambda/\psi)'_0 = 0$; and thus follows $(\lambda/\psi)' \geqslant 0$ for all s, and λ/ψ is non-decreasing. Consequently,

$$\left(\frac{\lambda}{\psi}\right)_s \leqslant \left(\frac{\lambda}{\psi}\right)_l = 1;$$

that is, $\lambda(s) \leqslant \psi(s)$. This completes the second part.

LEMMA 4. *For every value of t in the interval $0 \leqslant t \leqslant 1$ the function ψ; defined by $\psi(t, \alpha) = \sin t\alpha/\sin \alpha$ when $\alpha \neq 0$, and by $\psi(t, 0) = t$ when $\alpha = 0$, is a monotone non-decreasing function of α in the interval $0 \leqslant \alpha < \pi$.*

Proof. The cases $t = 0$ and $t = 1$ are trivial. Take $0 < t < 1$; $0 < \alpha < \pi$, then

$$\frac{\partial\psi(t, \alpha)}{\partial\alpha} = \frac{N}{\sin^2 \alpha},$$

where

$$N = t \cos t\alpha \sin \alpha - \sin t\alpha \cos \alpha = \sin (1 - t)\alpha - (1 - t) \cos t\alpha \sin \alpha$$

$$\geqslant \sin \alpha(1 - t) - (1 - t) \sin \alpha = \alpha(1 - t)\left[\frac{\sin (1 - t)\alpha}{(1 - t)\alpha} - \frac{\sin \alpha}{\alpha}\right] > 0,$$

because $\sin \alpha/\alpha$ is a decreasing function of α in the interval $0 \leqslant \alpha \leqslant \pi$. Hence, $\partial\psi/\partial\alpha > 0$ when $0 < \alpha < \pi$; since $\psi(t, \alpha)$ is continuous in α at $\alpha = 0$, it is monotone non-decreasing in α in the interval $0 \leqslant \alpha < \pi$.

3. Geodesic segments connecting a fixed point with the points of another curve. We return to the geometric situation of § 1. In this section, the results are true under the assumption $kD^2 < \pi^2$. Not until the proof in § 4 do we need the stronger condition $kD^2 < \pi^2/4$. When $k = 0$, $D = +\infty$, the expression kD^2 is interpreted as zero.

LEMMA 5. *In a convex set V in M, whose diameter and maximum curvature k satisfy $kD^2 < \pi^2$, let x be any point, and y_τ ($0 \leqslant \tau \leqslant 1$) a C^1 path. The geodesic segment xy_τ is parametrized by t from 0 to 1 proportional to arc length: $(t, \tau) \to f(t, \tau)$; $f(0, \tau) = x$, $f(1, \tau) = y_\tau$.* If

$$v(t, \tau) = \frac{\partial f(t, \tau)}{\partial \tau}$$

denotes the tangent vector to the path $\tau \to f(t, \tau)$, for fixed t; and l_τ denotes the length of xy_τ, then

$$|v(t, \tau)| \leqslant \frac{\sin tl_\tau k^{1/2}}{\sin l_\tau k^{1/2}} |v(1, \tau)|$$

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if $k > 0$; *and*

$$|v(t,\tau)| \leqslant t|v(1,\tau)| \quad \text{if } k \leqslant 0.$$

Proof. If s denotes the arc length function on xy_τ, measured from x to y_τ, and $\delta/\delta s$ denotes covariant differentiation with respect to s, then the components v^κ of v satisfy the classical Jacobi equation

$$\frac{\delta^2}{\delta s^2} v^\kappa + v^\nu K_{\nu\mu\lambda}^{\cdots\kappa} i_\mu i_\lambda = 0,$$

where i is the unit tangent vector to xy_τ. Choose along xy_τ a self-parallel orthonormal moving frame $(e_1, \ldots, e_n)$, with $i = e_1$. Denote the components of v with respect to this frame by $\mathbf{y} = (v^1, \ldots, v^n)$, and let $K(s)$ be the linear transformation

$$\mathbf{y} \to K(s)\mathbf{y} = (v^\nu K_{\nu 11}^{\cdots 1}, v^\nu K_{\nu 11}^{\cdots 2}, \ldots, v^\nu K_{\nu 11}^{\cdots n}),$$

where now all components of $K_{\nu\mu\lambda}^{\cdots\kappa}$ are taken with respect to $(e_1, \ldots, e_n)$. Then $\mathbf{y} + K(s)\mathbf{y} = 0$; with, of course, $\mathbf{y}(0) = 0$. We claim that $m = \max\{k, 0\}$ is an upper bound for the inner product $(\mathbf{u}, K(s)\mathbf{u})$, with $|\mathbf{u}| = 1$. Decompose $\mathbf{u}$ into $\mathbf{u}'$ along i, and $\mathbf{u}''$ perpendicular to i. Then

$$\begin{aligned}(\mathbf{u}, K(s)\mathbf{u}) = K_{\kappa\lambda\mu\nu} u^\kappa i^\lambda i^\mu u^\nu = -K_{\kappa\lambda\mu\nu} u^\kappa i^\lambda u^\mu i^\nu = K(\mathbf{u}, \mathbf{i})\\ = K(\mathbf{u}'', \mathbf{i}) \leqslant |\mathbf{u}''| \,.\, k \leqslant |\mathbf{u}''| \,.\, m \leqslant m.\end{aligned}$$

Since $l_\tau < D$, we have $m l_\tau^2 < \pi^2$, and thus, by Lemma 3, we have

$$|\mathbf{y}(s)| \leqslant \frac{\sin s m^{\frac{1}{2}}}{\sin l_\tau m^{\frac{1}{2}}} |\mathbf{y}(l_\tau)|, \qquad |\mathbf{y}(s)| \leqslant \frac{s}{l_\tau} |\mathbf{y}(l_\tau)|$$

respectively for $k = m > 0$ and $m = 0$ $(k \leqslant 0)$. In terms of the parameter $t = s/l_\tau$ this is precisely the statement of the lemma.

LEMMA 6. *Let $V, k, D, x, y_\tau, f(t,\tau)$ be as in Lemma 5, but let $y_0 y_1$: $\tau \to f(1,\tau) = y_\tau$ be a geodesic segment of length L, parametrized by τ proportional to arc length. Then*

$$\rho(f(t,0), f(t,1)) \leqslant \frac{\sin tDk^{\frac{1}{2}}}{\sin Dk^{\frac{1}{2}}} L, \qquad \rho(f(t,0), f(t,1)) \leqslant tL$$

respectively in the cases $k > 0$ and $k \leqslant 0$.

Proof. We have $\rho(y_0, y_1) = L$, $|v(1,t)| = L$; and $\rho(f(t,0), f(t,1))$ is majorized by the length of the path $\tau \to f(t,\tau)$, $\tau \in [0,1]$. The latter equals $\int_0^1 |v(t,\tau)| d\tau$. In case $k = m > 0$, we have $l_\tau k^{\frac{1}{2}} < Dk^{\frac{1}{2}} < \pi$. Lemmas 4 and 5 (with $\alpha = l_\tau k^{\frac{1}{2}}$ and $\alpha = Dk^{\frac{1}{2}}$) are now applied:

$$\rho(f(t,0), f(t,1)) \leqslant \int_0^1 |v(t,\tau)| d\tau \leqslant L \int_0^1 \frac{\sin t l_\tau k^{\frac{1}{2}}}{\sin l_\tau k^{\frac{1}{2}}} d\tau \leqslant \frac{\sin tDk^{\frac{1}{2}}}{\sin Dk^{\frac{1}{2}}} L,$$

$$\rho(f(t,0), f(t,1)) \leqslant L \int_0^1 t d\tau = tL$$

(cases $k > 0$ and $k \leqslant 0$ respectively).

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4. Proof of Theorem II. Choose a positive number ϵ. Let $y_1, y_2, z_1, z_2 \ni V$; and $\rho(y_1, y_2) < \epsilon$, $\rho(z_1, z_2) < \gamma\epsilon$, where $\gamma = \cos Dk^{\frac{1}{2}}$ if $k > 0$, and $\gamma = 1$ if $k \leqslant 0$. If $\phi, \chi, \psi: [0, 1] \to V$ are parametrizations proportional to arc length of the geodesic segments y_1z_1, y_2z_1, y_2z_2 respectively, then by Lemma 6 we have, for $k > 0$, $Dk^{\frac{1}{2}} < \frac{1}{2}\pi$:

$$\begin{aligned}\rho(\phi(t), \psi(t)) &\leqslant \rho(\phi(t), \chi(t)) + \rho(\phi(t), \chi(t)) \\ &\leqslant \rho(y_1, y_2) \frac{\sin (1-t)Dk^{\frac{1}{2}}}{\sin Dk^{\frac{1}{2}}} + \rho(z_1, z_2) \frac{\sin tDk^{\frac{1}{2}}}{\sin Dk^{\frac{1}{2}}} \\ &< \frac{\epsilon \sin (1-t)Dk^{\frac{1}{2}} + \epsilon \cos Dk^{\frac{1}{2}} \sin tDk^{\frac{1}{2}}}{\sin Dk^{\frac{1}{2}}} \\ &= \epsilon \frac{\sin Dk^{\frac{1}{2}} \cos tDk^{\frac{1}{2}} - \cos Dk^{\frac{1}{2}} \sin tDk^{\frac{1}{2}} + \cos Dk^{\frac{1}{2}} \sin tDk^{\frac{1}{2}}}{\sin Dk^{\frac{1}{2}}} \\ &= \epsilon \cos tDk^{\frac{1}{2}} \leqslant \epsilon;\end{aligned}$$

and for $k \leqslant 0$: $\rho(\phi(t), \psi(t)) < (1-t)\epsilon + t\epsilon = \epsilon$. This completes the proof.

Final Remark. The lemmas derived in §§ 2 and 3, can be used to deduce more results on upper bounds for the distance between corresponding points of geodesic segments in convex sets with either $kD^2 < \pi^2$ or $kD^2 < \frac{1}{4}\pi^2$. The approach to take seems so obvious that there is no need here to amplify this point with a number of examples.

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