ON SUCCESSIVE APPROXIMATIONS FOR NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Let $X$ be a Banach space and $K$ a convex subset of $X$. A mapping $T$ of $K$ into $K$ is called a nonexpansive mapping if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in K$.

In general, it is not the case for nonexpansive mappings $T$ that the sequences of Picard iterates $\{T^n(x)\}$ converge to fixed points of $T$, and thus when such fixed points exist other approximation techniques are needed. One such technique is to form the mapping

$$S_\lambda = \lambda I + (1 - \lambda)T \quad (0 < \lambda < 1),$$

and then show that under certain circumstances the Picard iterates of $S_\lambda$ converge to a fixed point of $T$. The first such result was obtained by Krasnoselskii [7], who proved that if $K$ is a closed convex subset of a uniformly convex Banach space and if $T$ is a nonexpansive mapping of $K$ into a compact subset of $K$, then for any $x \in K$ the sequence of iterates $\{S_\lambda^n(x)\}$, for $\lambda = 1/2$, converges to a fixed point of $T$. It was noted by Schaefer [8] that this theorem holds for arbitrary $\lambda \in (0, 1)$ and subsequently Edelstein [4] proved the corresponding result in a strictly convex Banach space. Even more recently, Browder and Petryshyn have obtained Krasnoselskii’s theorem as a corollary of their results in [3].

Our purpose in this note is to observe that mappings more general than those of type $S_\lambda$ yield similar convergence theorems.

**Theorem 1.** Let $K$ be a convex subset of a Banach space and $T$ a nonexpansive mapping of $K$ into itself. Define the mapping $S: K \to K$ by

$$S = \alpha_0 I + \alpha_1 T + \alpha_2 T^2 + \ldots + \alpha_k T^k,$$

where $\alpha_i \geq 0$, $\alpha_0 > 0$, and $\sum_{i=0}^{k} \alpha_i = 1$. Then $S(x) = x$ if and only if $T(x) = x$.

**Proof.** Suppose $S(x) = x$. Then

$$x = \sum_{i=1}^{k} \beta_i T^i(x),$$

where $\beta_i = \alpha_i(1 - \alpha_0)$. Thus $x \in \text{conv}\{T(x), T^2(x), \ldots, T^k(x)\}$. Let

$$\delta = \sup \{ \|u - v\| : u, v \in \{x, T(x), T^2(x), \ldots, T^k(x)\} \}.$$ 

Because $T$ is nonexpansive, for some integer $p \geq 1$,

$$\|x - T^p(x)\| = \delta. \quad (*)$$

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Assume \( \delta > 0 \), and let \( p \) be the smallest positive integer such that (*) holds. Since \( \alpha_1 > 0 \),
\[
x = \beta_1 T(x) + (1 - \beta_1)z,
\]
where \( z \in \text{conv}\{T^2(x), T^3(x), \ldots, T^k(x)\} \) and \( 0 < \beta_1 \leq 1 \); thus
\[
\delta = \|x - T^p(x)\| = \|\beta_1 T(x) + (1 - \beta_1)z - T^p(x)\|
\]
\[
\leq \beta_1 \|T(x) - T^p(x)\| + (1 - \beta_1) \|z - T^p(x)\|
\]
\[
\leq \beta_1 \delta + (1 - \beta_1) \delta = \delta.
\]
This implies \( \|T(x) - T^p(x)\| = \delta \), yielding \( \|x - T^{p-1}(x)\| \geq \delta \). This gives a contradiction if \( p > 1 \). However, if \( p = 1 \) the preceding argument yields \( \|T(x) - T(x)\| \geq \delta > 0 \), which is absurd. Thus, \( \delta = 0 \) and \( x = T(x) \). Since the converse is obvious, the theorem is proved. (We should remark that the stipulation \( x_0 > 0 \) in Theorem 1 is necessary to rule out the possibility that a fixed point of \( S \) is merely a point at which \( T \) is periodic.)

Next we prove that in uniformly convex spaces the mapping \( S \) is asymptotically regular; that is,
\[
\lim_{n \to \infty} \|S^{n+1}(x) - S^n(x)\| = 0 \quad (x \in K).
\]
This result is patterned after Theorem 5 in Browder and Petryshyn [3].

**THEOREM 2.** Let \( X \) be uniformly convex and let \( T \) and \( S \) be as defined in Theorem 1. If \( T \) has at least one fixed point then the mapping \( S \) is asymptotically regular.

**Proof.** Let \( x \in K \). Define the sequence \( \{x_n\} \) by \( x_n = S^n x, n = 1, 2, \ldots \). Suppose \( u \) is a fixed point of \( T \) in \( K \). Then the sequence \( \{\|x_n - u\|\} \) is nonincreasing (since \( S \) is nonexpansive and \( S(u) = u \)), and we may suppose\( \lim_{n \to \infty} \|x_n - u\| = d \geq 0 \). Assume \( d > 0 \). (If \( d = 0 \) there is clearly nothing to prove.) Then (adopting the notation \( T^0 = I \)) we have
\[
x_{n+1} - u = S(x_n) - u
\]
\[
= \sum_{i=0}^{k} \alpha_i T^i(x_n) - u
\]
\[
= \alpha_0(x_n - u) + (1 - \alpha_0)z_n,
\]
where
\[
z_n = \frac{1}{1 - \alpha_0} \sum_{i=1}^{k} \alpha_i (T^i(x_n) - u).
\]
Since
\[
\|T^i(x_n) - u\| = \|T^i(x_n) - T^i(u)\| \leq \|x_n - u\|
\]
and \( \sum_{i=0}^{k} \alpha_i = 1 \) it follows that \( \limsup_{n \to \infty} \|z_n\| \leq d \). Also \( \lim_{n \to \infty} \|x_n - u\| = d \), \( \lim_{n \to \infty} \|x_{n+1} - u\| = d \).
Because $X$ is uniformly convex it must be the case that

$$\lim_{n \to \infty} ||x_n - u - z_n|| = 0.$$ 

However, $x_{n+1} - x_n = (1 - \alpha_0)(x_n - u - z_n)$ and so $\lim_{n \to \infty} (x_{n+1} - x_n) = 0$, completing the proof.

The above results and Theorem 6 of [3] yield the following corollary.

**Corollary.** Let $X$ be a uniformly convex Banach space and $T$ a nonexpansive compact mapping of $X$ into $X$ (i.e., $T$ maps bounded subsets of $X$ into precompact subsets of $X$) which has at least one fixed point. Then if the mapping $S$ is defined as in Theorem 1, for each $x_0 \in X$ the sequence $\{S^n(x_0)\}$ converges to a fixed point of $T$.

**Proof.** Since $S$ is asymptotically regular and has the same fixed points as $T$, the conclusion is a direct consequence of Theorem 6 of Browder–Petryshyn [3] if it is the case that $I - S$ maps bounded closed subsets of $X$ into closed subsets of $X$. Let $H$ be a bounded closed subset of $X$ and suppose $\lim (h_n - Sh_n) = z$, $h_n \in H$. We need to show that $z \in (I - S)[H]$. Since $T$ is a compact mapping, some subsequence $\{T(h_{nj})\}$ of $\{T(h_n)\}$ converges; say $T(h_{nj}) \to v$ as $j \to \infty$. Fix $i$ between 1 and $k$. Continuity of $T$ implies $T^j(h_{nj}) \to T^{j-1}(v)$ as $j \to \infty$. Thus by repeatedly choosing subsequences, we may obtain a subsequence $\{h_{n_j}\}$ of $\{h_n\}$ which has the property:

$$\lim_{n \to \infty} T^i(h_{n_j}) = w_i \in X \quad (i = 1, \ldots, k).$$

Now

$$(I - S)(h_n) = h_n - \sum_{i=0}^{k} \alpha_i T^i(h_n)$$

$$= (1 - \alpha_0)h_n - \sum_{i=1}^{k} \alpha_i T^i(h_n).$$

Since $h_n - S(h_n) \to z$ as $n \to \infty$ it follows that

$$\lim_{n \to \infty} (1 - \alpha_0)h_n = z + \sum_{i=1}^{k} \alpha_i w_i.$$

This implies that $\{h_n\}$ converges, say to $h \in H$ (since $H$ is closed). Hence $h - Sh = z$, which completes the proof.

We conclude by giving an analogue of Theorem 7 of Browder [2].

**Theorem 3.** Let $X$ be a uniformly convex Banach space, $K$ a closed bounded convex subset of $X$, and $T$ a nonexpansive mapping of $K$ into $K$. Let

$$S = \sum_{i=0}^{k} \alpha_i T^i$$

where $\alpha_i \geq 0$, $\alpha_0 > 0$, and $\sum_{i=0}^{k} \alpha_i = 1$. Suppose $T$ has at most one fixed point $y$ in $K$. Then for each $x_0$ in $K$ the sequence $\{S^n(x_0)\}$ converges weakly to $y$ in $K$.

**Proof.** Since $S$ is nonexpansive, Theorem 3 of [2] implies that $I - S$ is demiclosed. This means that if $\{u_j\}$ converges weakly to $u_0$ in $K$ and $(I - S)(u_j)$ converges strongly to $w$, then $(I - S)(u_0) = w$. 

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Now let \( x_n = S^n(x_0), \ n = 1, 2, \ldots, \) and suppose \( \{x_n\} \) converges weakly to \( u_0. \) By Theorem 2, \( S \) is asymptotically regular so
\[
\lim_{i \to \infty} (I - S)(x_n) = \lim_{i \to \infty} (S^n(x_0) - S^{n+1}(x_0)) = 0
\]
and thus demiclosedness of \( I - S \) implies
\[
(I - S)(u_0) = 0.
\]
Thus \( u_0 \) is a fixed point of \( S. \) However, by Theorem 1 the fixed points of \( S \) and \( T \) coincide. Therefore \( u_0 \) is the unique fixed point of \( T \) and it follows that every weakly convergent subsequence of \( \{x_n\} \) converges weakly to \( u_0. \) If \( \{x_n\} \) does not converge weakly to \( u_0 \) then there exists a weak neighborhood \( W \) of \( u_0 \) and a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) with the property that \( x_{n_k} \notin W, \ k = 1, 2, \ldots. \) However, reflexivity of \( X \) and boundedness of \( \{x_n\} \) imply that some subsequence of \( \{x_{n_k}\} \) converges weakly, and by what we have just shown, \( \text{this weakly convergent subsequence must converge to } u_0. \) This implies that terms of the sequence \( \{x_{n_k}\} \) must lie in \( W \) — a contradiction. Therefore, \( \{S^n(x_0)\} \) converges weakly to \( u_0. \)

We might remark that the existence of at least one fixed point for \( T \) in \( K \) follows from a theorem proved independently by Browder [1], Gohde [5], and Kirk [6]. In general, this fixed point is not unique, but it will be unique for strictly contractive mappings (i.e., mappings \( T \) for which \( \|T(x) - T(y)\| < \|x - y\| \) when \( x \neq y \)).

ADDED IN PROOF. Using Theorem 1, one may also obtain Theorems 2 and 3 as direct consequences of their analogues in [2] and [3] by applying the original theorems to the mapping
\[
R = \left( \frac{1}{1 - \alpha_0} \right) \sum_{i=1}^{k} \alpha_i T^i.
\]

REFERENCES


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