BULL. AUSTRAL. MATH. SOC. 42C40
VOL. 67 (2003) [285–295]

LOCAL ANALYSIS OF FRAME MULTIRESOLUTION ANALYSIS WITH A GENERAL DILATION MATRIX

HONG OH KIM, RAE YOUNG KIM AND JAE KUN LIM

A multivariate semi-orthogonal frame multiresolution analysis with a general integer dilation matrix and multiple scaling functions is considered. We first derive the formulas of the lengths of the initial (central) shift-invariant space V_0 and the next dilation space V_1, and, using these formulas, we then address the problem of the number of the elements of a wavelet set, that is, the length of the shift-invariant space $W_0 := V_1 \ominus V_0$. Finally, we show that there does not exist a 'genuine' frame multiresolution analysis for which V_0 and V_1 are quasi-stable spaces satisfying the usual length condition.

1. INTRODUCTION

The orthonormal dyadic multiresolution analysis of $L^2(\mathbb{R})$ with a single scaling function was introduced by Mallat and Meyer in order to construct an orthonormal wavelet basis of $L^2(\mathbb{R})$ [**17, 18**]. Benedetto and Li considered the dyadic semi-orthogonal frame multiresolution analysis of $L^2(\mathbb{R})$ with a single scaling function, and successfully applied the theory in the analysis of narrow band signals [**1**]. We refer to [**9**] for the basic definitions and properties of frames and Riesz bases of a Hilbert space. Unlike the multiresolution analysis of Mallat and Meyer, where there always exists a wavelet set consisting of a single element whose dyadic dilations of the integer translates form an orthonormal basis of $L^2(\mathbb{R})$, the multiresolution analysis of Benedetto and Li has a wavelet set whose cardinality may be one or two [**14**]. The exact definition of a wavelet set of a multiresolution analysis is found in Section 3. The characterisation of the dyadic semi-orthogonal frame multiresolution analysis with a single scaling function admitting a single frame wavelet whose dyadic dilations of the integer translates form a frame for $L^2(\mathbb{R})$ was obtained, independently, by Benedetto and Treiber by a direct method [**2**], and by Kim and Lim by using the theory of shift-invariant spaces [**14**]. The dyadic multivariate generalisation (with a single scaling function) of the multiresolution analysis of Mallat and Meyer were considered by several authors. See [**3**], for example. Lim, among other things, addressed the problem of the cardinality of a wavelet set in the setting of

Received 10th September, 2002
The first and the second author were supported by KRF-2002-070-000004.

the dyadic multivariate frame multiresolution analysis with a single scaling function [**16**], and Kim and Lim gave an analysis of dyadic multivariate frame multiresolution analysis with multiple scaling functions [**15**]. As the theory and applications of wavelets evolve, many authors considered more general dilations other than the dyadic ones (for example, [**8**]).

In this paper we consider a multivariate frame multiresolution analysis with a general integer dilation matrix and multiple scaling functions by extending the method and results of [**15**]. We first analyse the local dimension of the initial, that is, central, shift-invariant space V_0 and the next dilation space V_1 of the multiresolution analysis. Using this, we derive the formulas of the lengths of the shift-invariant spaces V_0 and V_1, and address the problem of the number of the elements of a wavelet set. Finally, we show that there does not exist a 'genuine' frame multiresolution analysis for which V_0 and V_1 are quasi-stable spaces satisfying the usual length condition by applying the local dimension analysis and the ergodicity of the dilation matrix (Theorem 9). This result improves Theorem 3.9 in [**15**] in the sense that we do not presuppose that the spectrums of V_0 and V_1 coincide.

The organisation of this paper is as follows: Preliminary discussions on the dilation matrix and shift-invariant spaces and the definition of the multiresolution analysis we consider are given in Section 2, and our main results, along with an analysis of the local dimensions of V_0 and V_1, are given in Section 3.

2. Preliminary discussion

Suppose that M is a $d \times d$ integer dilation matrix, that is, the entries of M are integers and the moduli of the eigenvalues of M are strictly greater than one. It is known that the order of the quotient group $\mathbb{Z}^d/M\mathbb{Z}^d$ is $|\det M|$ [**8**, Lemma 2]. Let $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ denote the d-dimensional torus which is identified with $[-1/2, 1/2)^d$. For $x \in \mathbb{R}^d$, let $x \,(\mathrm{mod}\, 1)$ denotes the standard representative of $x + \mathbb{Z}^d$ in $[-1/2, 1/2)^d$. Suppose that T is a $d \times d$ invertible matrix with integer entries such that the moduli of the eigenvalues of T are all different from 1. Then the map $\widetilde{T} : \mathbb{T}^d \to \mathbb{T}^d$, defined via $\widetilde{T}x := Tx \,(\mathrm{mod}\, 1)$, is ergodic [**20**, Theorem 0.15, Corollary 1.10.1]. We note that $M^t = M^*$, where t and $*$ denote the transpose and the adjoint of a matrix with complex entries, respectively. For notational convenience we let $Q := \mathbb{Z}^d/M\mathbb{Z}^d$ and let $Q^* := \mathbb{Z}^d/M^*\mathbb{Z}^d$. Let $D := D_M : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ denote the unitary dilation operator defined via $Df(x) := |\det M|^{1/2} f(Mx)$. For $y \in \mathbb{R}^d$, $T_y : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ denotes the unitary translation operator such that $T_y f(x) := f(x - y)$. In this paper we adapt the following definition of multiresolution analysis.

DEFINITION 1: $\{V_k\}_{k\in\mathbb{Z}}$ is said to be a frame multiresolution analysis if each V_k is a closed subspace of $L^2(\mathbb{R}^d)$ such that:

(1) $V_k \subset V_{k+1}, \quad k \in \mathbb{Z}$;
(2) $\overline{\bigcup_{k\in\mathbb{Z}} V_k} = L^2(\mathbb{R}^d), \ \bigcap_{k\in\mathbb{Z}} V_k = \{0\}$;
(3) $D(V_k) = V_{k+1}, \quad k \in \mathbb{Z}$;
(4) There exists a finite set of scaling functions $\Phi \subset V_0$ such that $\{T_\alpha\varphi : \alpha \in \mathbb{Z}^d, \varphi \in \Phi\}$ is a frame for V_0.

Various examples and applications of multiresolution analyses are found in the references cited in Section 1.

The following form of the Fourier transform is used throughout this paper: for $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $t \in \mathbb{R}^d$ $\widehat{f}(t) := \int_{\mathbb{R}^d} f(x)e^{-2\pi i t\cdot x}\,dx$, where $\cdot$ denotes the d-dimensional real inner product. It is, of course, extended to be a unitary transform from $L^2(\mathbb{R}^d)$ onto $L^2(\mathbb{R}^d)$ via the Plancherel theorem.

Suppose that $\{f_i : 1 \leqslant i \leqslant n\}$ is a finite family of elements of a Hilbert space $\mathcal{H}$ with an inner product $\langle\cdot,\cdot\rangle$. We frequently use the following simple observation: $\dim\operatorname{span}\{f_i : 1 \leqslant i \leqslant n\} = \operatorname{rank}(\langle f_i, f_j\rangle)_{1\leqslant i,j\leqslant n}$.

Our analysis uses the theory of (multi-integer) shift-invariant spaces developed in [**4, 5, 6, 10, 11, 12, 19**] and the references therein. We briefly review the theory and uses the existing results freely. A closed subspace S of $L^2(\mathbb{R}^d)$ is said to be *shift-invariant* if $T_\alpha f \in S$ whenever $f \in S$ and $\alpha \in \mathbb{Z}^d$. If $\Phi \subset L^2(\mathbb{R}^d)$, then $S := \mathcal{S}(\Phi) := \overline{\operatorname{span}}\{T_\alpha\varphi : \alpha \in \mathbb{Z}^d\}$ is a shift-invariant space. In this case, Φ is called a *generator* of S. If Φ is finite, then S is called a *finite shift-invariant space.* We write $S = \mathcal{S}(\varphi)$ instead of $\mathcal{S}(\{\varphi\})$ if $\Phi = \{\varphi\}$ is a singleton. In this case, we call S a *principal shift-invariant space.* It is known that any shift-invariant space has a countable generator. The *length* of a shift-invariant space is defined to be

$$\operatorname{len} S := \inf\{\#\Phi : S = \mathcal{S}(\Phi), \Phi \subset L^2(\mathbb{R}^d)\},$$

where $\#$ denotes the cardinality. Let $\widehat{f}_{||x}$ be the sequence $\big(\widehat{f}(x+\alpha)\big)_{\alpha\in\mathbb{Z}^d}$ which is in $\ell^2(\mathbb{Z}^d)$ for almost every $x \in \mathbb{T}^d$. If $A \subset L^2(\mathbb{R}^d), x \in \mathbb{T}^d$, then we let $\widehat{A}_{||x} := \{\widehat{f}_{||x} \in \ell^2(\mathbb{Z}^d) : f \in A\}$, which is called the *fibre* of A at x. It is a subspace of $\ell^2(\mathbb{Z}^d)$ if A is a shift-invariant space. The following theorem is used frequently in our discussion.

THEOREM 2. ([**4, 6, 10, 11**].) *Let S be a closed, not necessarily shift-invariant, subspace of $L^2(\mathbb{R}^d)$ and Φ a countable subset of $L^2(\mathbb{R}^d)$. Then $S = \mathcal{S}(\Phi)$ if and only if $\widehat{f}_{||x} \in \overline{\operatorname{span}}\{\widehat{\varphi}_{||x} : \varphi \in \Phi\}$ for almost every $x \in \mathbb{T}^d$ and for each $f \in S$.*

The *spectrum* of a shift-invariant space is defined to be $\sigma(S) := \{x \in \mathbb{T}^d : \widehat{S}_{||x} \neq \{0\}\}$. A finite subset Φ of $L^2(\mathbb{R}^d)$ is said to be a *quasi-stable generator* for the shift-invariant space $\mathcal{S}(\Phi)$ if, in addition to the condition that the family of the integer translates of Φ is a frame for $\mathcal{S}(\Phi)$, $\dim\operatorname{span}\{\widehat{\varphi}_{||x} : \varphi \in \Phi\} = \#\Phi$ or 0 for almost every $x \in \mathbb{T}^d$. If Φ is a quasi-stable generator, then there is a convenient 'local' formula for the orthogonal projection onto $\mathcal{S}(\Phi)$ [**4, 19**]. The *stable generator* is a quasi-stable generator such that

https://doi.org/10.1017/S000497270003375X Published online by Cambridge University Press

the spectrum of the shift-invariant space it generates is $\mathbb{T}^d$. It turns out that if Φ is a stable generator, then the family of the integer translates of Φ is a Riesz basis for $\mathcal{S}(\Phi)$ [**4, 19**]. We say that a shift-invariant space S is *quasi-stable*, if $\dim \widehat{S}_{||x} = n$ or 0 for some non-negative integer n almost everywhere. It is said to be *stable* if $\dim \widehat{S}_{||x} = n$ almost everywhere. It is known that a quasi-stable/stable shift-invariant space has a quasi-stable/stable generator [**4, 19**].

We need the following results:

THEOREM 3. ([4].) *For a shift-invariant subspace S of $L^2(\mathbb{R}^d)$*

$$\operatorname{len} S = \operatorname{ess-sup}\{\dim \widehat{S}_{||x} : x \in \mathbb{T}^d\}.$$

THEOREM 4. ([4].) *Let S_1 be a shift-invariant subspace of a shift-invariant space S and let $S_2 := S \ominus S_1$. Then S_2 is also a shift-invariant subspace of S and $\widehat{S}_{||x} = \widehat{S}_{1||x} \oplus \widehat{S}_{2||x}$ for almost every $x \in \mathbb{T}^d$.*

Suppose that $S = \mathcal{S}(\Phi)$ for a finite set $\Phi := \{\varphi_1, \varphi_2, \dots, \varphi_n\}$. Then the $n \times n$ matrix

$$G(x) := G_\Phi(x) := (\langle \widehat{\varphi}_{i||x}, \widehat{\varphi_j}_{||x} \rangle_{\ell^2(\mathbb{Z}^d)})_{1 \leqslant i,j \leqslant n}$$

is the *Gramian* of Φ at $x \in \mathbb{T}^d$. Let $\lambda(x), \lambda^+(x)$ and $\Lambda(x)$ denote the smallest eigenvalue, the smallest non-negative eigenvalue and the largest eigenvalue of $G(x)$, respectively.

THEOREM 5. ([**4, 6, 19**].) *The family of the integer translates of Φ is a frame for S if and only if there exist positive constants A and B such that $A \leqslant \lambda^+(x) \leqslant \Lambda(x) \leqslant B$ for almost every $x \in \sigma(S)$. It is a Riesz basis for S if and only if $A \leqslant \lambda(x) \leqslant \Lambda(x) \leqslant B$ for almost every $x \in \mathbb{T}^d$. Moreover, A and B are a pair of frame (Riesz) bounds of the frame (Riesz basis), respectively.*

3. Frame multiresolution analysis

Suppose that $\{V_j\}_{j \in \mathbb{Z}}$ is a frame multiresolution analysis. Then there exists a set of scaling functions $\Phi := \{\varphi_i : 1 \leqslant i \leqslant n\} \subset L^2(\mathbb{R}^d)$ such that $\{T_\alpha \varphi_i : \alpha \in \mathbb{Z}^d, 1 \leqslant i \leqslant n\}$ is a frame for V_0. We may assume that the length of V_0 is n. Then $V_0 = \mathcal{S}(\Phi)$ and $V_1 := D(V_0)$. Let

$$G(x) := G_\Phi(x) := (\langle \widehat{\varphi}_{i||x}, \widehat{\varphi_j}_{||x} \rangle_{\ell^2(\mathbb{Z}^d)})_{1 \leqslant i,j \leqslant n}$$

be the Gramian of Φ at $x \in \mathbb{T}^d$.

Since $\varphi_i \in V_1$ for each $1 \leqslant i \leqslant n$, and since $\{DT_\alpha \varphi_i : \alpha \in \mathbb{Z}^d, 1 \leqslant i \leqslant n\}$ is a frame for V_1, there exist $a_{ij} \in \ell^2(\mathbb{Z}^d), 1 \leqslant i,j \leqslant n$, such that $\varphi_i = \sum_{j=1}^{n} \sum_{\alpha \in \mathbb{Z}^d} a_{ij}(\alpha) DT_\alpha \varphi_j$. Hence

$$\begin{aligned}\widehat{\varphi}_i(x) &= \sum_{j=1}^{n} \sum_{\alpha \in \mathbb{Z}^d} a_{ij}(\alpha) |\det M|^{-1/2} e^{-2\pi i \alpha \cdot (M^{*-1}x)} \widehat{\varphi}_j(M^{*-1}x) \\ &= \sum_{j=1}^{n} m_{ij}(M^{*-1}x) \widehat{\varphi}_j(M^{*-1}x),\end{aligned}$$

where

$$m_{ij}(x) := \sum_{\alpha\in\mathbb{Z}^d} |\det M|^{-1/2} a_{ij}(\alpha) e^{-2\pi i\alpha\cdot x} \in L^2(\mathbb{T}^d).$$

For $x \in \mathbb{T}^d$, let

$$m(x) := \big(m_{ij}(x)\big)_{1\leqslant i,j\leqslant n}$$

and

$$\widehat{\Phi}(x) := \big(\widehat{\varphi}_1(x),\ \widehat{\varphi}_2(x), \dots, \widehat{\varphi}_n(x)\big)^t.$$

Then

$$\widehat{\Phi}(x) = m(M^{*-1}x)\widehat{\Phi}(M^{*-1}x). \tag{1}$$

This m, called a *mask* of the multiresolution analysis, may not be unique since $\{DT_\alpha\varphi_i : \alpha \in \mathbb{Z}^d, 1 \leqslant i \leqslant n\}$ is only assumed to be a frame, not necessarily a Riesz basis. Note that $DT_{My} = T_y D$ for $y \in \mathbb{R}^d$. Since each $\gamma \in \mathbb{Z}^d$ can be written uniquely as $\gamma = M\alpha + \beta$ for some $\alpha \in \mathbb{Z}^d$ and $\beta \in Q$,

$$\{DT_\gamma\varphi_i : \gamma \in \mathbb{Z}^d, 1 \leqslant i \leqslant n\} = \{T_\alpha DT_\beta\varphi_i : \alpha \in \mathbb{Z}^d, \beta \in Q, 1 \leqslant i \leqslant n\}.$$

Hence $V_1 = \mathcal{S}(\Pi)$, where

$$\Pi := \{DT_\beta\varphi_i : \beta \in Q, 1 \leqslant i \leqslant n\}. \tag{2}$$

This implies that the length of the shift-invariant space V_1 is less than or equal to $n|\det M|$. Since V_0 is a shift-invariant subspace of V_1, $\operatorname{len} V_1 \geqslant \operatorname{len} V_0 = n$. There is an example of a frame multiresolution analysis in which the length of V_1 is that of V_0. See Example 6 below. Let W_0 denote $V_1 \ominus V_0$, and let $W_j := D^j(W_0), j \in \mathbb{Z}$. Then Definition 1 implies that $L^2(\mathbb{R}^d) = \bigoplus_{j\in\mathbb{Z}} W_j$. W_0 is a shift-invariant space by Theorem 4. Since W_0 is a subspace of V_1, the length of W_0 is also less than or equal to $n|\det M|$. It cannot be zero. If it were zero, then $W_0 = \{0\}$; hence $V_0 = V_1$. Definition 1 implies that $L^2(\mathbb{R}) = V_0 = \mathcal{S}(\Phi)$. This contradicts a result in [7] which states roughly that there are no frames of $L^2(\mathbb{R}^d)$ consisting of the translates of a finite number of functions. Since W_0 is a finite shift-invariant space, there is a finite set Ψ, called a *wavelet set*, such that $W_0 = \mathcal{S}(\Psi)$. We may assume that the integer translates of the elements of Ψ form a frame for W_0 [**5**, **19**]. Then, obviously, $\{D^jT_\alpha\psi : j \in \mathbb{Z}, \alpha \in \mathbb{Z}^d, \psi \in \Psi\}$ is a frame for $L^2(\mathbb{R}^d)$. Since the minimal cardinality of such Ψ is $\operatorname{len} W_0$, the (minimal) number of the elements of a wavelet set is $\operatorname{len} W_0$.

Note that, for $\beta \in \mathbb{Z}^d$ and $x \in \mathbb{T}^d$,

$$\begin{aligned}(DT_\beta\varphi_i)^\wedge(x) &= |\det M|^{-1/2} e^{-2\pi i\beta\cdot(M^{*-1}x)}\widehat{\varphi}_i(M^{*-1}x),\\ (DT_\beta\varphi_i)^\wedge_{||x} &= |\det M|^{-1/2} e^{-2\pi i\beta\cdot(M^{*-1}x)}\Big(e^{-2\pi i\beta\cdot(M^{*-1}\alpha)}\widehat{\varphi}_i\big(M^{*-1}(x+\alpha)\big)\Big)_{\alpha\in\mathbb{Z}^d}.\end{aligned}$$

Hence, for almost every $x \in \mathbb{T}^d$,

$$\widehat{V}_{1||x} = \operatorname{span}\Big\{\Big(e^{-2\pi i\beta\cdot(M^{*-1}\alpha)}\widehat{\varphi}_i\big(M^{*-1}(x+\alpha)\big)\Big)_{\alpha\in\mathbb{Z}^d} : \beta\in Q, 1\leqslant i\leqslant n\Big\},$$

(3)
$$\widehat{V}_{0||x} = \operatorname{span}\Big\{\big(\widehat{\varphi}(x+\alpha)\big)_{\alpha\in\mathbb{Z}^d} : 1\leqslant i\leqslant n\Big\}$$
$$= \operatorname{span}\Big\{\sum_{j=1}^{n}\Big(m_{ij}\big(M^{*-1}(x+\alpha)\big)\widehat{\varphi}_j\big(M^{*-1}(x+\alpha)\big)\Big)_{\alpha\in\mathbb{Z}^d} : 1\leqslant i\leqslant n\Big\}.$$

For $\beta^*\in Q^*$ define $P_{\beta^*} : \ell^2(\mathbb{Z}^d)\to\ell^2(\mathbb{Z}^d)$ via

$$(P_{\beta^*}a)(\alpha) := \begin{cases} a(\alpha), & \text{if } \alpha\in\beta^*+M^*\mathbb{Z}^d,\\ 0, & \text{otherwise.}\end{cases}$$

Then $\ell^2(\mathbb{Z}^d) = \bigoplus_{\beta^*\in Q^*} P_{\beta^*}\big(\ell^2(\mathbb{Z}^d)\big)$. Define, for $x\in\mathbb{T}^d, 1\leqslant i\leqslant n, \beta^*\in Q^*$,

$$a_{x,i,\beta^*} := P_{\beta^*}\Big(\Big(\widehat{\varphi}_i\big(M^{*-1}(x+\alpha)\big)\Big)_{\alpha\in\mathbb{Z}^d}\Big).$$

Notice that a_{x,i,β^*} is the 'up-sampled' version of $\widehat{\varphi}_{i||M^{*-1}(x+\beta^*)}$, that is,

(4)
$$a_{x,i,\beta^*}(\beta^*+M^*\alpha) = \widehat{\varphi}_{i||M^{*-1}(x+\beta^*)}(\alpha), \quad \alpha\in\mathbb{Z}^d,$$

(5)
$$a_{x,i,\beta^*}(\gamma^*) = 0, \quad \gamma^*\notin\beta^*+M^*\mathbb{Z}^d.$$

Therefore

(6)
$$\|a_{x,i,\beta^*}\|_{\ell^2(\mathbb{Z})} = \big\|\widehat{\varphi}_{i||M^{*-1}(x+\beta^*)}\big\|_{\ell^2(\mathbb{Z})}.$$

We also have

$$\Big(e^{-2\pi i\beta\cdot(M^{*-1}\alpha)}\widehat{\varphi}_i\big(M^{*-1}(x+\alpha)\big)\Big)_{\alpha\in\mathbb{Z}} = \sum_{\gamma^*\in Q^*} e^{-2\pi i\beta\cdot(M^{*-1}\gamma^*)}a_{x,i,\gamma^*}.$$

Let $b_{x,i,\beta}$ be the right-hand side of the above equation. Then, for a fixed $x\in\mathbb{T}^d, 1,\leqslant i\leqslant n$, we have the following matrix relation:

$$(b_{x,i,\beta})^t_{\beta\in Q} = (e^{-2\pi i\beta\cdot(M^{*-1}\gamma^*)})_{\beta\in Q,\gamma^*\in Q^*}(a_{x,i,\gamma^*})^t_{\gamma^*\in Q^*}.$$

Recall that, for any $\beta\in\mathbb{Z}^d$, the map $\gamma^*\to e^{-2\pi i\beta\cdot(M^{*-1}\gamma^*)}$ is a character of the discrete group G^*. Hence the sum $\sum_{\gamma^*\in Q^*} e^{-2\pi i\beta\cdot(M^{*-1}\gamma^*)}$ is the order of Q^*, which is $|\det M|$, if the map is the identity character, and the sum is 0 if the map is not the identity character since the only discrete multiplicative subgroups of $\mathbb{T}$ are the groups of the p-th roots of unity ([**20**, Theorem 0.14]). Using this observation, it is easy to see that

$$(e^{-2\pi i\beta\cdot(M^{*-1}\gamma^*)})_{\beta\in Q,\gamma^*\in Q^*}(e^{2\pi i(M^{*-1}\delta^*)\cdot\eta})_{\delta^*\in Q^*,\eta\in Q} = |\det M|I_{|\det M|}.$$

In particular, for each $1 \leqslant i \leqslant n$ and $x \in \mathbb{T}^d$, $\operatorname{span}\{b_{x,i,\beta} : \beta \in Q\} = \operatorname{span}\{a_{x,i,\gamma^*} : \gamma^* \in Q^*\}$. This shows that:

$$\widehat{V}_{1||x} = \operatorname{span}\{a_{x,i,\gamma^*} : 1 \leqslant i \leqslant n, \gamma^* \in Q^*\}. \tag{7}$$

The 1-periodicity of the mask m and (4) imply that:

$$\widehat{V}_{0||x} = \operatorname{span}\left\{\sum_{j=1}^{n} \sum_{\alpha^* \in Q^*} m_{ij}\big(M^{*-1}(x+\alpha^*)\big)a_{x,j,\alpha^*} : 1 \leqslant i \leqslant n\right\}.$$

Note that, for almost every $x \in \mathbb{T}^d$, $\dim \widehat{V}_{1||x}$ equals the rank of the following $n|\det M| \times n|\det M|$ matrix

$$\big(\langle a_{x,i,\alpha^*}, a_{x,j,\beta^*}\rangle_{\mathbb{C}^{2^d}}\big)_{(i,\alpha^*),(j,\beta^*)}.$$

If we order the indices suitably, then (4) and (5) imply that the matrix is the block diagonal matrix

$$\operatorname{diag}\Big(G\big(M^{*-1}(x+\alpha^*)\big)\Big)_{\alpha^* \in Q^*}.$$

Recall that $\operatorname{rank} G\big(M^{*-1}(x+\alpha^*)\big) = \dim \widehat{V}_{0||M^{*-1}(x+\alpha^*)}$ for each $\alpha^* \in Q^*$. Hence, for almost every $x \in \mathbb{T}^d$,

$$\dim \widehat{V}_{1||x} = \sum_{\alpha^* \in Q^*} \operatorname{rank} G\big(M^{*-1}(x+\alpha^*)\big) = \sum_{\alpha^* \in Q^*} \dim \widehat{V}_{0||M^{*-1}(x+\alpha^*)}. \tag{8}$$

A direct calculation shows that

$$G(x) = \sum_{\alpha^* \in Q^*} m\big(M^{*-1}(x+\alpha^*)\big)G\big(M^{*-1}(x+\alpha^*)\big)m\big(M^{*-1}(x+\alpha^*)\big)^*.$$

Hence, for almost every $x \in \mathbb{T}^d$,

$$\dim \widehat{V}_{0||x} = \operatorname{rank} \sum_{\alpha^* \in Q^*} m\big(M^{*-1}(x+\alpha^*)\big)G\big(M^{*-1}(x+\alpha^*)\big)m\big(M^{*-1}(x+\alpha^*)\big)^*. \tag{9}$$

EXAMPLE 6. Let us first consider a dyadic univariate frame multiresolution analysis with a single scaling function, that is, $d = n = 1$ and $Q = \mathbb{Z}/2\mathbb{Z} = \{0,1\}$. Therefore $Df(x) = 2^{1/2}f(2x)$. Let V_0 be a Paley–Wiener space such that $\{f \in L^2(\mathbb{R}) : \operatorname{supp}(f) \subset [-a,a]\}$ with $0 < a < 1/4$, and let $V_j := D^j(V_0)$, $j \in \mathbb{Z}$. Then it is easy to see that $\{V_j\}_{j\in\mathbb{Z}}$ is a frame multiresolution analysis ([14]). Obviously, $V_0 = \mathcal{S}(\check{\chi}_{[-a,a]})$, where $\vee$ denotes the inverse Fourier transform. Hence, $V_1 = \mathcal{S}(\check{\chi}_{[-2a,2a]})$ is a shift-invariant space of length 1. This can be proved by using (8). Note that $G(x) = \chi_{[-a,a]+\mathbb{Z}}(x)$ for $x \in \mathbb{T}$. Hence, $\widehat{V}_{1||x} = \chi_{[-a,a]+\mathbb{Z}}(x/2) + \chi_{[-a,a]+\mathbb{Z}}(x/2+1/2)$ for $x \in \mathbb{T}$. Hence $\dim \widehat{V}_{1||x} = 1$ for $x \in [-2a,2a]$, and $\dim \widehat{V}_{1||x} = 0$ for $x \in \mathbb{T}\backslash[-2a,2a]$. Therefore $\operatorname{len} V_1 = 1$ by Theorem 3. Recall that $\operatorname{len} V_1$ is less than or equal to $n2^d = 2$. In this example, the length of V_1 is

https://doi.org/10.1017/S000497270003375X Published online by Cambridge University Press

that of V_0. Notice, however, that V_0 is a strict subspace of V_1. It is now easy to see that $\operatorname{len} V_2 = 2$.

The above example can be directly extended to the case where $d > 1$, $n = 1$ and $M = 2I_d$. Then $Q = Q^* = \{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d) : \varepsilon_i = 0 \text{ or } 1\}$. For the sake of simplicity, define $\widehat{\varphi} := \chi_{(-1/4,1/4)^d}$, $V_0^d := \mathcal{S}(\varphi)$ and $V_j^d := D^j(V_0^d)$. Then $\{V_j^d\}_{j\in\mathbb{Z}}$ is a frame multiresolution analysis. We show that $\operatorname{len} V_1^d = 1$. (8) implies that, for almost every $x \in \mathbb{T}^d = [-1/2, 1/2)^d$, $\dim \widehat{V_1^d}_{||x}$ is the number of the sets $2\sigma(V_0^d) - \beta^* + 2\mathbb{Z}^d, \beta^* \in Q^*$, to which x belongs. A direct calculation shows that x belongs only to $2\sigma(V_0^d) + 2\mathbb{Z}^d$ since $\sigma(V_0^d) = (-1/4, 1/4)^d$. Therefore $\operatorname{len} V_1^d = 1$ by Theorem 3. Since V_0^d is a strict subset of V_1^d, $\operatorname{len} W_0 = 1$.

Suppose, temporarily, that $\{T_\alpha \varphi_i : \alpha \in \mathbb{Z}, 1 \leqslant i \leqslant n\}$ is a Riesz basis of V_0. Then $\operatorname{rank} G(x) = n$ for almost every $x \in \mathbb{T}^d$ [4]. Hence $\dim \widehat{V}_{1||x} = n|\det M|$ for almost every $x \in \mathbb{T}^d$. Since $\widehat{V}_{1||x} = \widehat{V}_{0||x} \oplus \widehat{W}_{0||x}$ almost everywhere by Theorem 4, $\dim \widehat{W}_{0||x} = n(|\det M| - 1)$ almost everywhere. Therefore $\operatorname{len} W_0 = n(|\det M| - 1)$ by Theorem 3. Benedetto and Li [1] introduced the following concept: a frame multiresolution analysis admit a *standard (frame) wavelet set* if $\operatorname{len} W_0 \leqslant n(|\det M| - 1)$. The characterisations of dyadic frame multiresolution analyses admitting standard wavelet sets were given in [**2, 14**], independently, for $d = n = 1$, and in [**16**] for $d > 1$ and $n = 1$. Combining (8) and (9) with Theorems 3 and 4 yield the following general result on the admittance of a standard wavelet set.

Theorem 7. *The frame multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ admits a standard wavelet set if and only if, for almost every $x \in \mathbb{T}^d$,*

$$\sum_{\alpha^*\in Q^*} \operatorname{rank} G\big(M^{*-1}(x+\alpha^*)\big) - \operatorname{rank} \sum_{\alpha^*\in Q^*} m\big(M^{*-1}(x+\alpha^*)\big) G\big(M^{*-1}(x+\alpha^*)\big) m\big(M^{*-1}(x+\alpha^*)\big)^* \leqslant n\big(|\det M| - 1\big). \tag{10}$$

We now recover the previous results on the admittance of a standard wavelet set [**2, 14, 16**]. Suppose that $n = 1$ and $M = 2I_d$. Then $Q = Q^* = \mathbb{Z}^d/2\mathbb{Z}^d = \{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d) : \varepsilon_i = 0 \text{ or } 1\}$, $|\det M| = 2^d$, $\Phi = \{\varphi\} \subset L^2(\mathbb{R}^d)$, $G(x) = \|\widehat{\varphi}_{||x}\|^2_{\ell^2(\mathbb{Z}^d)}$, and $m(x)$ is also a scalar. (10) becomes

$$\sum_{\alpha\in Q} \operatorname{rank} \left\|\widehat{\varphi}_{||(x+\alpha)/2}\right\|^2_{\ell^2(\mathbb{Z}^d)} - \operatorname{rank} \sum_{\alpha\in Q} \left| m\Big(\frac{x+\alpha}{2}\Big)\right|^2 \left\|\widehat{\varphi}_{||(x+\alpha)/2}\right\|^2_{\ell^2(\mathbb{Z}^d)} \leqslant 2^d - 1,$$

where the rank of a scalar is the rank of the 1×1 matrix with the scalar entry. Notice that the left-hand side of the above inequality is less than or equal to 2^d. Hence the frame multiresolution analysis admits a standard wavelet set if and only if the left-hand side is not 2^d almost everywhere. The condition, now, is equivalent to the condition that E is of zero Lebesgue measure with

$$E := \left\{ x \in \mathbb{T}^d : \widehat{\varphi}_{||(x+\alpha)/2} \neq 0 \text{ for each } \alpha \in Q, \sum_{\alpha \in Q} \left| m\left(\frac{x+\alpha}{2}\right)\right|^2 = 0 \right\}.$$

This recovers Theorem 5 of [**16**] (see also [**2, 14**] for the univariate case).

We now observe a simple relationship between the spectrums of V_0 and V_1.

LEMMA 8. $\sigma(V_1) = M^*\sigma(V_0) \pmod 1$.

PROOF: Suppose that $x \in \sigma(V_0)$. Then $\widehat{\varphi}_{i||x} \neq 0$ for some i by (3). There exist $y \in \mathbb{T}^d, \alpha^* \in \mathbb{Z}^d$ such that $M^*x = y + \alpha^*$. Now $\alpha^* = \beta^* + M^*\gamma^*$ for some $\beta^* \in Q^*$ and $\gamma^* \in \mathbb{Z}^d$. (6) implies that

$$\|a_{y,i,\beta^*}\|_{\ell^2(\mathbb{Z}^d)} = \left\|\widehat{\varphi}_{i||M^{*-1}(M^*x-\beta^*-M^*\gamma^*+\beta^*)}\right\|_{\ell^2(\mathbb{Z})} = \|\widehat{\varphi}_{i||x-\gamma^*}\|_{\ell^2} \neq 0.$$

Therefore, $y = M^*x - \alpha^* \in \sigma(V_1)$ by (7). This shows that $M^*\sigma(V_0) \subset \sigma(V_1)$. Suppose, on the other hand, that $x \in \sigma(V_1)$. Then $a_{x,i,\beta^*} \neq 0$ for some $1 \leqslant i \leqslant n$ and $\beta^* \in Q^*$ by (7). Then $\widehat{\varphi}_{i||M^{*-1}(x+\beta^*)} \neq 0$ by (6). Hence $M^{*-1}(x+\beta^*) \pmod 1 \in \sigma(V_0)$ by (3). This shows that $\sigma(V_1) \subset M^*\sigma(V_0) \pmod 1$. ∎

Recall that a frame is a Riesz basis if it is, in a certain sense, 'globally' irredundant, that is, irredundant in the norm topology [**9, 13**]. Suppose that Φ is a quasi-stable generating set for V_0. Then the family of the integer translates of Φ, which is a frame for V_0, is 'locally', that is, fibre-wise, irredundant. We now show that: if Φ and Π are quasi-stable generating sets for V_0 and V_1, respectively, then the integer translates are 'globally' redundant. This result improves Theorem 3.9 [**15**] in the sense that we do not presuppose that $\sigma(V_0) = \sigma(V_1)$. More precisely, we show:

THEOREM 9. *Suppose that $\{V_j\}_{j\in\mathbb{Z}}$ is a frame multiresolution analysis. If V_0 and V_1 are both quasi-stable, and if $\operatorname{len} V_1 = |\det M| \operatorname{len} V_0$, then they are actually stable. In particular, $\sigma(V_0) = \sigma(V_1) = \mathbb{T}^d$.*

PROOF: We may assume that $V_0 = \mathcal{S}(\Phi)$ with $\Phi := \{\varphi_1, \varphi_2, \dots, \varphi_n\}$ a quasi-stable generator for V_0. Then $V_1 = \mathcal{S}(\Pi)$ with Π as in (2). Since $\#\Pi = \operatorname{len} V_1$, Π is a generator for V_1 with minimal length. Hence Π is actually a quasi-stable generator for V_1 [**5**, Theorem 3.12]. The length condition on V_1 and (8) show the following fact: If $x \in \sigma(V_1)$, then, for each $\alpha^* \in Q^*$, there exists $\delta_{x,\alpha^*} \in \mathbb{Z}^d$ such that $M^{*-1}(x+\alpha^*) + \delta_{x,\alpha^*} \in \sigma(V_0)$. It is obvious that for a fixed set of coset representatives Q^* the set of 'folding' multi-integers $\{\delta_{x,\alpha^*} : x \in \sigma(V_1), \alpha^* \in Q\}$ is a finite set. This implies that $\sigma(V_0)$ contains a measurable subset of Lebesgue measure $\left|M^{*-1}(\sigma(V_1) + \alpha^*)\right| = \left|M^{*-1}(\sigma(V_1))\right|$. We show that these subsets of $\sigma(V_0)$ do not overlap. Suppose that $M^{*-1}(x+\alpha^*) + \gamma = M^{*-1}(y+\beta^*) + \delta \in \mathbb{T}^d$ for $x, y \in \sigma(V_1), \alpha^*, \beta^* \in Q^*, \gamma, \delta \in \mathbb{Z}^d$. Then $x - y = \beta^* - \alpha^* + M^*(\delta - \gamma)$. Since the right-hand side is an integer and since $x, y \in \mathbb{T}^d = [-1/2, 1/2)^d$, $x - y = 0$. Hence $\alpha^* = \beta^*$ and

$\gamma = \delta$. Therefore $\sigma(V_0)$ contains $\#Q^* = |\det M|$ number of subsets of Lebesgue measure $\left|M^{*-1}(\sigma(V_1))\right|$. This shows that $|\sigma(V_0)| \geqslant |\sigma(V_1)|$. Since $V_0 \subset V_1$, $\sigma(V_0) \subset \sigma(V_1)$. Consequently, $\sigma(V_0) = \sigma(V_1)$. Lemma 8 implies that $\sigma(V_0) = M^*\sigma(V_0) \,(\mathrm{mod}\, 1)$. The ergodicity of the map $x \in \mathbb{T}^d \to M^*x \,(\mathrm{mod}\, 1)$ in $\mathbb{T}^d$ implies that $\sigma(V_0)$ is either $\mathbb{T}^d$ or empty. Since it is not empty, it is $\mathbb{T}^d$. ∎

The length condition in Theorem 9 is indispensable by Example 6.

References

[1] J.J. Benedetto and S. Li, 'The theory of multiresolution analysis frames and applications to filter banks', *Appl. Comput. Harm. Anal.* **5** (1998), 398–427.

[2] J.J. Benedetto and O.M. Treiber, 'Wavelet frames: multiresolution analysis and extension principle', in *Wavelet transforms and time-frequency signal analysis*, (L. Debnath, Editor) (Birkhaüser, Boston, 2000), **pp.** 3–36.

[3] C. de Boor, R. DeVore and A. Ron, 'On the construction of multivariate (pre)wavelets', *Constr. Approx.* **9** (1993), 123–166.

[4] C. de Boor, R. DeVore and A. Ron, 'Approximations from shift-invariant subspaces of $L^2(\mathbb{R}^d)$', *Trans. Amer. Math. Soc.* **341** (1994), 787–806.

[5] C. de Boor, R. DeVore and A. Ron, 'The structure of finitely generated shift-invariant spaces in $L_2(\mathbb{R}^d)$', *J. Funct. Anal.* **119** (1994), 37–78.

[6] M. Bownik, 'The structure of shift-invariant subspaces of $L^2(\mathbb{R}^n)$', *J. Funct. Anal.* **177** (2000), 282–309.

[7] O. Christensen, B. Deng and C. Heil, 'Density of Gabor frames', *Appl. Comput. Harmon. Anal.* **7** (1999), 292—304.

[8] K. Gröchenig and W. R. Madych, 'Multiresolution analysis, Haar bases, and self-similar tilings of R^n', *IEEE Trans. Inform. Theory* **38** (1992), 556–568.

[9] C. Heil and D. Walnut, 'Continuous and discrete wavelet transforms', *SIAM Rev.* **31** (1989), 628–666.

[10] H. Helson, *Lectures on invariant subspaces* (Academic Press, New York, 1964).

[11] R.-Q. Jia, 'Shift-invariant spaces and linear operator equations', *Israel J. Math.* **103** (1998), 258–288.

[12] R.-Q. Jia and C.A. Micchelli, 'Using the refinement equation for the construction of pre-wavelets, II: powers of two', in *Curves and Surfaces*, (P.J. Laurent, A. Le Méhauté and L. Schumaker, Editors) (Academic Press, New York), **pp.** 209–246.

[13] H.O. Kim and J.K. Lim, 'New characterizations of Riesz bases', *Appl. Comput. Harmon. Anal.* **4** (1997), 222–229.

[14] H.O. Kim and J.K. Lim, 'On frame wavelets associated with frame multi-resolution analysis', *Appl. Comput. Harmon. Anal.* **10** (2001), 61–70.

[15] H.O. Kim and J.K. Lim, 'Applications of shift-invariant space theory to some problems of multi-resolution analysis of $L^2(\mathbb{R}^d)$', in *Wavelet Analysis and Applications*, (D. Deng, D. Huang, R.-Q Jia, W. Lin and J. Wang, Editors) (American Mathematical Society/International Press, Boston, 2001), **pp.** 183–191.

[16] J.K Lim, 'Gramian analysis of multivariate frame multiresolution analyses', *Bull. Austral. Math. Soc.* **66** (2002), 291–300.

[17] S. Mallat, 'Multiresolution approximations and wavelets', *Trans. Amer. Math. Soc.* **315** (1989), 69–88.

[18] Y. Meyer, *Wavelets and operators*, Cambridge Studies in Advanced Mathematics **37** (Cambridge University Press, Cambridge, 1992).

[19] A. Ron and Z. Shen,, 'Frames and stable bases for shift-invariant subspaces of $L_2(\mathbb{R}^d)$', *Canad. J. Math.* **47** (1995), 1051–1094.

[20] P. Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics **79** (Springer-Verlag, New-York, Berlin, 1981).

Division of Applied Mathematics
KAIST
373-1 Guseong-dong, Yuseong-gu
Daejeon 305-701
Korea
e-mail: hkim@amath.kaist.ac.kr
rykim@amath.kaist.ac.kr

CHiPS
KAIST
373-1, Guseong-dong, Yuseong-gu
Daejeon, 305-701
Republic of Korea
e-mail: jaekun@ftn.kaist.ac.kr