LOCAL ANALYSIS OF FRAME MULTIRESOLUTION ANALYSIS
WITH A GENERAL DILATION MATRIX
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A multivariate semi-orthogonal frame multiresolution analysis with a general integer
dilation matrix and multiple scaling functions is considered. We first derive the
formulas of the lengths of the initial (central) shift-invariant space \( V_0 \) and the next
dilation space \( V_1 \), and, using these formulas, we then address the problem of the
number of the elements of a wavelet set, that is, the length of the shift-invariant
space \( W_0 := V_1 \ominus V_0 \). Finally, we show that there does not exist a 'genuine' frame
multiresolution analysis for which \( V_0 \) and \( V_1 \) are quasi-stable spaces satisfying the
usual length condition.

1. INTRODUCTION

The orthonormal dyadic multiresolution analysis of \( L^2(\mathbb{R}) \) with a single scaling
function was introduced by Mallat and Meyer in order to construct an orthonormal wavelet
basis of \( L^2(\mathbb{R}) \) [17, 18]. Benedetto and Li considered the dyadic semi-orthogonal frame
multiresolution analysis of \( L^2(\mathbb{R}) \) with a single scaling function, and successfully ap-
plied the theory in the analysis of narrow band signals [1]. We refer to [9] for the
basic definitions and properties of frames and Riesz bases of a Hilbert space. Unlike
the multiresolution analysis of Mallat and Meyer, where there always exists a wavelet
set consisting of a single element whose dyadic dilations of the integer translates form
an orthonormal basis of \( L^2(\mathbb{R}) \), the multiresolution analysis of Benedetto and Li has a
wavelet set whose cardinality may be one or two [14]. The exact definition of a wavelet
set of a multiresolution analysis is found in Section 3. The characterisation of the dyadic
semi-orthogonal frame multiresolution analysis with a single scaling function admitting
a single frame wavelet whose dyadic dilations of the integer translates form a frame for
\( L^2(\mathbb{R}) \) was obtained, independently, by Benedetto and Treiber by a direct method [2],
and by Kim and Lim by using the theory of shift-invariant spaces [14]. The dyadic mul-
tivariate generalisation (with a single scaling function) of the multiresolution analysis of
Mallat and Meyer were considered by several authors. See [3], for example. Lim, among
other things, addressed the problem of the cardinality of a wavelet set in the setting of

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the dyadic multivariate frame multiresolution analysis with a single scaling function [16], and Kim and Lim gave an analysis of dyadic multivariate frame multiresolution analysis with multiple scaling functions [15]. As the theory and applications of wavelets evolve, many authors considered more general dilations other than the dyadic ones (for example, [8]).

In this paper we consider a multivariate frame multiresolution analysis with a general integer dilation matrix and multiple scaling functions by extending the method and results of [15]. We first analyse the local dimension of the initial, that is, central, shift-invariant space \( V_0 \) and the next dilation space \( V_1 \) of the multiresolution analysis. Using this, we derive the formulas of the lengths of the shift-invariant spaces \( V_0 \) and \( V_1 \), and address the problem of the number of the elements of a wavelet set. Finally, we show that there does not exist a ‘genuine’ frame multiresolution analysis for which \( V_0 \) and \( V_1 \) are quasi-stable spaces satisfying the usual length condition by applying the local dimension analysis and the ergodicity of the dilation matrix (Theorem 9). This result improves Theorem 3.9 in [15] in the sense that we do not presuppose that the spectrums of \( V_0 \) and \( V_1 \) coincide.

The organisation of this paper is as follows: Preliminary discussions on the dilation matrix and shift-invariant spaces and the definition of the multiresolution analysis we consider are given in Section 2, and our main results, along with an analysis of the local dimensions of \( V_0 \) and \( V_1 \), are given in Section 3.

2. PRELIMINARY DISCUSSION

Suppose that \( M \) is a \( d \times d \) integer dilation matrix, that is, the entries of \( M \) are integers and the moduli of the eigenvalues of \( M \) are strictly greater than one. It is known that the order of the quotient group \( \mathbb{Z}^d/M\mathbb{Z}^d \) is \( |\det M| \) [8, Lemma 2]. Let \( T^d := \mathbb{R}^d/\mathbb{Z}^d \) denote the \( d \)-dimensional torus which is identified with \([-1/2, 1/2)^d\). For \( x \in \mathbb{R}^d \), let \( x \pmod{1} \) denotes the standard representative of \( x + \mathbb{Z}^d \) in \([-1/2, 1/2)^d\). Suppose that \( T \) is a \( d \times d \) invertible matrix with integer entries such that the moduli of the eigenvalues of \( T \) are all different from 1. Then the map \( \tilde{T} : T^d \to T^d \), defined via \( \tilde{T}x := T x \pmod{1} \), is ergodic [20, Theorem 0.15, Corollary 1.10.1]. We note that \( M^t = M^\ast \), where \( t \) and \( * \) denote the transpose and the adjoint of a matrix with complex entries, respectively. For notational convenience we let \( Q := \mathbb{Z}^d/M\mathbb{Z}^d \) and let \( Q^* := \mathbb{Z}^d/M^\ast \mathbb{Z}^d \). Let \( D := D_M : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \) denote the unitary dilation operator defined via \( Df(x) := |\det M|^{1/2} f(Mx) \). For \( y \in \mathbb{R}^d \), \( T_y : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \) denotes the unitary translation operator such that \( T_y f(x) := f(x - y) \). In this paper we adapt the following definition of multiresolution analysis.

**Definition 1:** \( \{V_k\}_{k \in \mathbb{Z}} \) is said to be a frame multiresolution analysis if each \( V_k \) is a closed subspace of \( L^2(\mathbb{R}^d) \) such that:
Constructions of multivariate wavelet frames

(1) \( V_k \subset V_{k+1}, \quad k \in \mathbb{Z} \);

(2) \( \bigcup_{k \in \mathbb{Z}} V_k = L^2(\mathbb{R}^d), \quad \bigcap_{k \in \mathbb{Z}} V_k = \{0\} \);

(3) \( D(V_k) = V_{k+1}, \quad k \in \mathbb{Z} \);

(4) There exists a finite set of scaling functions \( \Phi \subset V_0 \) such that \( \{ T_\alpha \varphi : \alpha \in \mathbb{Z}^d, \varphi \in \Phi \} \) is a frame for \( V_0 \).

Various examples and applications of multiresolution analyses are found in the references cited in Section 1.

The following form of the Fourier transform is used throughout this paper: for \( f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \) and \( t \in \mathbb{R}^d \),

\[
T(t) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i t \cdot x} \, dx,
\]

where \( \cdot \) denotes the \( d \)-dimensional real inner product. It is, of course, extended to be a unitary transform from \( L^2(\mathbb{R}^d) \) onto \( L^2(\mathbb{R}^d) \) via the Plancherel theorem.

Suppose that \( \{ f_i : 1 \leq i \leq n \} \) is a finite family of elements of a Hilbert space \( \mathcal{H} \) with an inner product \( \langle \cdot, \cdot \rangle \). We frequently use the following simple observation:

\[
\dim \text{span}\{ f_i : 1 \leq i \leq n \} = \text{rank}( (f_i, f_j) )_{1 \leq i, j \leq n}.
\]

Our analysis uses the theory of (multi-integer) shift-invariant spaces developed in [4, 5, 6, 10, 11, 12, 19] and the references therein. We briefly review the theory and uses the existing results freely. A closed subspace \( S \) of \( L^2(\mathbb{R}^d) \) is said to be shift-invariant if \( T_\alpha f \in S \) whenever \( f \in S \) and \( \alpha \in \mathbb{Z}^d \). If \( \Phi \subset L^2(\mathbb{R}^d) \), then \( S := S(\Phi) := \text{span}\{ T_\alpha \varphi : \alpha \in \mathbb{Z}^d \} \) is a shift-invariant space. In this case, \( \Phi \) is called a generator of \( S \). If \( \Phi \) is finite, then \( S \) is called a finite shift-invariant space. We write \( S = S(\varphi) \) instead of \( S(\{ \varphi \}) \) if \( \Phi = \{ \varphi \} \) is a singleton. In this case, we call \( S \) a principal shift-invariant space. It is known that any shift-invariant space has a countable generator. The length of a shift-invariant space is defined to be

\[
\text{len} S := \inf\{ \# \Phi : S = S(\Phi), \Phi \subset L^2(\mathbb{R}^d) \},
\]

where \( \# \) denotes the cardinality. Let \( \widehat{f}_{||x||} \) be the sequence \( (\widehat{f}(x+\alpha))_{\alpha \in \mathbb{Z}^d} \) which is in \( \ell^2(\mathbb{Z}^d) \) for almost every \( x \in \mathbb{T}^d \). If \( A \subset L^2(\mathbb{R}^d), x \in \mathbb{T}^d \), then we let \( \widehat{A}_{||x||} := \{ \widehat{f}_{||x||} \in \ell^2(\mathbb{Z}^d) : f \in A \} \), which is called the fibre of \( A \) at \( x \). It is a subspace of \( \ell^2(\mathbb{Z}^d) \) if \( A \) is a shift-invariant space. The following theorem is used frequently in our discussion.

**Theorem 2. ([4, 6, 10, 11]).** Let \( S \) be a closed, not necessarily shift-invariant, subspace of \( L^2(\mathbb{R}^d) \) and \( \Phi \) a countable subset of \( L^2(\mathbb{R}^d) \). Then \( S = S(\Phi) \) if and only if

\[ \widehat{f}_{||x||} \in \text{span}\{ \widehat{\varphi}_{||x||} : \varphi \in \Phi \} \quad \text{for almost every } x \in \mathbb{T}^d \text{ and for each } f \in S. \]

The spectrum of a shift-invariant space is defined to be \( \sigma(S) := \{ x \in \mathbb{T}^d : \widehat{S}_{||x||} \neq \{0\} \} \). A finite subset \( \Phi \) of \( L^2(\mathbb{R}^d) \) is said to be a quasi-stable generator for the shift-invariant space \( S(\Phi) \) if, in addition to the condition that the family of the integer translates of \( \Phi \) is a frame for \( S(\Phi) \), \( \dim \text{span}\{ \widehat{\varphi}_{||x||} : \varphi \in \Phi \} = \# \Phi \) or 0 for almost every \( x \in \mathbb{T}^d \). If \( \Phi \) is a quasi-stable generator, then there is a convenient 'local' formula for the orthogonal projection onto \( S(\Phi) \) [4, 19]. The stable generator is a quasi-stable generator such that
the spectrum of the shift-invariant space it generates is $T^d$. It turns out that if $\Phi$ is a stable generator, then the family of the integer translates of $\Phi$ is a Riesz basis for $S(\Phi)$ [4, 19]. We say that a shift-invariant space $S$ is quasi-stable, if $\dim \hat{S}_{||x||} = n$ or 0 for some non-negative integer $n$ almost everywhere. It is said to be stable if $\dim \hat{S}_{||x||} = n$ almost everywhere. It is known that a quasi-stable/stable shift-invariant space has a quasi-stable/stable generator [4, 19].

We need the following results:

**Theorem 3.** ([4].) For a shift-invariant subspace $S$ of $L^2(\mathbb{R}^d)$

$$\text{len } S = \text{ess-sup}\{\dim \hat{S}_{||x||} : x \in T^d\}.$$ 

**Theorem 4.** ([4].) Let $S_1$ be a shift-invariant subspace of a shift-invariant space $S$ and let $S_2 := S \ominus S_1$. Then $S_2$ is also a shift-invariant subspace of $S$ and $\hat{S}_{||x||} = \hat{S}_1_{||x||} \ominus \hat{S}_2_{||x||}$ for almost every $x \in T^d$.

Suppose that $S = S(\Phi)$ for a finite set $\Phi := \{\varphi_1, \varphi_2, \ldots, \varphi_n\}$. Then the $n \times n$ matrix

$$G(x) := G_{\Phi}(x) := ((\hat{\varphi}_{i||x||}, \hat{\varphi}_{j||x||})_p(z^d))_{1 \leq i, j \leq n}$$

is the Gramian of $\Phi$ at $x \in T^d$. Let $\lambda(x)$, $\lambda^+(x)$ and $\Lambda(x)$ denote the smallest eigenvalue, the smallest non-negative eigenvalue and the largest eigenvalue of $G(x)$, respectively.

**Theorem 5.** ([4, 6, 19].) The family of the integer translates of $\Phi$ is a frame for $S$ if and only if there exist positive constants $A$ and $B$ such that $A \leq \lambda^+(x) \leq \Lambda(x) \leq B$ for almost every $x \in \sigma(S)$. It is a Riesz basis for $S$ if and only if $A \leq \lambda(x) \leq \Lambda(x) \leq B$ for almost every $x \in T^d$. Moreover, $A$ and $B$ are a pair of frame (Riesz) bounds of the frame (Riesz basis), respectively.

### 3. Frame Multiresolution Analysis

Suppose that $\{V_j\}_{j \in \mathbb{Z}}$ is a frame multiresolution analysis. Then there exists a set of scaling functions $\Phi := \{\varphi_i : 1 \leq i \leq n\} \subset L^2(\mathbb{R}^d)$ such that $\{T_{\alpha} \varphi_i : \alpha \in \mathbb{Z}^d, 1 \leq i \leq n\}$ is a frame for $V_0$. We may assume that the length of $V_0$ is $n$. Then $V_0 = S(\Phi)$ and $V_1 := D(V_0)$. Let

$$G(x) := G_{\Phi}(x) := ((\hat{\varphi}_{i||x||}, \hat{\varphi}_{j||x||})_p(z^d))_{1 \leq i, j \leq n}$$

be the Gramian of $\Phi$ at $x \in T^d$.

Since $\varphi_i \in V_1$ for each $1 \leq i \leq n$, and since $\{DT_{\alpha} \varphi_i : \alpha \in \mathbb{Z}^d, 1 \leq i \leq n\}$ is a frame for $V_1$, there exist $a_{ij} \in c^2(\mathbb{Z}^d), 1 \leq i, j \leq n$, such that $\varphi_i = \sum_{j=1}^{n} \sum_{\alpha \in \mathbb{Z}^d} a_{ij}(\alpha) DT_{\alpha} \varphi_j$. Hence

$$\hat{\varphi}_i(x) = \sum_{j=1}^{n} \sum_{\alpha \in \mathbb{Z}^d} a_{ij}(\alpha) |\det M|^{-1/2} e^{-2\pi i \alpha \cdot (M^{*-1} x)} \hat{\varphi}_j(M^{*-1} x)$$

$$= \sum_{j=1}^{n} m_{ij}(M^{*-1} x) \hat{\varphi}_j(M^{*-1} x),$$
where
\[ m_{ij}(x) := \sum_{\alpha \in \mathbb{Z}^d} |\det M|^{-1/2} a_{ij}(-\alpha) e^{-2\pi i \alpha \cdot x} \in L^2(\mathbb{T}^d). \]

For \( x \in \mathbb{T}^d \), let
\[ m(x) := (m_{ij}(x))_{1 \leq i,j \leq n} \]
and
\[ \vec{\varphi}(x) := (\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x))^t. \]

Then
\[ (1) \quad \vec{\varphi}(x) = m(M^{-1}x) \vec{\varphi}(M^{-1}x). \]

This \( m \), called a mask of the multiresolution analysis, may not be unique since \( \{DT_\alpha \varphi_i : \alpha \in \mathbb{Z}^d, 1 \leq i \leq n\} \) is only assumed to be a frame, not necessarily a Riesz basis. Note that \( DT_y = T_y D \) for \( y \in \mathbb{R}^d \). Since each \( \gamma \in \mathbb{Z}^d \) can be written uniquely as \( \gamma = M\alpha + \beta \) for some \( \alpha \in \mathbb{Z}^d \) and \( \beta \in \mathbb{Q} \),
\[ \{DT_\gamma \varphi_i : \gamma \in \mathbb{Z}^d, 1 \leq i \leq n\} = \{T_\alpha DT_\beta \varphi_i : \alpha \in \mathbb{Z}^d, \beta \in \mathbb{Q}, 1 \leq i \leq n\}. \]

Hence \( V_i = \mathcal{S}(\Pi) \), where
\[ (2) \quad \Pi := \{DT_\beta \varphi_i : \beta \in \mathbb{Q}, 1 \leq i \leq n\}. \]

This implies that the length of the shift-invariant space \( V_i \) is less than or equal to \( n|\det M| \). Since \( V_0 \) is a shift-invariant subspace of \( V_1 \), \( \text{len} V_1 \geq \text{len} V_0 = n \). There is an example of a frame multiresolution analysis in which the length of \( V_1 \) is that of \( V_0 \). See Example 6 below. Let \( W_0 \) denote \( V_1 \cap V_0 \), and let \( W_j := D^j(W_0), j \in \mathbb{Z} \). Then Definition 1 implies that \( L^2(\mathbb{R}^d) = \bigoplus_{j \in \mathbb{Z}} W_j \). \( W_0 \) is a shift-invariant space by Theorem 4.

Since \( W_0 \) is a subspace of \( V_1 \), the length of \( W_0 \) is also less than or equal to \( n|\det M| \). It cannot be zero. If it were zero, then \( W_0 = \{0\} \); hence \( V_0 = V_1 \). Definition 1 implies that \( L^2(\mathbb{R}) = V_0 = \mathcal{S}(\psi) \). This contradicts a result in [7] which states roughly that there are no frames of \( L^2(\mathbb{R}^d) \) consisting of the translates of a finite number of functions. Since \( W_0 \) is a finite shift-invariant space, there is a finite set \( \Psi \), called a wavelet set, such that \( W_0 = \mathcal{S}(\Psi) \). We may assume that the integer translates of the elements of \( \Psi \) form a frame for \( W_0 \) [5, 19]. Then, obviously, \( \{D^jT_\alpha \psi : j \in \mathbb{Z}, \alpha \in \mathbb{Z}^d, \psi \in \Psi\} \) is a frame for \( L^2(\mathbb{R}^d) \). Since the minimal cardinality of such \( \Psi \) is \( \text{len} W_0 \), the (minimal) number of the elements of a wavelet set is \( \text{len} W_0 \).

Note that, for \( \beta \in \mathbb{Z}^d \) and \( x \in \mathbb{T}^d \),
\[
(DT_\beta \varphi_i)^\wedge(x) = |\det M|^{-1/2} e^{-2\pi i \beta \cdot (M^{-1}x)} \hat{\varphi}_i(M^{-1}x),
\]
\[
(DT_\beta \varphi_i)_\parallel^\wedge = |\det M|^{-1/2} e^{-2\pi i \beta \cdot (M^{-1}x)} \left( e^{-2\pi i \beta \cdot (M^{-1}_0) \hat{\varphi}_i(M^{-1}_0(x + \alpha))} \right)_{\alpha \in \mathbb{Z}^d}. 
\]
Hence, for almost every $x \in \mathbb{T}^d$,
\[
\hat{V}_{1|z} = \text{span}\left\{ \left( e^{-2\pi i \beta(M^{*-1}\alpha)} \hat{\varphi}_i(M^{*-1}(x + \alpha)) \right)_{\alpha \in \mathbb{Z}^d} : \beta \in Q, 1 \leq i \leq n \right\},
\]
\[
\hat{V}_{0|z} = \text{span}\left\{ \left( \hat{\varphi}(x + \alpha) \right)_{\alpha \in \mathbb{Z}^d} : 1 \leq i \leq n \right\}
= \text{span}\left\{ \sum_{j=1}^{n} \left( m_{ij} \left( M^{*-1}(x + \alpha) \right) \hat{\varphi}_j \left( M^{*-1}(x + \alpha) \right) \right)_{\alpha \in \mathbb{Z}^d} : 1 \leq i \leq n \right\}.
\]

For $\beta^* \in Q^*$ define $P_{\beta^*} : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ via
\[
(P_{\beta^*} \alpha)(\alpha) := \begin{cases} a(\alpha), & \text{if } \alpha \in \beta^* + M^* \mathbb{Z}^d, \\ 0, & \text{otherwise}. \end{cases}
\]

Then $\ell^2(\mathbb{Z}^d) = \bigoplus_{\beta^* \in Q^*} P_{\beta^*}(\ell^2(\mathbb{Z}^d))$. Define, for $x \in \mathbb{T}^d, 1 \leq i \leq n, \beta^* \in Q^*$,
\[
a_{x,i,\beta^*} := P_{\beta^*} \left( \left( \hat{\varphi}_i(M^{*-1}(x + \alpha)) \right)_{\alpha \in \mathbb{Z}^d} \right).
\]

Notice that $a_{x,i,\beta^*}$ is the ‘up-sampled’ version of $\hat{\varphi}_{i|M^{*-1}(x + \beta^*)}$, that is,
\[
a_{x,i,\beta^*}(\beta^* + M^* \alpha) = \hat{\varphi}_{i|M^{*-1}(x + \beta^*)}(\alpha), \quad \alpha \in \mathbb{Z}^d,
\]
\[
a_{x,i,\beta^*}(\gamma^*) = 0, \quad \gamma^* \notin \beta^* + M^* \mathbb{Z}^d.
\]

Therefore
\[
\|a_{x,i,\beta^*}\|_{\ell^2(\mathbb{Z})} = \|\hat{\varphi}_{i|M^{*-1}(x + \beta^*)}\|_{\ell^2(\mathbb{Z})}.
\]

We also have
\[
\left( e^{-2\pi i \beta(M^{*-1}\alpha)} \hat{\varphi}_i(M^{*-1}(x + \alpha)) \right)_{\alpha \in \mathbb{Z}} = \sum_{\gamma^* \in Q^*} e^{-2\pi i \beta(M^{*-1}\gamma^*)} a_{x,i,\gamma^*}.
\]

Let $b_{x,i,\beta}$ be the right-hand side of the above equation. Then, for a fixed $x \in \mathbb{T}^d, 1 \leq i \leq n$, we have the following matrix relation:
\[
(b_{x,i,\beta})_{\beta \in Q} = (e^{-2\pi i \beta(M^{*-1}\gamma^*)})_{\beta \in Q, \gamma^* \in Q^*} (a_{x,i,\gamma^*})_{\gamma^* \in Q^*}^t.
\]

Recall that, for any $\beta \in \mathbb{Z}^d$, the map $\gamma^* \mapsto e^{-2\pi i \beta(M^{*-1}\gamma^*)}$ is a character of the discrete group $G^*$. Hence the sum $\sum_{\gamma^* \in Q^*} e^{-2\pi i \beta(M^{*-1}\gamma^*)}$ is the order of $Q^*$, which is $|\det M|$, if the map is the identity character, and the sum is 0 if the map is not the identity character since the only discrete multiplicative subgroups of $\mathbb{T}$ are the groups of the $p$-th roots of unity ([20, Theorem 0.14]). Using this observation, it is easy to see that
\[
\left( e^{-2\pi i \beta(M^{*-1}\gamma^*)} \right)_{\beta \in Q, \gamma^* \in Q^*} (e^{2\pi i (M^{*-1}\delta^*)})_{\delta^* \in Q^*} = |\det M| \hat{I}_{|\det M|}.
\]
In particular, for each $1 \leq i \leq n$ and $x \in \T^d$, span\{$b_{x,i,\beta} : \beta \in Q \}$ = span\{$a_{x,i,\gamma^*} : \gamma^* \in Q^* \}$. This shows that:

$$\hat{V}_{1||x} = \text{span}\{a_{x,i,\gamma^*} : 1 \leq i \leq n, \gamma^* \in Q^* \}.$$  

The 1-periodicity of the mask $m$ and (4) imply that:

$$\hat{V}_{0||x} = \text{span}\left\{ \sum_{j=1}^{n} \sum_{\alpha^* \in Q^*} m_{ij}(M^{*-1}(x + \alpha^*))a_{x,j,\alpha^*} : 1 \leq i \leq n \right\}.$$ 

Note that, for almost every $x \in \T^d$, dim $\hat{V}_{1||x}$ equals the rank of the following $n|\det M| \times n|\det M|$ matrix

$$\left( a_{x,i,\alpha^*}, a_{x,i,\beta^*} \right)_{\alpha^*, \beta^*}.$$ 

If we order the indices suitably, then (4) and (5) imply that the matrix is the block diagonal matrix

$$\text{diag}\left( G(M^{*-1}(x + \alpha^*)) \right)_{\alpha^* \in Q^*}.$$ 

Recall that rank $G(M^{*-1}(x + \alpha^*)) = \text{dim} \hat{V}_{0||M^{*-1}(x+\alpha^*)}$ for each $\alpha^* \in Q^*$. Hence, for almost every $x \in \T^d$, 

$$\text{dim} \hat{V}_{1||x} = \sum_{\alpha^* \in Q^*} \text{rank} G(M^{*-1}(x + \alpha^*)) = \sum_{\alpha^* \in Q^*} \text{dim} \hat{V}_{0||M^{*-1}(x+\alpha^*)}.$$ 

A direct calculation shows that

$$G(x) = \sum_{\alpha^* \in Q^*} m(M^{*-1}(x + \alpha^*))G(M^{*-1}(x + \alpha^*))m(M^{*-1}(x + \alpha^*))^*.$$ 

Hence, for almost every $x \in \T^d$,

$$\text{dim} \hat{V}_{0||x} = \text{rank} \sum_{\alpha^* \in Q^*} m(M^{*-1}(x + \alpha^*))G(M^{*-1}(x + \alpha^*))m(M^{*-1}(x + \alpha^*))^*.$$ 

**Example 6.** Let us first consider a dyadic univariate frame multiresolution analysis with a single scaling function, that is, $d = n = 1$ and $Q = \Z/2\Z = \{0,1\}$. Therefore $Df(x) = 2^{1/2} f(2x)$. Let $V_0$ be a Paley–Wiener space such that \{f $\in L^2(\R)$ : supp(f) $\subset [-a,a]$\} with $0 < a < 1/4$, and let $V_j := D^j(V_0)$, $j \in \Z$. Then it is easy to see that \{V_j\}_j\in\Z is a frame multiresolution analysis ([14]). Obviously, $V_0 = S(\chi_{[-a,a]})$, where $\chi$ denotes the inverse Fourier transform. Hence, $V_1 = S(\chi_{[-2a,2a]})$ is a shift-invariant space of length 1. This can be proved by using (8). Note that $G(x) = \chi_{[-a,a]+Z}(x)$ for $x \in \T$. Hence, $\hat{V}_{1||x} = \chi_{[-a,a]+Z}(x/2) + \chi_{[-a,a]+Z}(x/2 + 1/2)$ for $x \in \T$. Hence dim $\hat{V}_{1||x} = 1$ for $x \in [-2a,2a]$, and dim $\hat{V}_{1||x} = 0$ for $x \in \T \setminus [-2a,2a]$. Therefore len $V_1 = 1$ by Theorem 3. Recall that len $V_1$ is less than or equal to $n2^d = 2$. In this example, the length of $V_1$ is
that of $V_0$. Notice, however, that $V_0$ is a strict subspace of $V_1$. It is now easy to see that \( \text{len} V_2 = 2 \).

The above example can be directly extended to the case where \( d > 1, \ n = 1 \) and \( M = 2I_d \). Then \( Q = Q^* = \{(\epsilon_1, \epsilon_2, \ldots, \epsilon_d) : \epsilon_i = 0 \text{ or } 1 \} \). For the sake of simplicity, define \( \tilde{\varphi} := \chi_{(-1/4,1/4)^d} \), \( V_0^d := S(\varphi) \) and \( V_j^d := D^j(V_0^d) \). Then \( \{V_j^d\}_{j \in \mathbb{Z}} \) is a frame multiresolution analysis. We show that \( \text{len} V_1^d = 1 \). \((8)\) implies that, for almost every \( x \in \mathbb{T}^d = [-1/2,1/2)^d \), \( \text{dim} \tilde{V}_1^d = n \left( |\text{det} M| - 1 \right) \) almost everywhere. Therefore \( \text{len} W_0 = n \left( |\text{det} M| - 1 \right) \) by Theorem 3.

Suppose, temporarily, that \( \{T_\alpha \varphi_i : \alpha \in \mathbb{Z}, 1 \leq i \leq n \} \) is a Riesz basis of \( V_0 \). Then \( \text{rank} G(x) = n \) for almost every \( x \in \mathbb{T}^d \) \([4]\). Hence \( \text{dim} \tilde{V}_1^d = n |\text{det} M| \) for almost every \( x \in \mathbb{T}^d \). Since \( \tilde{V}_1^d = \tilde{V}_0^d \oplus \tilde{W}_0^d \) almost everywhere by Theorem 4, \( \text{dim} \tilde{W}_0^d = n \left( |\text{det} M| - 1 \right) \) almost everywhere. Therefore \( \text{len} W_0 = n \left( |\text{det} M| - 1 \right) \) by Theorem 3.

Benedetto and Li \([1]\) introduced the following concept: a frame multiresolution analysis admit a standard (frame) wavelet set if \( \text{len} W_0 \leq n \left( |\text{det} M| - 1 \right) \). The characterisations of dyadic frame multiresolution analyses admitting standard wavelet sets were given in \([2, 14]\), independently, for \( d = n = 1 \), and in \([16]\) for \( d > 1 \) and \( n = 1 \). Combining \((8)\) and \((9)\) with Theorems 3 and 4 yield the following general result on the admittance of a standard wavelet set.

**Theorem 7.** The frame multiresolution analysis \( \{V_j\}_{j \in \mathbb{Z}} \) admits a standard wavelet set if and only if, for almost every \( x \in \mathbb{T}^d \),

\[
\sum_{\alpha^* \in Q^*} \text{rank} G(M^{*-1}(x + \alpha^*)) - \text{rank} \sum_{\alpha^* \in Q^*} m(M^{*-1}(x + \alpha^*)) G(M^{*-1}(x + \alpha^*))^* \leq n \left( |\text{det} M| - 1 \right). 
\]

We now recover the previous results on the admittance of a standard wavelet set \([2, 14, 16]\). Suppose that \( n = 1 \) and \( M = 2I_d \). Then \( Q = Q^* = \mathbb{Z}^d/2\mathbb{Z}^d = \{(\epsilon_1, \epsilon_2, \ldots, \epsilon_d) : \epsilon_i = 0 \text{ or } 1 \}, |\text{det} M| = 2^d, \phi = \{\varphi\} \subset L^2(\mathbb{R}^d), G(x) = \|\tilde{\varphi}||x||L^2(\mathbb{R}^d), \) and \( m(x) = \text{a scalar} \). \((10)\) becomes

\[
\sum_{\alpha \in Q} \|\tilde{\varphi}||x+\alpha||L^2(\mathbb{R}^d)\| - \text{rank} \sum_{\alpha \in Q} m\left(\frac{x + \alpha}{2}\right) \left\|\tilde{\varphi}||x+\alpha||L^2(\mathbb{R}^d)\right\|^2 \leq 2^d - 1,
\]

where the rank of a scalar is the rank of the \( 1 \times 1 \) matrix with the scalar entry. Notice that the left-hand side of the above inequality is less than or equal to \( 2^d \). Hence the frame multiresolution analysis admits a standard wavelet set if and only if the left-hand side is...
not $2^d$ almost everywhere. The condition, now, is equivalent to the condition that $E$ is of zero Lebesgue measure with

$$E := \left\{ x \in T^d : \varphi_{\|x+\alpha\|/2} \neq 0 \text{ for each } \alpha \in Q, \sum_{\alpha \in Q} \left| m \left( \frac{x + \alpha}{2} \right) \right|^2 = 0 \right\}.$$  

This recovers Theorem 5 of [16] (see also [2, 14] for the univariate case).

We now observe a simple relationship between the spectrums of $V_0$ and $V_1$.

**Lemma 8.** $\sigma(V_1) = M^* \sigma(V_0) \pmod{1}$.

**Proof:** Suppose that $x \in \sigma(V_0)$. Then $\varphi_{\|x\|} \neq 0$ for some $i$ by (3). There exist $y \in T^d, \alpha^* \in Z^d$ such that $M^*x = y + \alpha^*$. Now $\alpha^* = \beta^* + M^* \gamma^*$ for some $\beta^* \in Q^*$ and $\gamma^* \in Z^d$. (6) implies that

$$\|a_{x*,\beta^*}\varphi(x)\|_{L^2} = \|\varphi_{\|M^{-1}(M^*x - \beta^* - M^* \gamma^* + \beta^*)\|} = \|\varphi_{\|x\| - \gamma^*}\|_{L^2} \neq 0.$$  

Therefore, $y = M^*x - \alpha^* \in \sigma(V_1)$ by (7). This shows that $M^* \sigma(V_0) \subseteq \sigma(V_1)$. Suppose, on the other hand, that $x \in \sigma(V_1)$. Then $a_{x*,\beta^*} \neq 0$ for some $1 \leq i \leq n$ and $\beta^* \in Q^*$ by (7). Then $\varphi_{\|M^{-1}(x + \beta^*)\|} \neq 0$ by (6). Hence $M^{n-1}(x + \beta^*) \pmod{1} \in \sigma(V_0)$ by (3). This shows that $\sigma(V_1) \subseteq M^* \sigma(V_0) \pmod{1}$.

Recall that a frame is a Riesz basis if it is, in a certain sense, ‘globally’ irredundant, that is, irredundant in the norm topology [9, 13]. Suppose that $\Phi$ is a quasi-stable generating set for $V_0$. Then the family of the integer translates of $\Phi$, which is a frame for $V_0$, is ‘locally’, that is, fibre-wise, irredundant. We now show that: if $\Phi$ and $\Pi$ are quasi-stable generating sets for $V_0$ and $V_1$, respectively, then the integer translates are ‘globally’ redundant. This result improves Theorem 3.9 [15] in the sense that we do not presuppose that $\sigma(V_0) = \sigma(V_1)$. More precisely, we show:

**Theorem 9.** Suppose that $\{V_j\}_{j \in \mathbb{Z}}$ is a frame multiresolution analysis. If $V_0$ and $V_1$ are both quasi-stable, and if $\text{len} V_1 = |\det M| \text{len} V_0$, then they are actually stable. In particular, $\sigma(V_0) = \sigma(V_1) = T^d$.

**Proof:** We may assume that $V_0 = S(\Phi)$ with $\Phi := \{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ a quasi-stable generator for $V_0$. Then $V_1 = S(\Pi)$ with $\Pi$ as in (2). Since $\# \Pi = \text{len} V_1$, $\Pi$ is a generator for $V_1$ with minimal length. Hence $\Pi$ is actually a quasi-stable generator for $V_1$ [5, Theorem 3.12]. The length condition on $V_1$ and (8) show the following fact: If $x \in \sigma(V_1)$, then, for each $\alpha^* \in Q^*$, there exists $\delta_{x,\alpha^*} \in Z^d$ such that $M^{n-1}(x + \alpha^*) + \delta_{x,\alpha^*} \in \sigma(V_0)$. It is obvious that for a fixed set of coset representatives $Q^*$ the set of ‘folding’ multi-integers $\{\delta_{x,\alpha^*} : x \in \sigma(V_1), \alpha^* \in Q^*\}$ is a finite set. This implies that $\sigma(V_0)$ contains a measurable subset of Lebesgue measure $M^{n-1}(\sigma(V_1) + \alpha^*) = M^{n-1}(\sigma(V_1))$. We show that these subsets of $\sigma(V_0)$ do not overlap. Suppose that $M^{n-1}(x + \alpha^* + \gamma = M^{n-1}(y + \beta^*) + \delta \in T^d$ for $x, y \in \sigma(V_1), \alpha^*, \beta^* \in Q^*, \gamma, \delta \in Z^d$. Then $x - y = \beta^* - \alpha^* + M^*(\delta - \gamma)$. Since the right-hand side is an integer and since $x, y \in T^d = [-1/2, 1/2]^d$, $x - y = 0$. Hence $\alpha^* = \beta^*$ and
\( \gamma = \delta \). Therefore \( \sigma(V_0) \) contains \( \#Q^* = |\det M| \) number of subsets of Lebesgue measure \( |M^{-1}(\sigma(V_1))| \). This shows that \( |\sigma(V_0)| > |\sigma(V_1)| \). Since \( V_0 \subset V_1 \), \( \sigma(V_0) \subset \sigma(V_1) \). Consequently, \( \sigma(V_0) = \sigma(V_1) \). Lemma 8 implies that \( \sigma(V_0) = M^* \sigma(V_0) \text{ (mod 1)} \). The ergodicity of the map \( x \in \mathbb{T}^d \rightarrow M^* x \text{ (mod 1)} \) in \( \mathbb{T}^d \) implies that \( \sigma(V_0) \) is either \( \mathbb{T}^d \) or empty. Since it is not empty, it is \( \mathbb{T}^d \).

The length condition in Theorem 9 is indispensable by Example 6.

**REFERENCES**


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