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NORM INEQUALITIES FOR GENERATORS OF ANALYTIC SEMIGROUPS AND COSINE OPERATOR FUNCTIONS

BY

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ABSTRACT. We prove that if A is the infinitesimal generator of a bounded analytic semigroup in a sector $\{z \in \mathbf{C} : |\arg z| \leqq (\alpha\pi)/2\}$ of bounded linear operators on a Banach space, then the following inequalities hold:

$$\|A^k x\| \leqq M(\beta)S_{n,k}(\alpha)\|x\|^{1-(k/n)}\|A^n x\|^{k/n}\ (1 \leqq k \leqq n-1)$$

for any $x \in D(A^n)$ and for any $0 < \beta < \alpha$. This result helps us to answer in affirmative a question raised by M. W. Certain and T. G. Kurtz [3]. Similar inequalities are proved for cosine operator funtions.

1. **Introduction.** In 1939 A.N. Kolmogorov [7] proved that if $f \in C^{(n)}(\mathbf{R})$, $\|f\|_\infty < \infty$, $\|f^{(n)}\|_\infty < \infty$, then

$$(1) \qquad \|f^{(k)}\|_\infty \leqq C_{n,k}\|f\|_\infty^{1-(k/n)}\|f^{(n)}\|_\infty^{k/n}, \qquad (1 \leqq k \leqq n-1),$$

where

$$C_{n,k} = \frac{K_{n-k}}{K_n^{1-(k/n)}}, \quad K_j = \frac{4}{\pi}\sum_{p=0}^{\infty}\frac{(-1)^{p(j+1)}}{(2p+1)^{j+1}}.$$

This inequality is precise in the sense that there exists a function in $C^{(n)}(\mathbf{R})$ for which the above inequality becomes actually an equality. For the half-line $\mathbf{R}_+$, A. Cavaretta and I. J. Schoenberg [2] proved the existence of a minimal constant $D_{n,k}$ such that if $f \in C^{(n)}(\mathbf{R}_+)$, $\|f\|_\infty < \infty$, $\|f^{(n)}\|_\infty < \infty$ then

$$(2) \qquad \|f^{(k)}\|_n \leqq D_{n,k}\|f\|_\infty^{1-(k/n)}\|f^{(n)}\|_\infty^{k/n}, \qquad (1 \leqq k \leqq n-1)$$

However, they did not give this constant explicitly.

A number of authors have extended these inequalities to an abstract operator setting (cf. P. R. Chernoff [4] and references quoted in that paper). R. R. Kallman and G. C.

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Rota [6] were the first to prove such inequality for generators of strongly continuous contraction semigroups on a Banach space viz.,

$$\|Ax\|^2 \leqq 4\|x\|\,\|A_x^2\|, \quad x \in D(A^2).$$

Let $C^{(n)}(I,X)$ denote the space of n-times continuously differentiable functions on an interval $I \subset \mathbf{R}$ with values in the Banach space X and let $\|f^{(k)}\|_\infty = \sup\{\|f^{(k)}(x)\| : x \in I\}$ for $k = 0, 1, \ldots, n$, where $\|\cdot\|$ denotes the norm in X. Following an idea essentially due to E. M. Stein [11], Z. Ditzian [5] and M. W. Certain and T. G. Kurtz [3] (cf. also P. R. Chernoff [4]) proved the following theorems:

THEOREM A. *Suppose* $f \in C^{(n)}(\mathbf{R}_+, X)$. *If* $\|f\|_\infty < \infty$ *and* $\|f^{(n)}\|_\infty < \infty$, *then*

$$\|f^{(k)}\|_\infty \leqq D_{n,k}\|f\|^{1-(k/n)}\|f^{(n)}\|_\infty^{k/n}, \qquad (1 \leqq k \leqq n-1).$$

THEOREM B. *If A is the generator of a strongly continuous contraction semigroup,* $x \in D(A^n)$ *and* $n \geqq 2$ *is an integer, then*

$$\|A^k x\| \leqq D_{n,k}\|x\|^{1-(k/n)}\|A^n x\|^{k/n}, \qquad (1 \leqq k \leqq n-1).$$

THEOREM C. *Suppose* $f \in C^{(n)}(\mathbf{R}, X)$. *If* $\|f\|_\infty < \infty$, $\|f^{(n)}\|_\infty < \infty$, *then*

$$\|f^{(k)}\|_\infty \leqq C_{n,k}\|f\|_\infty^{1-(k/n)}\|f^{(n)}\|_\infty^{k/n}, \qquad (1 \leqq k \leqq n-1).$$

THEOREM D. *If A is the generator of a strongly continuous group of isometries and* $x \in D(A^n)$ *with* $n \geqq 2$ *an integer, then*

$$\|A^k x\| \leqq C_{n,k}\|x\|^{1-(k/n)}\|A^n\|^{k/n}, \qquad (1 \leqq k \leqq n-1).$$

In connection with Theorem B, M. W. Certain and T. G. Kurtz [3] raised the question as to whether the constants $D_{n,k}$ appearing in the inequalities of Theorem B can be improved if the strongly continuous semigroup is supposed to be also analytic.

In §2 of this paper we show that it is effectively so. In fact, we generalize Theorems A and B to vector-valued analytic functions defined in a sector $I_\alpha = \{z : |\arg z| < (\alpha\pi)/2\}$, where $0 < \alpha \leqq 1$ and to the generators of bounded analytic semigroups in I_α. The constants we obtain are smaller than those in Theorems A and B. Finally, in §3, we prove an analogue of Theorem D for the generators of strongly continuous cosine operator functions which contains as a particular case a sharpened version of a theorem of S. Kurepa [8].

2. **Analytic Semigroups.** We denote by A the space of analytic functions f from I_α to a Banach space X and by $\|f^{(k)}\|_\infty$ the supremum of $\|f^{(k)}(z)\|$ in z over I_α for $k = 0, 1, \ldots$

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THEOREM 1. *Let $f \in A$ be such that $\|f\|_\infty < \infty$ and $\|f^{(n)}\|_\infty < \infty$. Then setting $\gamma = 1 - \alpha$, we have*

$$\|f^{(k)}\|_\infty \leqq S_{n,k}(\alpha)\|f\|_\infty^{1-(k/n)}\|f^{(n)}\|_\infty^{k/n} \qquad (1 \leqq k \leqq n-1), \tag{3}$$

where

$$S_{n,k}(\alpha) = C_{n,k}\{\Gamma(\gamma n + 1)\}^{k/n}/\Gamma(\gamma k + 1).$$

PROOF. If $\alpha = 1$, by Theorem C applied to the function $y \longrightarrow f(x+iy)$, where $x > 0$ is fixed, we get

$$\|f^{(k)}(x+iy)\| \leqq C_{n,k}\|f\|_\infty^{1-(k/n)}\|f(n)\|_\infty^{k/n} \tag{4}$$

and this yields (3) with $S_{n,k}(1) = C_{n,k}$.

Suppose now that $0 < \alpha < 1$. Put

$$g_\alpha(z) = \int_{\Gamma_\phi} e^{-\gamma\omega} f(z_0 + z\omega^\gamma)d\omega,$$

where $|\phi| < \frac{\pi}{2}$, $z_0 \in I_\alpha$ and $\Gamma_\phi = \{\omega = re^{-i\phi},\ 0 \leqq r \leqq \infty\}$.

This integral converges uniformly in each sector $\{z : |\arg z - \gamma\phi| \leqq \alpha\frac{\pi}{2}\}$, where $0 < \alpha \leqq 1$, and is independent of ϕ. Since in the common part of the sector defined by two different ϕ's, $g_\alpha(z)$ coincide, g_α is analytic in $\{z : |\arg z| < \frac{\pi}{2}\}$. It is easily seen that

$$\|g_\alpha(z)\| \leqq \gamma^{-1}\|f\|_\infty$$

and

$$\|g_\alpha^{(n)}(z)\| \leqq \gamma^{-\gamma n - 1}\Gamma(\gamma n + 1)\|f^{(n)}\|_\infty.$$

But for $1 \leqq k \leqq n-1$

$$g_\alpha^{(k)}(z) = \int_{\Gamma_\phi} e^{-\gamma\omega}\omega^{k\gamma} f^{(k)}(z_0 + z\omega^\gamma)d\omega,$$

where the integral on the right converges uniformly in $\{z : |\arg z - \gamma\phi| < \frac{1}{2}\alpha\pi;\ |z| < \rho\}$. It follows that

$$\begin{aligned}\|g_\alpha^{(k)}(0)\| &= \|f^{(k)}(z_0)\| \int_0^\infty e^{-\gamma u}u^{k\gamma}du \\ &\geqq \|f^{(k)}(z_0)\|\gamma^{-k\gamma-1}\Gamma(\gamma k + 1).\end{aligned}$$

Applying the inequality (4) to g_α, we get

$$\|f^{(k)}(z_0)\| \leqq S_{n,k}(\alpha)\|f\|^{1-(k/n)}\|f^{(n)}\|_\infty^{k/n},$$

where

$$S_{n,k}(\alpha) = C_{n,k}\{\Gamma(\gamma n + 1)\}^{k/n}\{\Gamma(\gamma k + 1)\}^{-1}$$

and this yields (3).

We recall that a family $\{T(z) : z \in I_\alpha\}$ of operators in $L(X)$ is called an analytic semigroup in I_α if (i) $z \longrightarrow T(z)$ is analytic in I_α, (ii) $T(0) = I$ and $\lim_{z\to 0, z\in I_\alpha} T(z)x = x$ for every $x \in X$ and (iii) $T(z_1 + z_2) = T(z_1)T(z_2)$ for $z_1, z_2 \in I_\alpha$. The semigroup $\{T(z), z \in I_\alpha\}$ is called exponentially bounded if for each $0 < \beta < \alpha$, there exists constants $M = M_\beta$ and $\eta = \eta_\beta$ such that $\|T(z)\| \leqq Me^{\eta|z|}$ for all z in $\overline{I_\beta}$. It is called bounded if $\|T(z)\| \leqq M$ for all z in $\overline{I_\beta}$. As a corollary of Theorem 1, we get

THEOREM 2. *Let A be the infinitesimal generator of a bounded analytic semigroup in I_α of operators on a Banach space X. For each $n \geqq 2$, for each $x \in D(A^n)$, the domain of A^n, and for each $0 < \beta < \alpha$, there exists constants $M(\beta)$ and $S_{n,k}(\beta)$ such that*

$$\|A^k x\| \leqq M(\beta)S_{n,k}(\beta)\|x\|^{1-(k/n)}\|A^n x\|^{k/n}, \qquad (1 \leqq k \leqq n-1),$$

where $S_{n,k}(\beta)$ is defined in Theorem 1.

PROOF. Let $f(z) = T(z)x$, $x \in D(A^n)$, $z \in I_\alpha$. Let β be such that $0 < \beta < \alpha$. Since for all $z \in I_\beta$

$$\|f(z)\| \leqq M(\beta)\|x\|,$$

$$\|f^{(n)}(z)\| \leqq M(\beta)\|A^n x\|,$$

on applying Theorem 1, we get the result.

A semigroup $\{T(t), t \geqq 0\}$ is called analytic if it is analytic in some sector I_α containing the non-negative real axis. As a corollary, we have the following:

COROLLARY. *Let $\{T(t), t \geqq 0\}$ be a strongly continuous contraction semigroup and A be its infinitesimal generator. Suppose that $T(t)(X) \subset D(A)(t \geqq 0)$ and $t\|AT(t)\| \leqq \frac{1}{e\sin(\alpha\pi/2)}(t > 0)$ for some $\alpha : 0 < \alpha < 1$. Then for each $0 < \beta < \alpha$,*

$$\|A^k x\| \leqq \frac{\sin(\alpha\pi/2)}{\sin(\alpha\pi/2) - \sin(\beta\pi/2)} S_{n,k}(\beta)\|x\|^{1-(k/n)}\|A^n x\|^{k/n}, \qquad (1 \leqq k \leqq n-1).$$

PROOF. It is known (cf. e.g. [1], p. 16) that $\{T(t), t \geqq 0\}$ has an analytic extension in I_α. If $z \in \overline{I}_\beta\backslash\{0\}$, where $0 < \beta < \alpha$, writing $t = \operatorname{Re} z$, we see that

$$\frac{|z-t|}{t} \leqq \sin(\beta\pi/2).$$

Since

$$\|A^k T(t)\| \leqq \left\|AT\left(\frac{t}{k}\right)\right\|^k \leqq \frac{1}{(e\sin(\alpha\pi/2))^k}\left(\frac{k}{t}\right)^k.$$

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and

$$\|T(z)\| \leqq \|T(t)\| + \sum_{k=1}^{\infty} \frac{|z-t|^k}{k!}\|A^k T(t)\|$$

the assertion follows from Theorem 2.

The constants $S_{n,k}(\alpha)(0 < \alpha \leqq 1; k = 1, 2, \ldots, n-1)$ appearing in the inequalities of Theorems 1 and 2 are essentially sharper than the constants $D_{n,k}(k = 1, 2, \ldots, n-1)$ appearing in those of Theorems A and B.

As already remarked no explicit expression for the constants $D_{n,k}$ is known and this makes a precise comparison of the two constants difficult. However S. B. Stechkin (cf. [10]) has shown that

$$D_{n,k} \geqq \frac{k!}{(2k)!}\left\{\frac{(2n)!}{n!}\right\}^{k/n}.$$

Since the constants $C_{n,k}$ satisfy the inequalities (cf. [7]) $1 \leqq C_{n,k} \leqq \pi/2$ we have

$$\frac{D_{n,k}}{S_{n,k}(\alpha)} \geqq \frac{2}{\pi}\frac{k!\Gamma(\gamma k+1)}{(2k)!}\left\{\frac{(2n)!}{n!\Gamma(\gamma n+1)}\right\}^{k/n} = A_{n,k},$$

where, using the Stirling's formula, we see that

$$A_{n,k} \geqq \left(\frac{n}{k}\right)^{\alpha k} \quad \text{for } (k \geqq k_0(\alpha)).$$

Thus we conclude that the constants $S_{n,k}(\alpha)$ in norm inequalities for the infinitesimal generators of analytic semigroups can be smaller than the constants $D_{n,k}$ thereby answering in the affirmative the question raised by M. W. Certain and T. G. Kurtz already referred to in §1.

3. **Cosine Operator Functions.** Let X be a Banach space and let $c = \{c(t), t \in \mathbf{R}\}$ be a family of operators in $L(X)$ such that $c(0) = I$, $c(s+t) + c(s-t) = 2c(s)c(t)$ and the mapping $t \longrightarrow c(t)$ of $\mathbf{R}$ into $L(X)$ is strongly continuous. Such a family is called a strongly continuous cosine operator function on X (cf. e.g. [9]). The infinitesimal generator A of this family is defined by the rule

$$Ax = \lim_{t \to 0} 2t^{-2}(c(t)x - x),$$

where $x \in D(A)$, the set of all $x \in X$ for which this limit exits. We further suppose that $\sup_{t\in\mathbf{R}} \|c(t)\|_{L(X)} \leqq 1$.

S. Kurepa [8] proved an analogous of Kallman-Rota inequality for generators of cosine operator functions on a Banach space:

$$\|Ax\|^2 \leqq \frac{4}{3}\|x\| \, \|A^2 x\|, \quad x \in D(A^2).$$

We generalize this inequality as follows:

THEOREM 3. *If A is the infinitesimal generator of a strongly continuous cosine operator function $\{c(t), t \in \mathbf{R}\}$ such that $\sup_{t\in\mathbf{R}} \|c(t)\|_{L(X)} \leqq 1$, then for $x \in D(A^n)$ and $n \geqq 2$, we have*

$$\|A^k x\| \leqq C_{2n,2k}\|x\|^{1-(k/n)}\|A^n x\|^{k/n}, \qquad (1 \leqq k \leqq n-1),$$

where $C_{n,k}$ denote the constants in (1).

PROOF. We set $f(t) = c(t)x$; $x \in D(A^n)$. We have

$$f^{(2k)}(t) = c(t)A^k x$$

and

$$\|f\|_\infty \leqq \|x\|, \|f^{(2n)}\|_\infty \leqq \|A^n x\|.$$

Applying Theorem C, we obtain

$$\begin{aligned}\|A^k x\| &= \|f^{(2k)}(0)\| \leqq \|f^{(2k)}\|_\infty \\ &\leqq C_{2n,2k}\|f\|_\infty^{1-(k/n)}\|f^{(2k)}\|_\infty^{k/n} \\ &\leqq C_{2n,2k}\|x\|^{1-(k/n)}\|A^n x\|^{k/n}.\end{aligned}$$

We remark that Theorem 3 sharpens the result of S. Kurepa [8] since for $n = 2$ and $k = 1$, we get

$$\|Ax\|^2 \leqq \frac{6}{5}\|x\|\,\|A^2 x\|, x \in D(A^2).$$

It may be observed that the constants in Theorem 3 are the best possible since they are the best possible for $X = L^1(\mathbf{R})$ and $Af = f''$ (cf. [5]). This fact answers in negative the following question raised by Ditzian ([5], p. 148): Does there exist a Banach space for which the constant $(4/3)^{1/2}$ of Kurepa is the best possible?

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