GRAPHS ASSOCIATED WITH TRIANGULATIONS OF LATTICE POLYGONS

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(Received 23 March 1987; revised 17 June 1988)

Communicated by Louis Caccetta

Abstract

Two graphs, the edge crossing graph $E$ and the triangle graph $T$ are associated with a simple lattice polygon. The maximal independent sets of vertices of $E$ and $T$ correspond to the triangulations of the polygon into fundamental triangles. Properties of $E$ and $T$ are derived including a formula for the size of the maximal independent sets in $E$ and $T$. It is shown that $T$ is a factor graph of edge-disjoint 4-cycles, which gives corresponding geometric information, and is a partition graph as recently defined by the authors and F. Harary.


1. Introduction

If the values of a real valued function $f$ are known at a sequence of integral points $1,2,\ldots,n$ on the real line there is only one function which is affine on $[j,j+1], 1 \leq j \leq n - 1,$ and agrees with $f$ at each $j \in \{1,2,\ldots,n\}$. The two-dimensional analog is more interesting. Let $P$ be a simple polygon in the plane with vertices at lattice points and let a given real valued function $f$ defined on $P$ and its interior have values $y_{ij} = f((i,j))$ at lattice points $(i,j)$ inside and on the boundary of $P$. In general there will be many functions $\hat{f}$ which are piecewise affine approximations to $f$ in the following sense:

(i) $\hat{f}((i,j)) = f((i,j)) = y_{ij}$ for all lattice points $(i,j)$ inside and on the boundary of $P$, and
(ii) \( \hat{f} \) is affine on fundamental triangles (those with exactly three lattice points on the boundary at the vertices and no interior lattice points) in \( P \).

To emphasize the dependence of \( \hat{f} \) on the triangulation of \( P \), suppose the values \( y_{ij} \) are linearly independent in the vector space of real numbers over rational scalars. Let \( P \) be triangulated into fundamental triangles with node \((i,j)\) having valence \( a_{ij} \) in the graph which the triangulation induces. By Pick's Theorem [1] each of the triangles has area 1/2, so if \( \hat{f} \) satisfies (i) and (ii) we have

\[
\int_P \hat{f} = \frac{1}{6} \left( \sum_{(i,j) \text{ interior to } P} a_{ij} y_{ij} + \sum_{(i,j) \text{ on boundary of } P} (a_{ij} - 1) y_{ij} \right).
\]

Thus if \( \hat{f}_1 \) and \( \hat{f}_2 \) are determined by two triangulations of \( P \), \( \int_P \hat{f}_1 = \int_P \hat{f}_2 \) if and only if each lattice point has the same valence in the two triangulations. A natural question is how many triangulations are there for a simple lattice polygon \( P \)?

In the following work we relate the problem of determining all possible triangulations of \( P \) to the problem of determining all maximal independent sets of vertices in each of two related graphs. Properties of the graphs are discussed. One of the graphs is a special intersection graph and this naturally introduces partition graphs which have been studied in [2], [3], and [4].

Some related problems are known to be difficult. Given inputs of a graph \( G \) and arbitrary integer \( k \), determining whether or not \( G \) has an independent set with \( k \) or more vertices is \( NP \)-complete [5], and finding the number of maximal independent sets for an arbitrary graph is \#\( P \)-complete [8], so the results we give are likely more of theoretical rather than practical interest (except possibly in special instances). Also Gavril [6] has shown that determining whether or not a graph is the intersection graphs for a set of rectangles on an \( m \times n \) grid is \( NP \)-complete. The graphs \( T \) we consider below are intersection graphs for triangles inside lattice polygons.

We first formalize the terminology and introduce graphs \( E \) and \( T \). A segment joining two lattice points is fundamental if no other lattice point lies on the segment. A fundamental triangle is one which does not contain any lattice points in its interior and whose three sides are each a fundamental segment. A fundamental parallelogram is a parallelogram either of whose diagonals divides it into two fundamental triangles. A lattice polygon is one having all of its vertices at lattice points in the plane.

Let \( P \) be a simple lattice polygon, and suppose that all fundamental edges in \( P \) are drawn. The edge crossing graph \( E \) of \( P \) is constructed by letting each fundamental edge \( e' \) in \( P \) correspond to a vertex \( e \) in \( E \), with \( e_1 \) and \( e_2 \) adjacent if and only if the corresponding fundamental segments \( e'_1 \) and \( e'_2 \) in
$P$ intersect at a point interior to each segment. Figure 1 shows examples of edge crossing graphs for two different polygons. Note that all boundary edges correspond to isolated vertices in $E$; more generally $e$ is isolated in $E$ if and only if the corresponding segment $e'$ in $P$ is used in every triangulation of $P$. The examples show that $E$ need not be connected even after isolated vertices are removed.
Recall that a subset $M$ of the vertices of a graph is independent if no two are joined by an edge, and is maximal independent if it is not properly contained in a larger independent set of vertices. The following result is clear from the construction of the edge crossing graph, since each maximal independent set of vertices in $E$ determines a unique triangulation of $P$, and conversely.

**Theorem 1.** If $\tau$ is the set of all triangulations $P$ and $M_E$ is the set of all maximal independent sets in $E$, then there is a natural bijection $\tau \leftrightarrow M_E$.

Next we define the triangle graph $T$ associated with a given lattice polygon $P$. To each fundamental triangle $t'$ in $P$ corresponds a vertex $t$ in $T$, with $t_1$ and $t_2$ adjacent if and only if the corresponding triangles $t'_1$ and $t'_2$ share common interior points (again see Figure 1). Noting that each maximal independent set in $T$ corresponds to a unique triangulation of $P$ and conversely we have the following.

**Theorem 2.** If $\tau$ is the set of all triangulations $P$ and $M_T$ is the set of all maximal independent sets of vertices in $T$, then there is a natural bijection $\tau \leftrightarrow M_T$.

2. Properties of $E$ and $T$

Let $P$ have $b$ boundary lattice points and $i$ interior lattice points. Pick’s Theorem [1] states that the area $A$ of $P$ is given by $A = (b/2) + i - 1$.

**Theorem 3.** All maximal independent sets in $E$ have the same cardinality: if $M_E \in M_E$, then $|M_E| = 2b + 3i - 3$. All maximal independent sets in $T$ have the same cardinality: if $M_T \in M_T$, then $|M_T| = b + 2i - 2$.

**Proof.** By Pick’s Theorem we have $2A = b + 2i - 2 = |M_T|$ since each fundamental triangle has area 1/2. The expressions $3|M_T| + b = 2|M_E|$ count each edge twice so that $|M_E| = 3/2|M_T| + b/2 = 3/2(b + 2i - 2) + b/2 = 2b + 3i - 3$.

The examples of Figure 1 suggest that the edge crossing graph $E$ may in general be considerably simpler than the triangle graph $T$. That is in fact the case.

**Theorem 4.** There is a mapping $f$ which assigns to each edge of the graph $E$ a unique 4-cycle in $T$. Furthermore the collection of these 4-cycles forms a disjoint cover of all edges in $T$. 


Proof. The result depends, of course, on the geometry of the lattice structure inside $P$. We first construct a function $f$ which associated edges of the graph $E$ with well defined four cycles in $T$. It is then established that $f$ covers each edge of $T$ once and only once.

Let $e_1 e_2$ be an edge in $E$ indicating that fundamental segments $e'_1$ joining lattice points $A$ and $B$ and $e'_2$ joining $C$ and $D$ share a common interior point in $P$ (Figure 2). The convex hull of $e'_1 \cup e'_2$ is a quadrilateral we call the kite $k(e'_1, e'_2)$. From among the lattice points lying in or on triangle $ABC$ let $S$ be the one nearest but not on $e'_1$. Triangle $ABS$ has area $1/2$ since it is fundamental by the choice of $S$. Furthermore $S$ is uniquely determined in triangle $ABC$ since it must fall on the next line parallel to $e'_1$ which contains lattice points and the distance between successive lattice points on this parallel line is the same as the distance from $A$ to $B$.

Similarly lattice points $T, U$ and $V$ are uniquely determined in the kite so that triangles $ABT, CDU$ and $CDV$ are all fundamental. In fact $ASBT$ and $CUDV$ are fundamental parallelograms since vector $SA$ must equal vector $BT$ because of the way lattice points fall on the two parallel lines containing lattice points nearest to $AB$. Also these fundamental parallelograms must lie in the original polygon $P$. To justify this we note that $S$, for example, must
lie in $P$ because it falls on the line nearest $AB$ which contains lattice points, and segment $CD$ is given to lie in $P$.

Let $f$ assign to the pair $e'_1, e'_2$ the four cycle $f(e'_1, e'_2)$ in $T$ given by the intersection of triangles $ABS, CDU, ABT$ and $CDV$ taken in that order. The function $f$ is well defined by what precedes and it remains to be shown that each edge in $T$ is covered once and only once by $f$. To see that edges in $T$ are covered at most once, suppose the four cycle $f(e'_1, e'_2)$ contains the edge $t_1 t_2$ in $T$ generated by the intersection of triangles $t'_1$ and $t'_2$. Assume these triangles and edges are labeled as in Figure 2 where $t'_1 = \triangle ABS$ and $t'_2 = \triangle CDU$. If $f(e''_1, e''_2)$ also contains $t_1 t_2$ for a second pair of crossing fundamental edges $e''_1$ and $e''_2$, then $e''_1$ must be a side of $t'_1$ and $e''_2$ must be a side of $t'_2$ because of the way $f$ is defined. But if $e'_1$ is replaced by $e''_1$ or $e'_2$ is replaced by $e''_2$ then one of $A$ or $B$ is not an endpoint of $e''_1$ or one of $C$ or $D$ is not an endpoint of $e''_2$. Let us assume $A$ is not an endpoint of $e''_1$. Then the kite $k(e'_1,e'_2)$ does not contain $A$ and any triangle with $A$ as one of its vertices, in particular $t'_1$, cannot correspond to an endpoint of edges in $f(e''_1, e''_2)$.

Figure 3. The four types of intersections of pairs of fundamental triangles

Finally we must observe that any edge $t_1 t_2$ in $T$ is part of a four cycle $f(e'_1, e'_2)$. This requires that we identify the proper intersecting sides $e'_1$ of $t'_1$ and $e'_2$ of $t'_2$ so that both $t'_1$ and $t'_2$ lie in $k(e'_1, e'_2)$. For a pair $t'_1, t'_2$ there
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may be one, two, three, or four possibilities for the pair \( e'_1, e'_2 \) as illustrated in Figure 3. In every case, any two fundamental triangles sharing interior points lie in a unique kite determined by two of their intersecting sides, \( e'_1 \) and \( e'_2 \), making edge \( t_1t_2 \) lie on the four cycle \( f(e'_1, e'_2) \). That the convex hull of two intersecting fundamental triangles is a kite determined by two intersecting edges can be shown directly, or by observing that those two triangles are affine images of a variation of those shown in Figure 3.

**Corollary 1.** The graph \( T \) is a factor graph of edge-disjoint 4-cycles and isolated vertices.

**Corollary 2.** For a given simple lattice polygon, \( T \) has four times as many edges as \( E \).

**Corollary 3.** If \( P \) is a lattice polygon in the plane and all fundamental triangles in \( P \) are considered, the number of pairs of such triangles sharing interior points is a multiple of 4.

The graph \( T \) is an example of a partition graph as recently investigated in [2], [3] and [4]. A graph \( G \) is a *partition graph* if to each vertex \( v \) of \( G \) there can be assigned a set \( S_v \neq \emptyset \) so that the following properties hold:

(i) distinct vertices \( u, v \) of \( G \) are assigned distinct sets \( S_u, S_v \);
(ii) \( uv \) is an edge of \( G \) if and only if \( S_u \cap S_v \neq \emptyset \);
(iii) every maximal independent set of vertices, \( M \), of \( G \) gives a partition of \( S = \bigcup S_u \); that is, \( S = \bigcup_{w \in M} S_w \) [\( \bigcup \) denotes disjoint union].

Properties (i) and (ii) mean \( G \) is an intersection graph [7], and so a partition graph is a special type of intersection graph.

**Theorem 5.** The triangle graph \( T \) for a polygon is a partition graph.

**Proof.** The system of all fundamental edges in \( P \) divides \( P \) into disjoint regions \( x_i \). If \( t \in T \), let \( S_t = \{x_i : x_i \subset t'\} \), where \( t' \) is the corresponding fundamental triangle inside \( P \). It is clear the graph \( T \) is a partition graph for the family \( \{S_t : t \in T\} \).

Theorems 3 and 5 and Corollary 1 give three properties of triangle graphs \( T \). Conversely one can ask: if the graph \( G \) is a partition graph and a factor graph of edge-disjoint four-cycles and isolated vertices with all maximal independent sets of the same cardinality, is it the triangle graph for some lattice polygon \( P \)? It is interesting to note there are no such graphs with 5 or 6 vertices, so that after \( C_4 \) the next graph with the three properties is the one shown in Figure 1.
References


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