Y. Hashimoto and T. Matsuzawa
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ON A CLASS OF DEGENERATE ELLIPTIC EQUATIONS

YOSHIAKI HASHIMOTO AND TADATO MATSUZAWA

We shall prove in Chapter I the hypoellipticity[1] for a class of degenerate elliptic operators of higher order. Chapter II will be devoted to the consideration of the regularity at the boundary for the solutions of general boundary problems for the equations considered in Chapter I being restricted to the second order.

Chapter I. Hypoellipticity for a class of degenerate elliptic operators.

§ 1. Introduction.

In [5], Grušin has proved the hypoellipticity for a class of degenerate elliptic equations. Our aim in this chapter is to give a simple proof with some additional assumptions on the operators considered in [5].

First we state the main result obtained in [5]. Let $\boldsymbol{R}^N$ be N-dimensional Euclidean space regarded as a direct product of two Euclidean spaces $\boldsymbol{R}^k$ and $\boldsymbol{R}^n$ $(k + n = N)$. We consider a pair (ρ, σ) of N rational numbers $(\rho_1, \cdots, \rho_N)$, $(\sigma_1, \cdots, \sigma_N)$ such that $\rho_j \geqq 1$ and $\sigma_j \geqq 0$ $(1 \leqq j \leqq N)$ and that

$$\text{(a)} \qquad \rho_j = \sigma_j = 1 \qquad k + 1 \leqq j \leqq k + n = N$$

and for each $j, 1 \leqq j \leqq k$, one of the following conditions is satisfied:

$$\text{(b)} \qquad \rho_j > \sigma_j > 0 \, ,$$

$$\text{(c)} \qquad \sigma_j = 0 \, .$$

Suppose (ρ, σ) is given. The following notations are convenient for the later discussions:

$$x = (x_1, \cdots, x_N) \in \boldsymbol{R}_N \, ,$$

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[1] A differential operator P is said to be hypoelliptic, if any distribution u is infinitely differentiable in every open set where Pu is infinitely differentiable.

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$$x=(x',y),\ x'=(x_1,\cdot\cdot,x_k),\ y_j=x_{k+j}\qquad 1\leqq j\leqq n\,,$$
$$x'=(x'',x'''),\ x''=(x_1,\cdots,x_{k'}),\ x'''=(x_{k'+1},\cdots,x_k)$$

where k' is in agreement with the number of j satisfying (b).

Let m be a positive integer and set

$$\mathfrak{M}=\{(\gamma,\alpha);|\alpha|\leqq m,\langle\rho,\alpha\rangle\geqq\langle\sigma,\gamma\rangle\geqq\langle\rho,\alpha\rangle-m\}\,, \tag{1.1}$$

$$\mathfrak{M}_0=\{(\gamma,\alpha);|\alpha|\leqq m,\langle\sigma,\gamma\rangle=\langle\rho,\alpha\rangle-m\}\,, \tag{1.2}$$

where (γ,α) is a pair of N-tupled multi-indices of non-negative integers and $\gamma_j=0$ for j if $\sigma_j=0$ $(1\leqq j\leqq k)$. We use the following notations:

$$|\alpha|=\alpha_1+\cdots+\alpha_N\,,$$
$$\langle\rho,\alpha\rangle=\rho_1\alpha_1+\cdots+\rho_N\alpha_N\quad\text{etc}\,.$$

Now we consider a partial differential operator

$$L(x,D)=\sum_{(\gamma,\alpha)\in\mathfrak{M}}a_{\alpha\gamma}(x)x^\gamma D^\alpha\,, \tag{1.3}$$

where $a_{\alpha\gamma}(x)\in C^\infty(\boldsymbol{R}^N)$, $(\alpha,\gamma)\in\mathfrak{M}$ and

$$D=\left(\frac{1}{i}\frac{\partial}{\partial x_1},\cdots,\frac{1}{i}\frac{\partial}{\partial x_N}\right).$$

Associated with (1.3) we consider the partial differential operator with polynomial coefficients:

$$L_0(x'',y\,;D)=\sum_{(\gamma,\alpha)\in\mathfrak{M}_0}a_{\alpha\gamma}(0)x^\gamma D^\alpha\,. \tag{1.4}$$

Condition 1. $L_0(x'',y\,;D)$ is elliptic for $|x''|+|y|=1$.

Condition 2. The differential equation

$$\hat{L}_0(x'',y\,;\xi,D_y)v(y)=\sum_{\substack{(\gamma,\alpha)\in\mathfrak{M}_0\\ \alpha=(\alpha',\beta)}}a_{\alpha\gamma}(0)x^\gamma\xi^{\alpha'}D_y^\beta v(y)=0 \tag{1.5}$$

has no non-trivial solution in $\mathscr{S}(\boldsymbol{R}_y^n)$ for any fixed $\xi\in\boldsymbol{R}^k$, $\xi\neq0$, and x''.

Set

$$\rho_0=\min_{1\leqq j\leqq k}\rho_j\,,\qquad \sigma^0=\max_{1\leqq j\leqq k}\sigma_j\,.$$

Having established these notations the main result of Grušin can be stated as follows:

THEOREM 1.1 ([5, Theorem 1.1]). *Assume* $\rho_0>\sigma^0$. *Under the con-*

ditions 1 *and* 2, *the operator* $L(x, D)$ *is hypoelliptic in a neighborhood of the origin.*

With the additional assumptions on $L(x, D)$ stated below, we shall give an alternative proof of this theorem. Throughout this paper we always assume that the order of $L(x, D)$ is $2m$ ($m \geqq 1$). Moreover we substitute the conditions 1 and 2 by the following conditions 1′ and 2′ respectively:

Condition 1′. $L_0(x'', y; D)$ is strongly elliptic for $|x''| + |y| = 1$ i.e. there exists a positive constant δ such that

$$\text{Re}\, L_0^0(x'', y; \xi, \eta) \geqq \delta(|\xi|^2 + |\eta|^2)^{2m} \tag{1.6}$$

for all (x'', y) ($|x''| + |y| = 1$), $\xi \in \boldsymbol{R}^k$ and $\eta \in \boldsymbol{R}^n$. Here L_0^0 denotes the homogeneous part of L_0 of order $2m$.

Condition 2′. The differential equation

$$\hat{L}_0(x'', y; \xi, D_y)v(y) = 0 \tag{1.5′}$$

has no non-trivial solution in $L^2(\boldsymbol{R}_y^n)$ for all x'' and ξ ($|\xi| = 1$).

THEOREM 1.1′. *Assume that* $\rho_0 > \sigma^0$. *Under the conditions* 1′ *and* 2′, *the operator* $L(x, D)$ *is hypoelliptic in a neighborhood of the origin of* $\boldsymbol{R}^N$.

Remark. Grušin showed in [3] that Condition 2 is equivalent to Condition 2′ under Condition 1.

In order to show the hypoellipticity of the operator $L(x, D)$ it is essential to obtain the inequality of the type (2.13) as will be explained in § 2. In § 3, we shall get this inequality with the aid of the method suggested by that used in [13].

§ 2. The proof of Theorem 1.1′.

In this section, according to [5, § 2], we shall describe in several steps how the proof of Theorem 1.1′ (or Theorem 1.1) is deduced to prove the inequality (2.13).

(A) Let H_1 and H_2 be two Hilbertian spaces and we denote by $\mathscr{L}(H_1, H_2)$ the set of all continuous linear mappings from H_1 into H_2. $\mathscr{L}(H_1, H_2)$ is a Banach space with the usual norm (denoted by $\|\cdot\|$). Let $p(x, \xi)$ be $\mathscr{L}(H_1, H_2)$-valued infinitely differentiable function defined in $\Omega \times \boldsymbol{R}_\xi^k$, where Ω is an open set in $\boldsymbol{R}_x^k$. We suppose that for every com-

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pact set $K \subset \Omega$ and for all multi-indices α, β, the following inequality holds:

$$\|p_{(\beta)}^{(\alpha)}(x,\xi)\| \leqq C_{\alpha,\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|} \qquad x \in K,\ \xi \in \boldsymbol{R}^k, \tag{2.1}$$

where ρ and δ are non-negative numbers and

$$p_{(\beta)}^{(\alpha)}(x,\xi) = \frac{\partial^{\alpha+\beta}}{\partial x^\beta \partial \xi^\alpha} p(x,\xi) .$$

We denote by $S_{\rho,\delta}^m = S_{\rho,\delta}^m(\Omega \times \boldsymbol{R}_\xi^k; H_1, H_2)$ the set of all $p(x,\xi)$ satisfying the above conditions. We denote by $\mathscr{D}(\Omega; H_1)$ the set of H_1-valued infinitely differentiable functions with compact support in an open set $\Omega \subset \boldsymbol{R}_x^k$. Then we define a pseudo-differential operator:

$$p(x,D)u = (2\pi)^{-k} \int_{\boldsymbol{R}_\xi^k} p(x,\xi)\hat{u}(\xi) e^{i\langle x,\xi\rangle} d\xi , \tag{2.2}$$

where $\hat{u}(\xi)$ is the Fourier transform of $u \in \mathscr{D}(\Omega; H_1)$. For this pseudo-differential operator almost all theorems proved in [8] in the scaler case are true.

Now consider a function $\mu(\xi)$ infinitely differentiable except at $\xi = 0$ such that $\mathrm{grad}_\xi\, \mu(\xi) \to 0$ as $\xi \to \infty$ and that the following inequality holds:

$$C_1(1+|\xi|)^{\varepsilon_1} \leqq \mu(\xi) \leqq C_2(1+|\xi|)^{\varepsilon_2} , \qquad \xi \in \boldsymbol{R}^k , \tag{2.3}$$

where $0 < \varepsilon_1 \leqq \varepsilon_2 \leqq 1$ and $C_1, C_2 > 0$.

THEOREM 2.1 (cf. [5, Theorem 1.1]; [6], [8]). *Let $p(x,\xi)$ be a symbol in $S_{\rho,\delta}^m$ with $0 \leqq \delta < \varepsilon_1 \leqq \varepsilon_2 < \rho$. Suppose that there exist positive constants d, A and a real number a such that for all $x \in \Omega, \alpha, \beta$ and $\xi, |\xi| > A$, the following inequalities are valid:*

$$|p_{(\beta)}^{(\alpha)}(x,\xi)v|_{H_2} \leqq C_{\alpha,\beta}\mu(\xi)^{-|\alpha|+|\beta|}|\xi|^{-d|\alpha+\beta|}|p(x,\xi)v|_{H_2} \qquad v \in H_1 , \tag{2.4}$$

$$|\xi|^a |v|_{H_1} \leqq C|p(x,\xi)v|_{H_2} \qquad v \in H_1 . \tag{2.5}$$

Then the pseudo-differential operator $p(x,D)$ is hypoelliptic in Ω.

In what follows we shall use the symbols $C, A, \mu, \delta, \cdots$ to denote constants, and suffix or prime will also be used if necessary.

(B) Let Ω' be an open neighborhood of the origin in $\boldsymbol{R}_{x'}^k$ and let $B_\mu = \{y \in \boldsymbol{R}_y^n; |y| < \mu\}$. For every $x' \in \Omega'$ and $\xi \in \boldsymbol{R}_\xi^k$ we have

$$\hat{L}(x', y, \xi, D_y) \in \mathscr{L}(\mathring{H}^{2m}(B_\mu), L^2(B_\mu)) .$$

Hence if we put

(2.6) $$p(x',\xi)=\hat{L}(x',y;\xi,D_y)\ ,$$

then $p(x',\xi)\in S^{2m}_{1,0}(\Omega'\times \boldsymbol{R}^k_\xi;\mathring{H}^{2m}(B_\mu),L^2(B_\mu))$.

If the hypoellipticity of the pseudo-differential operator $p(x',D_{x'})$ in Ω' is proved, then we can derive the hypoellipticity of $L(x,D)$ in $\Omega\times B_\mu$ as follows:

LEMMA 2.1. *Assume that Condition* 1′ *is satisfied and* $p(x',D_{x'})$ *is hypoelliptic in* Ω'. *Then the differential operator* $L(x,D)=L(x',y,D_{x'},D_y)$ *is hypoelliptic in a neighborhood of the origin of* $\boldsymbol{R}^k_{x'}\times\boldsymbol{R}^n_y=\boldsymbol{R}^N$.

Proof. By Condition 1′ L is elliptic in $\Omega'\times B_\mu, y\neq 0$ (letting the sets Ω' and B_μ shrink if necessary), therefore L is hypoelliptic in $\Omega'\times B_\mu$, $y\neq 0$. For any vector $(0,\cdots,0,\eta_1,\cdots,\eta_n)=(0,\eta)\neq 0$, we can easily see the characteristic polynomial of L_0 at $y=0$ does not vanish by Condition 1′:

$$\sum_{\substack{|\alpha|=2m\\ \alpha=(0,\alpha_1,\cdots,\alpha_n)}} a_{\alpha,0}(0)\eta^\alpha\neq 0\ .$$

Thus L is partially hypoelliptic with respect to the plane $y=0$, whence we have the conclusion of Lemma 2.1. (cf. [7, Chap. 4].)

(C) We introduce several notations. First we set

(2.7) $$\theta=\langle\sigma,\gamma\rangle+2m-\langle\rho,\alpha\rangle\ .$$

Then $\theta=0$ if and only if $(\alpha,\gamma)\in\mathfrak{M}_0$. (In the definition of $\mathfrak{M}$: (1.1) and $\mathfrak{M}_0$: (1.2), the number m must be substituted by $2m$.) We set

$$x'=(x_1,\cdots,x_k)\ ,\qquad y=(x_{k+1},\cdots,x_N)=(y_1,\cdots,y_n)\ .$$
$$|x'|_{\sigma'}=|x_1|^{1/\sigma_1}+\cdots+|x_k|^{1/\sigma_k}\ ,$$
$$|x|_\sigma=|x'|_{\sigma'}+|y_1|+\cdots+|y_n|\ ,$$
$$|\xi|_\rho=|\xi_1|^{1/\rho_1}+\cdots+|\xi_k|^{1/\rho_k}\ ,$$

(2.8) $$h(x'',y;\xi)=|x|_\sigma^{\rho_1-1}|\xi_1|+\cdots+|x|_\sigma^{\rho_k-1}|\xi_k|$$

and

$$\rho_0=\min_{1\leq j\leq k}\rho_j\ ,\quad \rho^0=\max_{1\leq j\leq k}\rho_j\ ,\quad \sigma_0=\min_{1\leq j\leq k}\sigma_j\ ,\quad \sigma^0=\max_{1\leq j\leq k}\sigma_j\ .$$

Here $|x|_\sigma$ and $|x'|_{\sigma'}$ include only the terms corresponding to $\sigma_j\neq 0$.

LEMMA 2.2 ([5, Lemma 3.1]). *Let* $\alpha=(\alpha',\beta)=(\alpha_1,\cdots,\alpha_k,\beta_1,\cdots,\beta_n)$ *and assume that*

$$|\alpha| + \frac{\theta}{\rho_0} \leqq 2m\ . \tag{2.9}$$

Then we have the following inequality:

$$|x^\gamma \xi^{\alpha'}|\,|\xi|_\rho^\theta \leqq C(|\xi|_\rho + h(x'', y\,;\xi))^{2m-|\beta|}\ . \tag{2.10}$$

LEMMA 2.3 ([5, Lemma 3.2]). *Assume that*

$$|\alpha| + \frac{\theta}{\rho_0} \geqq 2m\ . \tag{2.11}$$

Then for arbitrary $\mu > 0$ there exists a constant $C > 0$ such that

$$|x^\gamma \xi^{\alpha'}|\,|\xi|_\rho^{(2m-|\alpha|)\rho_0} \leqq C(|\xi|_\rho + h(x'', y\,;\xi))^{2m-|\beta|}\ , \qquad |\xi| \geqq 1\ , \quad y \in B_\mu = \{y\,|\,|y| \leqq \mu\}\ . \tag{2.12}$$

(D) In § 3 we shall prove the following lemma.

LEMMA 2.4 (cf. [5, Lemma 3.5]). *Assume that the conditions in Theorem 1.1′ are satisfied. Then there exist positive constants A, C and μ and a neighborhood Ω' of the origin in $\boldsymbol{R}^k$ such that*

$$\begin{aligned}\sum_{|\beta|\leqq 2m} \|(|\xi|_\rho + h(x'', y, \xi))^{2m-|\beta|} D_y^\beta v(y)\|^2_{L^2(\boldsymbol{R}^n)} \\ \leqq C\,\|\hat{L}(x', y\,;\xi, D_y)v(y)\|^2_{L^2(\boldsymbol{R}^n)}\end{aligned} \tag{2.13}$$

for all $x' \in \Omega'$, $v(y) \in \mathring{H}^{2m}(B_\mu)$ and $\xi, |\xi|_\rho \geqq A$.

Now by virtue of the inequality (2.13) we can see that the operator $p(x', \xi') = \hat{L}(x', y\,;\xi, D_y)$: (2.6) satisfies the conditions of Theorem 2.1. In fact, we shall estimate $p^{(\alpha_1)}_{(\beta_1)}(x', \xi)v(y)$, where $v(y) \in \mathring{H}^{2m}(B_\mu)$ and α_1, β_1 are multi-indices in N^k. Since $p(x', \xi)$ is expressed as a sum of terms $a_{\alpha\gamma}(x)\xi^{\alpha'}D_y^\beta$, $p^{(\alpha_1)}_{(\beta_1)}$ is a sum of terms

$$\left(\frac{\partial}{\partial x'}\right)^{\beta_1}(a_{\alpha\gamma}(x)x^\gamma)\left(\frac{\partial}{\partial \xi}\right)^{\alpha_1}(\xi^{\alpha'})D_y^\beta v(y)\ . \tag{2.14}$$

Considering $\langle\sigma, \gamma\rangle - \langle\rho, \alpha\rangle + 2m \geqq 0$, the number θ corresponding to the term (2.14) is not smaller than $\langle\rho', \alpha_1\rangle - \langle\sigma^1, \beta_1\rangle$, where $\rho' = (\rho_1, \cdots, \rho_k)$ and $\sigma' = \cdots, \sigma_k)$. Then by Lemma 2.2 and Lemma 2.3 L^2-norm of (2.14) is estimated by

$$C\,|\xi|_\rho^{-\langle\rho', \alpha_1\rangle + \langle\sigma', \beta_1\rangle}\,\|(|\xi|_\rho + h)^{2m-|\beta|}D_y^\beta v(y)\|_{L^2(\beta_\mu)} \qquad |\xi|_\rho \geqq A\ .$$

Hence we have by (2.13)

$$\|(2.14)\|_{L^2(B_\mu)} \leqq C|\xi|_\rho^{-\rho_0|\alpha_1|+\alpha^0|\beta_1|}\|\hat{L}(x', y; \xi, D_y)v(y)\|_{L^2(B_\mu)}, \qquad |\xi|_\rho \geqq A.$$

From this we can easily see that $p(x', \xi)$ satisfies the condition of Theorem 2.1 by taking

$$\sigma^0 < \varepsilon < \rho_0,$$
$$d = \min(\rho_0 - \varepsilon, \varepsilon - \sigma_0)$$

and

$$\mu(\xi) = [(1 + \xi_1^2)^{1/\rho_1} + \cdots + (1 + \xi_k^2)^{1/\rho_k}]^\varepsilon.$$

Thus to complete the proof of Theorem 1.1′, it remains to prove Lemma 2.4.

§ 3. Proof of Lemma 2.4: Main estimate.

Lemma 2.4 will be proved as a consequence of the following lemma.

LEMMA 3.1. *Let $L_0(x'', y, D)$ be given as in Theorem 1.1′. Then there exists a positive constant C such that*

$$\sum_{|\beta|\leqq 2m}\|(|\xi|_\rho + h(x'', y; \xi))^{2m-|\beta|}D_y^\beta v(y)\|^2_{L^2(R^n)} \leqq C\|\hat{L}_0(x'', y; \xi, D_y)v(y)\|^2_{L^2(R^n)} \tag{3.1}$$

for all $v(y) \in H^{2m}(R^n) \cap \mathscr{E}'(R^n)$ and $\xi \in R^k$.

Proof. In (3.1) substitute ξ by $\lambda^{\rho'}\xi$, x by $\lambda^{-\sigma}x$ and D_y by λD_y respectively ($\lambda > 0$). Then we see that quasi-homogeneous order of both sides and is $2m$. Hence it is sufficient to show the inequality (3.1) for $|\xi|_\rho = 1$. Let $t(r)$ be an infinitely differentiable function in $r \geqq 0$ such that

$$t(r) = \begin{cases} 0 & 0 \leqq r \leqq 1 \\ r & r \geqq 2. \end{cases}$$

Define

$$h_1(x'', y; \xi) = (|x'|_{\sigma'} + t(|y|))^{\rho_1 - 1}|\xi_1| + \cdots + (|x'|_{\sigma'} + t(|y|))^{\rho_k - 1}|\xi_k|. \tag{3.2}$$

Then h_1 has the same order as h in $|y|$ for large $|y|$ and is constant when x'', ξ are fixed and y runs through the sphere in R_y^n. Furtheremore for $j \geqq 1$,

$$(3.3)\qquad \begin{aligned} &\left|\frac{\partial^j h_1}{\partial r^j}\right| h_1^{-j} && \text{is bounded ,}\\ &\left|\frac{\partial^j h_1}{\partial r^j}\right| h_1^{-j-1} \to 0 && \text{as } |x''| + |y| \to \infty \,. \end{aligned}$$

Substituting h by h_1 in (3.1) we obtain the inequality equivalent to (3.1).

Put

$$g(x'', r; \xi) = \int_0^r (1 + h_1(x'', r; \xi))dr \,,$$

which has the same order as rh_1 as $r \to \infty$.

Now we shall use the coordinate transformation introduced in [5], [16]:

$$(3.4)\qquad z = \frac{y}{|y|} g(x'', |y|; \xi) \,.$$

Then we have

$$(3.5)\qquad \frac{\partial z_j}{\partial y_i} = \frac{g(x'', y; \xi)}{|y|}\delta_{ij} + \left(y_i y_j - \frac{g(x'', |y|; \xi)}{|h|^3} + \frac{1 + h_1(x'', |y|; \xi)}{|y|^2}\right),$$

and the Jacobian of the transformation is

$$(3.6)\qquad \frac{\partial(z_1 \cdots z_n)}{\partial(y_1, \cdots, y_n)} = \frac{g(x'', |y|; \xi)^{n-1}}{|y|^{n-1}}(1 + h_1(x'', |y|; \xi)) \,.$$

We now pass from the variables y to the new variables z in the equation

$$(3.7)\qquad \hat{L}_0(x'', y; \xi, D_y)v(y) = f(y) \,.$$

It will have the form:

$$(3.8)\qquad M(z, D_z)v_1(z) = f_1(z) \,,$$

where

$$(3.9)\qquad \begin{cases} v_1 = (1 + h_1)^{m-n/2} v \,, \\ f_1 = (1 + h_1)^{-m-n/2} f \,. \end{cases}$$

In the following we shall show the equation (3.8) is uniformly elliptic in $\boldsymbol{R}_z^n$ and all the coefficients belong to $\mathscr{B}(\boldsymbol{R}_z^n)$. From Condition 1′ it follows that for the principal part of L_0 (denoted by L_0^0):

$$(3.10)\qquad \operatorname{Re} L_0^0(x'', y; \xi, \eta) \geqq \delta(|\eta|^{2m} + h(x'', y; \xi)^{2m}) \,.$$

Considering

$$M(z, D_z)v_1 = (1 + h_1)^{-m-n/2}L_0(1 + h_1)^{-m+n/2}v_1 = f_1$$

we have for the principal part of M (denoted by M^0):

$$\text{Re}\, M^0(z, \zeta) \geqq \delta\Big(\sum_{i=1}^{n}\Big|\sum_{j=1}^{n}(1 + h_1)^{-1}\frac{\partial z_i}{\partial y_j}\zeta_j\Big|^2\Big)^m \tag{3.11}$$

Furthermore by (3.5) and (3.6) we have

$$\Big|\frac{\partial z_j}{\partial y_i}(1 + h_1)^{-1}\Big| \leqq C\,, \qquad z \in \boldsymbol{R}_z^n\,,$$

$$C'^{-1} \leqq (1 + h_1)^{-n}\frac{\partial(z_1 \cdots z_n)}{\partial(y_1, \cdots, y_n)} \leqq C'$$

for some positive constants C and C'. Hence we have

$$\text{Re}\, M^0(z, \zeta) \geqq \delta' |\zeta|^{2m} \qquad \zeta \in \boldsymbol{R}_\zeta^n \tag{3.12}$$

for some constant $\delta' > 0$. Hence $M(z, D_z)$ is uniformly strongly elliptic in $\boldsymbol{R}_z^n$. We remark that $M(z, D_z)$ depends on (x'', ξ) which is viewed as a parameter.

By the above consideration all the coefficients of $2m$-th order in $M(z, D_z)$ are bounded in $\boldsymbol{R}_z^n$. The first derivatives of them are given in the form:

$$\begin{aligned}&(1 + h_1)^{-2m-1}\frac{\partial h_1}{\partial z_k}a_\alpha(0)\sum C_{\alpha, i_1 \cdots i_{2m}, j_1 \cdots j_{2m}}\prod_{\nu=1}^{2m}\Big(\frac{\partial z_{j_\nu}}{\partial y_{i_\nu}}\Big)\\ &\quad + (1 + h_1)^{-2m}a_\alpha(0)\sum C_{\alpha, i_1 \cdots i_{2m}, j_1 \cdots j_{2m}}\prod_{\nu=1}^{2m}\Big(\frac{\partial z_{j_\nu}}{\partial y_{i_\nu}}\Big)\frac{\partial}{\partial z_k}\Big(\frac{\partial z_{j_\nu}}{\partial y_{i_\nu}}\Big)\,,\end{aligned} \tag{3.13}$$

where the summations are taken for $i_1, \cdots, i_{2m}, j_1, \cdots, j_{2m}$ which run through $1, \cdots, n$. By using the following inequalities derived by (3.5) and (3.6):

$$(1 + h_1)\Big|\frac{\partial h_1}{\partial z_k}\Big| \leqq C(1 + h_1)^{-2}\Big|\frac{\partial h_1}{\partial r}\Big|\,,$$

$$\Big|\frac{\partial z_j}{\partial y_j}\Big| \leqq C'(1 + h_1)\,,$$

and by using (3.3) we can finally see that (3.13) is bounded in $\boldsymbol{R}_z^n$. The boundedness of the higher derivatives are shown as above. Similarly, we can show recurcively that all the coefficients of $2m - j$-th order ($j =$

$1, 2, \cdots, 2m$) belong to $\mathscr{B}(R_z^n)$ using (2.10), (3.3), (3.4), (3.5), (3.6) and (3.12). Hence the Gårding inequality for $M(z, D_z)$ can be derived; namely there exist two constants $C_1 > 0$ and $C_2 \geqq 0$ such that

$$\operatorname{Re}(M(z, D_z)v_1(z), v_1(z)) \geqq C_1 \|v_1(z)\|^2_{H^m(R_z^n)} - C_2 \|v_1(z)\|_{L^2(R_z^n)} \qquad v_1 \in H^m(R_z^n) \cap \mathscr{E}'(R_z^n) \text{ (cf. [14], p. 240)}. \tag{3.14}$$

Here the constants C_1 and C_2 are independent of (x'', ξ), $|\xi|_\rho = 1$.

We can substitute v_1 in (3.14) by $(1 + h_1)^m v_1(z)$:

$$\operatorname{Re}(M(z, D_z)(1 + h_1)^m v_1(z), (1 + h_1)^m v_1(z)) \geqq C_1 \|(1 + h_1)^m v_1(z)\|^2_{H^m(R^n)} - C_2 \|(1 + h_1)^m v_1(z)\|^2_{L^2(R^n)} . \tag{3.15}$$

Using Leibniz' formula and integration by parts, the left-hand side of (3.15) is estimated from above by

$$C\{\|(1 + h_1)^m M(z, D_z)v_1\| \cdot \|(1 + h_1)^m v_1\| + \|(1 + h_1)^m v_1\|_{H^m} \cdot \|(1 + h_1)^m v_1\|_{H^{m-1}}\} . \tag{3.16}$$

By (3.15), (3.16) and by interpolating the inequality of the form

$$\|(1 + h_1)^m v_1\|_{H^{m-1}} \leqq \varepsilon \|(1 + h_1)^m v_1\|_{H^m} + C(\varepsilon) \|(1 + h_1)^m v_1\|_{L^2} , \tag{3.17}$$

we have

$$\|(1 + h_1)^m v_1\|_{H^m} \leqq C(\|(1 + h_1)^m M(z, D_z)v_1\|_{L^2} + \|(1 + h_1)^m v_1\|_{L^2}) \tag{3.18}$$

for some constant $C > 0$.

Now starting from (3.18) and (3.14), we can show, by induction in $|\beta|$, that the following inequality is valid:

$$\sum_{|\beta| \leqq m} \|(1 + h_1)^m D_z^\beta v_1\|_{H^m} \leqq C'\{\|(1 + h_1)^m M(z, D_z)v_1\|_{L^2} + \|(1 + h_1)^m v_1\|_{L^2}\} . \tag{3.19}$$

Going over to the variables y we have

$$\sum_{|\beta| \leqq 2m} \|(1 + h_1)^{2m-|\beta|} D_y^\beta v(y)\|_{L^2(R^n)} \leqq C''\{\|\hat{L}_0(x'', y; \xi, D_y)v\|_{L^2} + \|(1 + h_1)^{2m} v\|_{L^2}\} \qquad v \in H^{2m}(R^n) \cap \varepsilon'(R^n), |\xi|_\rho = 1 . \tag{3.20}$$

The second term of the right hand side of (3.20) can be dropped out by Condition 2′, which completes the proof of Lemma 3.1.

Proof of Lemma 2.4. The differential operator $\hat{L}(x', y; D_y)$ is expressed as a sum of the terms $a_{\alpha\gamma}(x)x^\gamma\xi^{\alpha'}D_y^\beta$, $(\gamma,\alpha)\in\mathfrak{M}$, $\alpha=(\alpha',\beta)$. For $v\in\mathring{H}^{2m}(B_\mu)$ we have

$$\begin{aligned}\hat{L}v &= \hat{L}_0v+(\hat{L}-\hat{L}_0)v\\ &= \hat{L}_0v+\sum_{\theta=0}(a_{\alpha\gamma}(x)-a_{\alpha\gamma}(0))x^\gamma\xi^{\alpha'}D_y^\beta v+\sum_{\theta>0}a_{\alpha\gamma}(x)x^\gamma\xi^{\alpha'}D_y^\beta v\,.\end{aligned}\tag{3.21}$$

For any $\varepsilon>0$, if we take $\mu>0$ and if the diameter of $\Omega'\subset \boldsymbol{R}_x^k$ is sufficiently small, we have

$$\begin{aligned}&\sum_{\theta=0}\|(a_{\alpha\gamma}(x)-a_{\alpha\gamma}(0))x^\alpha\xi^{\alpha'}D_y^\beta v(y)\|_{L^2(B_\mu)}\\ &\leqq \varepsilon\sum_{|\beta|\leqq 2m}\|(|\xi|_\rho+h(x'',y;\xi))^{2m-|\beta|}D_y^\beta v(y)\|_{L^2(B_\mu)}\,,\qquad v\in\mathring{H}^{2m}(B_\mu)\,.\end{aligned}$$

For the last sum in the right hand side of (3.21) we have by Lemma 2.2 and Lemma 2.3,

$$\begin{aligned}&\|a_{\alpha\gamma}(x)x^\gamma\xi^{\alpha'}D_y^\beta v\|_{L^2(B_\mu)}\\ &\leqq C_{\alpha,\gamma,\mu}|\xi|_\rho^{-\tau}\|(|\xi|_\rho+h(x'',y,\xi))^{2m-|\beta|}D_y^\beta v(y)\|_{L^2(B_\mu)}\,,\qquad \tau>0\,.\end{aligned}$$

If we take a sufficiently large number A, then we have

$$\begin{aligned}&\sum_{\theta=0}\|a_{\alpha\gamma}(x)x^\gamma\xi^{\alpha'}D_y^\beta v(y)\|_{L^2(B_\mu)}\\ &\leqq \varepsilon\sum_{|\beta|\leqq 2m}\|(|\xi|_\rho+h(x'',y,\xi))^{2m-|\beta|}D_y^\beta v(y)\|_{L^2(B_\mu)}\,,\qquad |\xi|_\rho\geqq A\,.\end{aligned}$$

Thus by using Lemma 3.1, we are given desired inequality:

$$\begin{aligned}\|\hat{L}v\|_{L^2(B_\mu)} &\geqq \|\hat{L}_0v\|_{L^2(B_\mu)}-\|(\hat{L}-\hat{L}_0)v\|_{L^2(B_\mu)}\\ &\geqq (C-2\varepsilon)\sum_{|\beta|\leqq 2m}\|(|\xi|_\rho+h(x'',y,\xi))^{2m-|\beta|}D_y^\beta v(y)\|_{L^2(B_\mu)}\,,\\ &\qquad\qquad v\in\mathring{H}^{2m}(B_\mu),\ |\xi|_\rho\geqq A\,.\end{aligned}$$

Since ε can be taken arbitrarily the inequality (2.13) is obtained.

§ 4. Examples.

The following examples have been given in [5].

1. $L=\dfrac{\partial^2}{\partial y^2}+y^2\dfrac{\partial^2}{\partial x^2}+\lambda\dfrac{\partial}{\partial x}$ Re $\lambda\neq 0$ or Re $\lambda=0$ and $|\lambda|<1$.

2. $L=\dfrac{\partial^2}{\partial y^2}+(x^2+y^2)\dfrac{\partial^2}{\partial x^2}$

These operators satisfy the conditions in Therem 1.1 and also 1.1′, hence they are hypoelliptic in (x,y)-plane.

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Chapter II. Boundary value problems.

§ 5. Introduction.

As an application of the method developed in Chapter I, we shall consider the regularity of solutions of boundary value problems for the degenerate elliptic equations being restricted to the second order.

Consider the case where $n = 1$ and $m = 2$. Set

$$\boldsymbol{R}_+^N = \boldsymbol{R}^k \times \boldsymbol{R}_+^1 = \{(x', y) \mid x' \in \boldsymbol{R}^k, y > 0\} .$$

The letters $\rho, \sigma, \mathfrak{M}$ and $\mathfrak{M}_0$ will be used as in §1. Thus we shall consider a partial differential operator

$$(5.1) \qquad L(x, D) = \sum_{\substack{|\alpha| \leq 2 \\ (\gamma,\alpha) \in \mathfrak{M}}} a_{\alpha\gamma}(x) x^\gamma D^\alpha , \qquad a_{\alpha,\gamma}(x) \in C^\infty(\boldsymbol{R}_+^N)$$

with one of the following boundary operators:

$$(5.2)_1 \qquad B_1(x', D_x) \equiv 1 ,$$

$$(5.2)_2 \qquad B_2(x', D_x) = D_y + b(x', D_{x'}) + c(x) .$$

Here $b(x', D_{x'})$ is a pseudo-differential operator defined in $\boldsymbol{R}^k$ with its symbol given by $\sum b_{\gamma'}(x') x'^{\gamma'} B_{\gamma'}(\xi)$, and $c(x')$ is a complex valued smooth function defined in $\boldsymbol{R}^k$. We shall, therefore, investigate the regularity of the solutions of the boundary value problem:

$$(5.3) \qquad L(x, D_x) u(x) = f(x) \qquad \text{in } \boldsymbol{R}_+^N ,$$

$$(5.4)_j \qquad B_j(x', D_x) u(x', 0) = 0 \qquad \text{on } \boldsymbol{R}_{x'}^k \quad (j = 1 \text{ or } 2) .$$

We freeze as in §1 the coefficients of the principal part of L and B_2 and introduce the following notations:

$$\hat{L}_0(x'', y\,; \xi, D_y) = \sum_{\substack{|\alpha| \leq 2 \\ (\gamma,\alpha) \in \mathfrak{M}_0 \\ \alpha = (\alpha', \alpha_N)}} a_{\alpha,\gamma}(0) x^\gamma \xi^{\alpha'} D_y^{\alpha_N} ,$$

$$\hat{B}_{2,0}(x'', \xi, D_y) = D_y + \sum b_{\gamma'}(0) x'^{\gamma'} B_{\gamma'}(\xi) .$$

Now we impose the following conditions.

Condition II. 1. $L_0(x'', y\,; D) = \sum_{\substack{|\alpha| \leq 2 \\ (\gamma,\alpha) \in \mathfrak{M}_0}} a_{\alpha,\gamma}(0) x^\gamma D^\alpha$

is strongly elliptic for $|x''| + y = 1$ with $y \geqq 0$.

Condition II. 2_j $(j = 1$ or $2)$. The homogeneous boundary value problem on the half line $\boldsymbol{R}^1_+$:

$$\hat{L}_0(x'', y; D_y)v(y) = 0 , \tag{5.5}$$

$$\hat{B}_{j,0}(x'', y; \xi, D_y)v(0) = 0 \tag{5.6$_j$}$$

has no non-trivial solution in $H^2(\boldsymbol{R}^1_+)$ for all x'' and ξ ($|\xi|_\rho = 1$).

Condition II. 3. The symbol $b_0(x'', \xi) = \sum b_{\gamma'}(0)x'^{\gamma'}B_{\gamma'}(\xi)$ is quasi-homogeneous of order one, that is,

$$b_0(\lambda^{-\sigma''}x'', \lambda^{\rho'}\xi) = \lambda b_0(x'', \xi) \qquad \lambda > 0 .$$

Condition II. 4. $b(x', \xi)$ is real valued in $\boldsymbol{R}^k_{x'}$.

THEOREM 5.1. *Assume that $\rho_0 > \sigma^0$ and that boundary value problem* (5.3), (5.4)$_j$ ($j = 1$ *or* 2) *satisfies either*

case 1. *the conditions* II. 1 *and* II. 2_1

or

case 2. *the conditions* II. 1, II. 2_2, II. 3 *and* II. 4 *corresponding to* $j = 1$ *or* 2.

If $f \in C^\infty(\overline{\boldsymbol{R}^N_+} \cap U)$ (where U is a neighborhood of the origin in $\boldsymbol{R}^N$) and if $u \in H^2_{\text{loc}}(\overline{\boldsymbol{R}^N_+})$ with (5.3), (5.4)$_j$ *in $\boldsymbol{R}^N_+ \cap U$* ($j = 1$ *or* 2), *then there exists a neighborhood V of the origin such that $u \in C^\infty(\overline{\boldsymbol{R}^N_+} \cap V)$.*

The proof of Theorem 5.1 can be reduced to that of the inequality (6.1) which will be proved in § 6. The inequality (6.1) corresponds to (2.13) proved in Chapter I. By applying (6.1) and Theorem 2.1, we can immediately prove the regularity of u at the boundary as in the case treated in Chapter I. In case the symbol $b(x', \xi)$ of the boundary opeyator B_2 is complex valued we need another method to obtain the estimate of the type (6.1). This method, which will be discussed in § 7, can be applied for higher order operators. To illustrate our discussions we shall give in § 8 examples of boundary systems satisfying the conditions in Theorem 5.1. Those conditions might be thought to be as close enough to a necessary condition.

§ 6. Boundary estimates.

As explained previously the proof of Theorem 5.1 can be reduced to the following lemma.

LEMMA 6.1 (cf. Lemma 2.4). *Assume that the conditions in Theorem* 5.1 *are satisfied.* (*i.e.* $\rho_0 > \sigma^0$, *either the case* 1: *Conditions* II. 1 *and* II. 2_1 *for the boundary system* $\{L, B_1\}$, *or the case* 2: *Conditions* II. 1, II. 2_2, II. 3 *and* II. 4 *for* $\{L, B_2\}$). *Then there exists positive constants* A, C *and* μ *and a neighborhood* Ω' *of the origin in* $\boldsymbol{R}^k$ *such that the following inequality holds*:

$$\sum_{\beta=0}^{2} \|(|\xi|_\rho + h(x'', y; \xi))^{2-\beta} D_y^\beta v(y)\|_{L^2(I_\mu)}^2 \leqq C \|\hat{L}(x', y; \xi, D_y) v(y)\|_{L^2(I_\mu)}^2 \tag{6.1}$$

for all $v \in \mathring{H}^2(I_\mu)$, $I_\mu = [0, \mu)$, *satisfying*

$$\hat{B}_j(x'; \xi, D_y) v(y)|_{y=0} = 0 \qquad (j = 1 \text{ or } 2)\,, \tag{6.2}$$

and for all $x' \in \Omega'$ *and* ξ, $|\xi|_\rho \geqq A$. *Here*

$$h(x'', y; \xi) = |x|_\sigma^{\rho_1 - 1} |\xi_1| + \cdots + |x|_\sigma^{\rho_k - 1} |\xi_k| \qquad (\text{see } (2.8))\,.$$

The proof will be obtained in several steps. Denote by $\mathscr{D}_j[\hat{L}_0]$ ($j =$ 1 or 2) the set of all $v = v(y) \in H^2(\boldsymbol{R}_+^1)$ such that $\operatorname{supp} v$ is compact in $[0, \infty)$ and that

$$\hat{B}_{j,0}(x'', \xi; D_y) v(y)|_{y=0} = 0 \qquad (j = 1 \text{ or } 2, \text{ respectively})\,. \tag{6.3}$$

Lemma 6.1 will be proved as a consequence of the following two lemmas 6.2 and 6.3.

LEMMA 6.2. *Assume that the conditions in Theorem* 5.1 *are satisfied. Then there exist a constant* C *and a neighborhood* Ω' *of the origin in* $\boldsymbol{R}^k$ *such that*

$$\sum_{\beta=0}^{2} \|(|\xi|_\rho + h(x'', y; \xi))^{2-\beta} D_y^\beta v(y)\|_{L^2(\boldsymbol{R}_+^1)}^2 \leqq C \|\hat{L}_0(x'', y; \xi, D_y) v(y)\|_{L^2(\boldsymbol{R}_+^1)}^2 \tag{6.4}$$

for all $v \in \mathscr{D}_j[\hat{L}_0]$, $x' \in \Omega'$ *and* $\xi \in \boldsymbol{R}^k$.

Proof. As in the proof of Lemma 3.1, the substitution: $\xi \to \lambda^\rho \xi$, $x \to \lambda^{-\sigma} x$ and $D_y \to \lambda D_y$ ($\lambda > 0$) reduces the proof of (6.4) to the case $|\xi|_\rho = 1$. We shall use the coordinate transformation as in § 3 and use the same notations $h_1, g, \cdots$. In particular,

$$z = g(x'', y; \xi) \qquad y \geqq 0, \ |\xi| = 1: \text{ see } (3.4)\,.$$

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We bring the equation

$$\hat{L}_0(x'', y; \xi, D_y)v(y) = f(y) \qquad y \geqq 0: \text{ see } (3.7)$$

into the form

$$M(z, D_z)v_1 = M(x'', \xi; z, D_z)v_1(z) = f_1(z) \qquad z \geqq 0, \ |\xi|_\rho = 1: \text{ see } (3.8)$$

where $v_1(z) = (1 + h_1)^{1-1/2}v$ and $f_1(z) = (1 + h_1)^{-1-1/2}v$ as in (3.9). Considering a pair (x'', ξ) ($|\xi|_\rho = 1$) as a parameter, we observe that

$$M(z, D_z) = a_1(z, x'')D_z^2 + a_2(z, x'', \xi)D_z + a_3(z, x'', \xi)$$
$$\operatorname{Re} a_1(z, x'', \xi) \geqq \delta > 0: \qquad \text{see } (3.10)\ ,$$
$$a_j \in B(\boldsymbol{R}^1_+) \qquad j = 1, 2, 3\ .$$

The boundary condition $(5.6)_j$ ($j = 1$ or 2) will change into

$$(6.5)_1 \qquad N_1 v_1(0) = v_1(0) = 0\ ,$$

$$(6.5)_2 \qquad N_2 v_1(0) = (D_z - (1 + h_1(x'', 0; \xi))^{-1} b_0(x'', \xi))v_1(0) = 0 \qquad (|\xi|_\rho = 1)\ .$$

Since the boundary systems $\{a_1 D_z^2 + 1, 1\}$ and $\{a_1 D_z^2 + 1, D_z\}$, considered in $\boldsymbol{R}^1_{+z}$, are stably variational (stablement variationnel) in the sense of Shimakura (cf. [15]), the Gårding inequality holds for $\{M(z, D_z), N_j\}$ ($j = 1$ or 2):

$$(6.6) \qquad \begin{aligned} \|v_1(z)\|^2_{H^1(\boldsymbol{R}^1_+)} &\geqq C_0 \operatorname{Re}(M(z, D_z)v_1(z), v_1(z)) \\ &\geqq C_1 \|v_1(z)\|^2_{H^1(\boldsymbol{R}^1_+)} - C_2 \|v_1(z)\|^2_{L^2(\boldsymbol{R}^1_+)} \qquad v \in \mathscr{D}_j(\hat{L}_0)\ . \end{aligned}$$

Here the positive constants C_0, C_1 and C_2 can be chosen independently of (x'', ξ) when (x'', ξ) runs through a compact set. We shall prove the inequality (6.4) starting with (6.6). Let $v \in \mathscr{D}_j[\hat{L}_0]$ ($j = 1$ or 2), then we have $v_1(z) \in H^2(\boldsymbol{R}^1_+)$, $\operatorname{supp} v_1$ is compact in $[0, \infty)$ and

$$(1 + h_1)v_1|_{z=1} = 0 \qquad \text{if } v \in \mathscr{D}_1[\hat{L}_0]: \text{ see } (6.5)_1\ ,$$
$$N_2(1 + h_1)v_1|_{z=0} = (1 + h_1(x'', 0; \xi))N_2(x''\xi, D_z)v_1|_{z=0} = 0$$
$$\text{if } v \in \mathscr{D}_2[\hat{L}_0]: \text{ see } (6.5)_2\ .$$

Thus we can substitute v_1 in (6.6) by $(1 + h_1)v_1(z)$ to prove

$$(6.7) \qquad \begin{aligned} &\operatorname{Re}(M(z, D_z)(1 + h_1)v_1(z), (1 + h_1)v_1(z)) \\ &\qquad \geqq C_1 \|(1 + h_1)v_1(z)\|^2_{H^1(\boldsymbol{R}^1_+)} - C_2 \|(1 + h_1)v_1(z)\|^2_{L^2(\boldsymbol{R}^1_+)}\ . \end{aligned}$$

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Continuing the similar argument as in the proof of the inequality (3.18), we obtain

$$\begin{aligned} &\|(1+h_1)v_1(z)\|_{H^1(R^1_+)} \\ (6.8)\qquad &\leqq C\{\|(1+h_1)M(z,D_z)v_1(z)\|_{L^2(R^1_+)} + \|(1+h_1)v_1(z)\|_{L^2(R^1_+)}\} \\ &\qquad v \in \mathscr{D}_j[\hat{L}_0] \ (j=1,2)\,. \end{aligned}$$

Since $a_1D_z^2v_1 = f_1 - (a_2D_zv_1 + a_3v_1)$, $\mathrm{Re}\, a_1 \geqq \delta > 0$, the L^2-norm of $(1+h_1)D_z^2v_1$ can be estimated by the right hand side of (6.8). Thus we have

$$\begin{aligned} (6.9)\qquad &\sum_{\beta=0}^{2}\|(1+h_1)D_z^\beta v(z)\|_{L^2(R^1_+)} \\ &\leqq C'\{\|(1+h_1)M(z,D_z)v_1(z)\|_{L^2(R^1_+)} + \|(1+h_1)v_1(z)\|_{L^2(R^1_+)}\} \\ &\qquad v \in \mathscr{D}_j[\hat{L}_0] \ (j=1 \text{ or } 2)\,, \end{aligned}$$

where the constant C' can be taken independently of (x'', ξ) when it runs through a compact set. Now going over to the variable y, as in the proof of (3.18), and using Condition II. 2, we have the inequality (6.4).

LEMMA 6.3. *Assume that* $\rho_0 > \sigma^0$ *and that the boundary system* $\{\hat{L}_0(x'', \xi; D_y), \hat{B}_0(x'', \xi; D_y)\}$ *satisfies the Conditions* II. 1, II. 2_2, II. 3 *and* II. 4. *Consider the perturbed boundary system* $\{\hat{L}_0(x'', \xi; D_y), \hat{B}_0(x'', \xi; D_y) + c(x')\}$ *with parameter* (x', ξ) *and with* $c(x')$ *given in* $(5.2)_2$. *Then for any* $\mu > 0$ *and for any neighborhood* Ω' *of the origin in* $\boldsymbol{R}^k$, *there exist positive constants* A *and* C *such that*

$$\begin{aligned} (6.10)\qquad &\sum_{\beta=0}^{2}\|(|\xi|_\rho + h(x'', y; \xi))^{2-\beta}D_y^\beta v(y)\|^2_{L^2(I_\mu)} \\ &\leqq C\,\|\hat{L}_0(x'', y; \xi, D_y)v(y)\|^2_{L^2(I_\mu)} \end{aligned}$$

for all $v \in H^2(I_\mu)$, $I_\mu = [0, \mu)$, *satisfying*

$$(6.11)\qquad [\hat{B}_{2,0}(x'', \xi, D_y) + c(x')]v(0) = [D_y + b_0(x'', \xi) + c(x')]v(0) = 0$$

and for all $x' \in \Omega'$ *and* $\xi, |\xi|_\rho \geqq A$.

Proof. Take $v \in \mathring{H}^2(I_\mu)$ satisfying the boundary condition (6.11). Put

$$u(y) = v(y)\exp[ic(x')y]\,.$$

Then we have $u(y) \in \mathring{H}^2(I_\mu)$ and

$$\hat{B}_{2,0}(x'', \xi, D_y)u(y)|_{y=0} = 0\,.$$

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By Lemma 6.2, we have

$$(6.12)\quad \sum_{\beta=0}^{2} \|(|\xi|_\rho + h)^{2-\beta} D_y^\beta u\|_{L^2(I_\mu)} \leqq C' \|\hat{L}_0(x'', y; \xi, D_y)u\|_{L^2(I_\mu)}, \qquad |\xi|_\rho \geqq A'$$

for some positive constants C' and A' independent of such u. From (6.12) we easily obtain the following inequality:

$$(6.13)\quad \begin{aligned} &\sum_{\beta=0}^{2} \|(|\xi|_\rho + h)^{2-\beta} D_y^\beta v\|_{L^2(I_\mu)} \\ &\leqq C_1 \|\hat{L}_0(x'', y; \xi, D_y)v\|_{L^2(I_\mu)} + C_2 \|(|\xi|_\rho + h)v\|_{L^2(I_\mu)}, \\ &\qquad |\xi|_\rho \geqq A' \end{aligned}$$

for some constants C_1 and C_2 which depend only on $\Omega'(c(x'))$. Since $h \to \infty$ as $|\xi|_\rho \to \infty$, there exist positive constants C and A such that

$$\sum_{\beta=0}^{2} \|(|\xi|_\rho + h)^{2-\beta} D_y^\beta v\|_{L^2(I_\mu)} \leqq C \|\hat{L}_0 v\|_{L^2}, \qquad |\xi|_\rho \geqq A,$$

which proves the inequality (6.10). Q.E.D.

Proof of Lemma 6.1. A) First we consider the boundary system $\{L, B_1\} = \{L, 1\}$ under Conditions II. 1 and II. 2_1. By Lemma 6.2, we have the following inequality for any $\mu > 0$:

$$(6.14)\quad \begin{aligned} &\sum_{\beta=0}^{2} \|(1 + h)^{2-\beta} D_y^\beta v\|_{L^2(I_\mu)} \leqq C \|\hat{L}_0(x'', y; \xi, D_y)v(y)\|^2_{L^2(I_\mu)} \\ &\qquad v \in \mathring{H}^2(I_\mu),\ v(0) = 0,\ I_\mu = [0, \mu). \end{aligned}$$

For $v \in \mathring{H}^2(I_\mu)$, with $v(0) = 0$, we have

$$(6.15)\quad \begin{aligned} L(x', y; \xi, D_y)v &= \hat{L}_0 v + (L - \hat{L}_0)v \\ &= \hat{L}_0 v + \sum_{\theta=0} (a_{\alpha\gamma}(x) - a_{\alpha\gamma}(0)) x^\gamma \xi^{\alpha'} D_y^\beta v \\ &\quad + \sum_{\theta>0} a_{\alpha\gamma}(x) x^\gamma \xi^{\alpha'} D_y^\beta v. \end{aligned}$$

Thus in a similar manner to the proof of Lemma 2.4 we have the inequality (6.1) for the boundary system $\{L, 1\}$.

B) We consider boundry system $\{L, B_2\}$ under Conditions II. 1, II. 2_2, II. 3 and II. 4. Let μ be a positive number determined later and let

$$v \in \mathring{H}^2(I_\mu), \qquad \hat{B}(x', \xi, D_y)v|_{y=0} = 0.$$

If we put

$$u(y) = v(y) \exp[i(b(x', \xi) - b_0(x'', \xi))y].$$

Then we have $u(y) \in \mathring{H}^2(I_\mu)$ and

$$[\hat{B}_0(x'', \xi, D_y) + c(x')]u(y)|_{y=0} = 0 .$$

We can apply Lemma 6.3 to u to obtain

$$\sum_{\beta=0}^{2} \|(|\xi|_\rho + h)^{2-\beta} D_y^\beta u\|_{L^2(I_\mu)} \leqq C \|\hat{L}_0 u\|_{L^2(I_\mu)} \qquad x' \in \Omega',\ |\xi|_\rho \geqq A \tag{6.16}$$

for some constants C and A. Since $\hat{L}_0 = \hat{L} - (\hat{L} - \hat{L}_0)$, we have

$$\|\hat{L}_0 u\|_{L^2(I_\mu)} \leqq \|\hat{L} u\|_{L^2(I_\mu)} + \|(\hat{L} - \hat{L}_0)u\|_{L^2(I_\mu)} . \tag{6.17}$$

The first term in the right hand side of (6.17) is estimated by

$$\begin{gathered} \text{const}\,(\|e^{-id(x',\xi)y}\hat{L}v\|_{L^2(I_\mu)} + \|d(x',\xi)^2 v\|_{L^2(I_\mu)} + \|d(x',\xi)D_y v\|_{L^2(I_\mu)}) , \\ d(x',\xi) = b(x',\xi) - b_0(x'',\xi) . \end{gathered} \tag{6.18}$$

By Condition II. 4, $d(x',\xi) = b(x',\xi) - b_0(x'',\xi)$ is real valued. For any $\varepsilon > 0$, if we choose the diameter of Ω' sufficiently small and apply the inequality (2.10), we have

$$\begin{aligned} &\|d(x',\xi)^2 v\|_{L^2(I_\mu)} + \|d(x',\xi)D_y v\|_{L^2(I_\mu)} \\ &\qquad \leqq \varepsilon \sum_{\beta=0}^{2} \|(|\xi|_\rho + h(x'', y\,; \xi))^{2-\beta} D_y^\beta v\|_{L^2(I_\mu)} . \end{aligned} \tag{6.19}$$

To the second term in the right hand side of (6.17), we can apply the the similar manner to the proof of Lemma 2.4. Consequently we can choose μ and Ω' sufficiently small and A sufficiently large so that

$$\|(\hat{L} - \hat{L}_0)u\|_{L^2(I_\mu)} \leqq \varepsilon \sum_{\beta=0}^{2} \|(|\xi|_\rho + h)^{2-\beta} D_y^\beta v\|_{L^2(I_\mu)} , \qquad x' \in \Omega',\ |\xi|_\rho \geqq A . \tag{6.20}$$

Similarly we have

$$\sum_{\beta=0}^{2} \|(|\xi|_\rho + h)^{2-\beta} D_y^\beta u\|_{L^2(I_\mu)} \geqq (1 - \varepsilon) \sum_{\beta=0}^{2} \|(|\xi|_\rho + h)^{2-\beta} D_y^\beta v\|_{L^2(I_\mu)} . \tag{6.21}$$

Summing up the results (6.16) ~ (6.21) we have the inequality of the form

$$(1 - 2\varepsilon) \sum_{\beta=0}^{2} \|(|\xi|_\rho + h)^{2-\beta} D_y^\beta v\|_{L^2(I_\mu)} \leqq C \|\hat{L} v\|_{L^2(I_\mu)} ,$$

where $\varepsilon > 0$ can be taken arbitrarily small if we choose $\mu, A > 0$ and Ω' adequately. This completes the proof of Lemma 6.1.

§ 7. Alternative method.

Let $L(x, D) = \sum_{\substack{|\alpha| \leq 2 \\ (\gamma, \alpha) \in \mathfrak{M}}} a_{\alpha\gamma}(x) x^{\gamma} D^{\alpha}$ and let $B_2(x', D_x) = D_y + b(x', D_{y'}) + c(x')$. We assume the conditions II. 1, II. 2_1, II. 2_2 and II. 3. In this section the symbol $b(x', \xi)$ can be complex-valued, since Conditions II. 2_1 and II. 2_2 are assumed simultaneously. We shall say that the boundary system $\{L(x, D), B_2(x'D)\}$ is hypoelliptic at the boundary if the conclusion of Theorem 5.1 holds.

LEMMA 7.1. *Under the conditions* II. 1, II. 2_1, II. 2_2 *and* II. 3, *we have the same conclusion as in Lemma* 6.1, *that is, we have the inequality of the type* (6.1).

As a consequences of Lemma 7.1 we have the following.

THEOREM 7.2. *Assume that* $\rho_0 > \sigma^0$. *Under the same conditions as in Lemma* 7.1 *the boundary system* $\{L(x, D), B_2(x', D)\}$ *is hypoelliptic at the boundary.*

We shall first give the outline of the proof of Lemma 7.1 in several steps.

A) By the inequality (6.6) and the condition II. 2_2 we have the following inequality for some positive constants C and A:

$$\text{Re}\,(\hat{L}_0(x'', y; \xi, D_y)v(y), v(y)) \geqq C \sum_{\beta=0}^{1} \|(|\xi|_\rho + h)^{1-\beta} D_y^\beta v\|^2_{L^2(0,\infty)}\,, \qquad |\xi|_\rho \geqq A,\ v \in \mathscr{D}_2[\hat{L}_0]\,. \tag{7.1}$$

B) From the inequality (7.1) we derive the following inequality for some positive constants C, A and μ:

$$\text{Re}\,(\hat{L}(x', y; \xi, D_y)v(y), v(y))_{L^2(0,\mu)} \geqq C \sum_{\beta=0}^{1} \|(|\xi|_\rho + h)^{1-\beta} D_y^\beta v(y)\|^2_{L^2(0,\mu)} \tag{7.2}$$

for $|\xi|_\rho \geqq A$, $|x'| \leqq \mu$ and for $v \in \mathscr{D}_{2,\mu}[\hat{L}]$ which means that $v \in H^2(0, \mu)$ with supp $v \subset [0, \mu)$ and satisfies

$$[D_y + b(x', \xi) + c(x')]v(0) = 0\,. \tag{7.3}$$

The inequality (7.2) might be viewed as an "coercive inequality" for the boundary system $\{L(x, D), B_2(x', D)\}$.

C) If $v \in \mathscr{D}_{2,\mu}[\hat{L}]$, then we obviously have $(|\xi|_\rho + h(x'', 0; \xi))v(y) \in \mathscr{D}_{2,\mu}[\hat{L}]$. Substiting v in (7.2) with $(|\xi|_\rho + h(x'', 0; \xi))v(y)$, we have

$$(7.4)\quad \sum_{\beta=0}^{1}\|(|\xi|_\rho + h(x'',y;\xi))^{1-\beta}(|\xi|_\rho + h(x'',0;\xi))D_y^\beta v(y)\|_{L^2(0,\mu)} \leqq C\|\hat{L}v\|_{L^2(0,\mu)}$$

for $|\xi|_\rho \geqq A$ and $v \in \mathscr{D}_{2,\mu}[\hat{L}]$.

D) We have the following trace formula:

$$(7.5)\quad (|\xi|_\rho + h(x'',0;\xi))^{1/2}|v(0)| \leqq C\sum_{\beta=0}^{1}\|(|\xi|_\rho + h(x'',y;\xi)^{1-\beta}D_y^\beta v(y)\|_{L^2(0,\infty)}$$

for $v \in H^1(0,\infty)$ with compact support.

E) By the steps C) and D) the following inequality for some positive constants C, A and μ is obtained:

$$(7.6)\quad (|\xi|_\rho + h(x'',0;\xi))^{3/2}|v(0)| \leqq C\|\hat{L}v\|_{L^2(0,\mu)}\,, \qquad |\xi|_\rho \geqq A,\ v \in \mathscr{D}_{2,\mu}[\hat{L}]\,.$$

F) For any $v \in \mathscr{D}_{2,\mu}[\hat{L}]$ there exists a function $w \in H^2(0,\mu)$ such that such that $w(0) = v(0)$, $\operatorname{supp} w \subset [0,\mu)$ and

$$(7.7)\quad \sum_{\beta=0}^{2}\|(|\xi|_\rho + h(x'',y;\xi))^{2-\beta}D_y^\beta w\|_{L^2(0,\mu)} \leqq C(|\xi|_\rho + h(x'',0;\xi))^{3/2}|v(0)|\,, \qquad |x''| \leqq \mu\,,$$

where the constant C is independent of $v \in \mathscr{D}_{2,\mu}[\hat{L}]$.

G) By (7.6) and (7.7) we have for another constant C

$$(7.8)\quad \sum_{\beta=0}^{2}\|(|\xi|_\rho + h(x'',y;\xi))^{2-\beta}D_y^\beta w\|_{L^2(0,\mu)} \leqq C\|\hat{L}v\|_{L^2(0,\mu)}\,, \qquad |\xi|_\rho \geqq A,\ v \in \mathscr{D}_{2,\mu}[\hat{L}]\,.$$

H) For $v \in \mathscr{D}_{2,\mu}[\hat{L}]$, we take w defined in the step F). Then $u = v - w$ satisfies the Dirichlet condition: $u(0) = 0$. By Condition II. 2_1 and by the conclusion of Lemma 6.1 for the boundary system $\{L, 1\}$, we have the following inequality for some positive constants C, A and μ:

$$(7.9)\quad \|u\|_{2,\rho} \equiv \sum_{\beta=0}^{2}\|(|\xi|_\rho + h(x'',y\xi;\xi))^{2-\beta}D_y^\beta u(y)\|_{L^2(0,\mu)} \leqq C\|\hat{L}u\|_{L^2(0,\mu)}\,, \qquad |\xi|_\rho \geqq A\,.$$

From this we have

$$\|v\|_{2,\rho} - \|w\|_{2,\rho} \leqq C(\|\hat{L}v\|_{L^2(0,\mu)} + \|\hat{L}w\|_{L^2(0,\mu)})$$

and

$$\|v\|_{2,\rho} \leqq C'(\|\hat{L}v\|_{L^2(0,\mu)} + \|w\|_{2,\rho})\,.$$

By (7.8) we have

$$\sum_{\beta=0}^{2}\|(|\xi|_\rho + h(x'',y;\xi))^{2-\beta}D_y^\beta v(y)\|_{L^2(0,\mu)} \leqq C''\|\hat{L}v\|_{L^2(0,\mu)}\,, \qquad |\xi|_\rho \geqq A,\ v \in \mathscr{D}_{2,\mu}[\hat{L}]\,. \tag{7.10}$$

This proves Lemma 7.1.

Next we shall prove the inequalities left unproved in the above steps.

Proof of (7.2). Set $\mathscr{D}_{2,\mu}[\hat{L}_0] = \{v\,|\,v \in \mathscr{D}_2[\hat{L}_0],\ \operatorname{supp} v \subset [0,\mu)\}$ and write

$$\hat{L}_0(x'',\xi,D_y) = a_0(0)D_y^2 + a_{1,0}(x'',\xi)D_y + a_{2,0}(x'',\xi)\,,$$
$$\hat{L}(x,\xi,D_y) = a_0(x)D_y^2 + a_1(x,\xi)D_y + a_2(x,\xi)\,.$$

For $v \in \mathscr{D}_{2,\mu}[\hat{L}_0]$ ($\mu > 0$ being determined later), we have

$$\begin{aligned}
&\mathrm{Re}\,(\hat{L}(x',y;\xi,D_y)v(y),v(y))\\
&\quad= \mathrm{Re}\,((\hat{L}-\hat{L}_0)v,v) + \mathrm{Re}\,(\hat{L}_0 v,v)\\
&\quad= \mathrm{Re}\,\{((a_0(x',y)-a_0(0))v'(y),v'(y))\\
&\qquad\qquad + ((i(\partial/\partial y)a_0(x',y) + a_1(x',y;\xi) - a_{1,0}(x'',y;\xi))D_y v,v)\\
&\qquad\qquad + ((a_2(x',y;\xi) - a_2(x'',y;\xi))v(y),v(y))\}\\
&\qquad + \mathrm{Re}\,\{(a_0(0)v'(y),v'(y)) + (a_{1,0}(x'',y;\xi)D_y v,v) + (a_{2,0}(x'',y;\xi)v,v)\}\\
&\qquad - \mathrm{Re}\,ib_0(x'',\xi)a_0(0)\,|v(0)|^2 - \mathrm{Re}\,ib_0(x'',\xi)(a_0(x',0)-a_0(0))\,|v(0)|^2
\end{aligned}$$

where we denote the inner product in $L^2(0,\mu)$ by $(\cdot,\cdot)$ and the norm in $L^2(0,\mu)$ by $\|\cdot\|$. We denote the last expression by $L[v,v]$. Taking $\mu > 0$ sufficiently small and using the inequalities (7.1), (7.5), (2.10) and (2.12), we have for some positive constants C and A

$$L[v,v] \geqq C\sum_{\beta=0}^{1}\|(|\xi|_\rho + h(x''y\,\xi))^{1-\beta}D_y^\beta v(y)\|^2_{L^2(0,\mu)} \tag{7.11}$$

for all $v \in \mathscr{D}_{2,\mu}[\hat{L}_0]$ and $|x'| \leqq \mu$. This inequality can be extended to all $v \in H^1(0,\mu)$ with $\operatorname{supp} v \subset [0,\mu]$. On the other hand, (7.11) is valid for all $u \in \mathscr{D}_{2,\mu}[\hat{L}]$ since $\mathscr{D}_{2,\mu}[\hat{L}] \subset H^1(0,\mu)$ and $\operatorname{supp} u \subset [0,\mu)$. Now for $u \in \mathscr{D}_{2,\mu}[\hat{L}]$, we have

$$L[u,u] = \mathrm{Re}\,(\hat{L}u,u) - \mathrm{Re}\,i(a_0(x',0)(b_0(x'',\xi) - b(x',\xi) - c(x'))\,|u(0)|^2\,.$$

Again taking $\mu > 0$ sufficiently small and by (7.5) we have the inequality (7.2).

Proof of F). It is sufficient to prove the case where $|\xi|_\rho = 1$. Take a function $\psi \in C_0^\infty[0,\mu)$ such that $\psi(y) \equiv 1$ $0 \leqq y \leqq \mu/2$. We make use of a transformation $y \to t$ given by

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$$t = \int_0^y (1 + h_1(x'', s; \xi))ds \qquad y \geqq 0 .$$

For $v \in H^2[0, \infty)$ set

$$w(y) = \psi(t)v(0) .$$

Then we have $w(0) = v(0)$ and

$$\begin{aligned} &\sum_{\beta=0}^{2} \|(1 + h_1(x'', y; \xi))^{2-\beta} D_y^\beta w(y)\|^2 \\ &\leqq C|v(0)|^2 \sum_{\beta=0}^{2} \|(1 + h_1(x'', y; \xi))^{3/2} D_t^\beta \psi(t)\|^2 \\ &\leqq C'|v(0)|^2 (1 + h_1(x'', 0; \xi))^3 \sum_{\beta=0}^{2} \|D_t^\beta \psi(t)\|^2_{L_t^2(0,\mu)} \\ &\leqq C''(1 + h_1(x'', 0; \xi))^3 |v(0)|^2 , \qquad |x''| \leqq \mu . \end{aligned}$$

From these the inequality (7.7) can be derived.

§ 8. Examples.

EXAMPLE 1. Consider the boundary value problem:

$$\text{(8.1)} \qquad Lv = (D_y^2 + (x^{2k} + y^\ell)D_x^2)u(x, y) \quad y > 0 , \quad -\infty < x < \infty , \quad (k, \ell = 1, 2, \cdots) ,$$

$$\text{(8.2)} \qquad B_2 u|_{y=0} = (D_y + b(x)x^k D_x + c(x)u|_{y=0} = 0 ,$$

where $b(x)$ and $c(x)$ are complex-valued smooth functions. If $|\text{Im}\, b(x)| < 1$, then the boundary system $\{L, B_2\}$ satisfies the conditions in Theorem 7.2. If, in particular, $b(x)$ is a real valued function, then $\{L, B_2\}$ satisfies the conditions in Theorem 5.1. Hence $\{L, B_2\}$ is hypoelliptic at the boundary.

Proof. We can choose $\rho = (\ell/2 + 1, 1)$ and $\sigma = (\ell/2k, 1)$. Then $\rho_0 = \ell/2 + 1 > \sigma^0 = \ell/2k$. Conditions II. 1, II. 2_1 and II. 3 are obvioulу verified. As for Condition II. 2_2, the equalities (5.5), $(5.6)_2$ now have the forms:

$$\text{(8.3)} \qquad \begin{cases} \left[-\dfrac{d^2}{dy^2} + (x^{2k} + y^\ell)\xi^2\right]v(y) = 0 & y > 0,\ \xi \neq 0 , \\ \left[\dfrac{1}{i}\dfrac{d}{dy} + b(x)x^k\xi\right]v(0) = 0 . \end{cases}$$

It follows that

$$ib(x)x^k\xi|v(0)|^2 + \int_0^\infty |v'(y)|^2 dy + \int_0^\infty (x^{2k} + y^\ell)\xi^2 v(y)^2 dy = 0 .$$

In case $|\text{Im}\, b(x)| < 1$, we have

$$|\text{Im}\, b(x)x^k\xi||v(0)|^2 \leqq \int_0^\infty |v'(y)|^2\, dy + x^{2k}\xi^2 \int_2^\infty |v(y)|^2\, dy \,.$$

Hence we have $v(y) \equiv 0$.

EXAMPLE 2. Consider the boundary value problem:

$$Lu = (D_y^2 + y^\ell D_x^2)u(x, y) = f(x, y) \qquad y > 0 \qquad (\ell = 0, 1, 2, \cdots)\,, \tag{8.4}$$

$$B_2 u|_{y=0} = [D_y + b(x)|D_x|^{2/(\ell+2)} + c(x)]u|_{y=0} = 0\,, \tag{8.5}$$

where $b(x)$ is a real valued smooth function and $c(x)$ is a complex valued smooth function. We can easily verify that $\{L, B_2\}$ satisfies the conditions in Theorem 5.1.

Remark 1. In the Example 2, we have $\rho = (\ell/2 + 1, 1)$ and $\sigma = (0, 1)$. For the operators of the type $\sigma = (0, \cdots, 0, 1, \cdots, 1)$, general boundary value problems have been investigated in [16].

Remark 2. Let $\rho = (\rho_1, \cdots, \rho_k, \rho_{k+1})$ be $(\rho_1, \cdots, \rho_k, 1)$ and $\sigma = (\sigma_1, \cdots, \sigma_k, \sigma_{k+1})$ be $(0, \cdots, 0, 1)$. For such a pair (ρ, σ) we consider the Dirichlet boundary value problem

$$L(x, D)u(x) = f(x) \qquad \text{in } x_N = y > 0 \tag{8.5}$$

$$u(x', 0) = 0 \tag{8.6}$$

Suppose that under the Condition II. 1 the Condition II. 2_1 were not satisfied. Then we see, by using the same method as in [3; Theorem 1.1] that the problem (8.5), (8.6) is not hypoelliptic in the upper half plane including the boundary. This shows that the Condition II. 2_1 or II. 2_2 is necessary to obtain the hypoellipticity.

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Nagoya University