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Abstract

Lazard showed in his seminal work (*Groupes analytiques p-adiques*, Publ. Math. Inst. Hautes Études Sci. **26** (1965), 389–603) that for rational coefficients, continuous group cohomology of p-adic Lie groups is isomorphic to Lie algebra cohomology. We refine this result in two directions: first, we extend Lazard's isomorphism to integral coefficients under certain conditions; and second, we show that for algebraic groups over finite extensions $K/\mathbb{Q}_p$, his isomorphism can be generalized to K-analytic cochains and K-Lie algebra cohomology.

Contents

1. Introduction

One of the main results of Lazard's magnum opus [Laz65] on p-adic Lie groups is a comparison isomorphism between continuous group cohomology, analytic group cohomology and Lie algebra cohomology. This comparison isomorphism is an important tool in the cohomological study of Galois representations in arithmetic geometry. It has also appeared more recently in topology and homotopy theory in connection with the formal groups associated to cohomology theories and, in particular, with topological modular forms.

Lazard's comparison theorem holds for $\mathbb{Q}_p$-vector spaces, and the isomorphism between continuous cohomology and Lie algebra cohomology is obtained from a difficult isomorphism between the saturated group ring and the saturated universal enveloping algebra. For some applications (e.g. the connection with the Bloch–Kato exponential map in [HK06]), it is important to have a version for integral coefficients and a better understanding of the map between the cohomology theories.

In this paper we extend and complement the comparison isomorphism in two directions.

Our first result is an integral version of the isomorphism, assuming some technical conditions (Theorem 3.1.1). For uniform pro-p-groups one gets a clean result with only a mild condition

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on the module (Theorem 3.3.3). To our knowledge, and with the notable exception of Totaro's work [Tot99], this has been the first progress made on a problem which Lazard described as 'reste à faire' more than forty years ago [Laz65, Introduction, § 7, paragraph C)].

Our second result concerns the definition of the isomorphism in the case of smooth group schemes. Here one can directly define a map from analytic group cohomology to Lie algebra cohomology with constant coefficients by differentiating cochains (see Definition 4.2.1). We showed in [HK06] (see Proposition 4.2.4) that the resulting map is Lazard's comparison isomorphism modulo the identification of continuous cohomology with analytic cohomology. Serre mentioned to us that this was clear to him at the time Lazard's paper was written; however, it was not included in the published results. Unfortunately, so far we have not been able to use this simple map to obtain an independent proof of Lazard's comparison result.

The advantage of this description of the map is not only its simplicity but also that it carries over to K-Lie groups for finite extensions $K/\mathbb{Q}_p$. In Theorem 4.3.1, we prove that this map is also an isomorphism in the case of K-Lie groups attached to smooth group schemes with connected generic fiber over the integers of K. This theorem generalizes the results in [HK06] for GL_n and complements the result of Lazard, who dealt only with $\mathbb{Q}_p$-analytic groups.

The paper is organized as follows. In § 2 we give a quick tour of the concepts from [Laz65] that we need. It is our hope that this section will also prove to be a useful overview of the central notions and results in [Laz65]. In § 3 we prove our integral refinement of Lazard's isomorphism. Finally, § 4 considers the isomorphism over a general base in the case of group schemes.

2. Review of some results of Lazard

In this section we recall the basic notions about groups and group rings that we need to formulate our main results. As we proceed, we shall illustrate the main concepts with the example of separated smooth group schemes. We hope that this section can serve as a guide through the long and difficult paper by Lazard.

2.1 Saturated groups

Definition 2.1.1 [Laz65, ch. II, §§ 1.1 and 1.2.10, ch. III, Definition 2.1.2]. A *filtration* on a group G is a map

$$\omega : G \to \mathbb{R}_+^* \cup \{\infty\}$$

such that:

(i) for all $x, y \in G$, $\omega(xy^{-1}) \geqslant \inf\{\omega(x), \omega(y)\}$;

(ii) for all $x, y \in G$, $\omega(x^{-1}y^{-1}xy) \geqslant \omega(x) + \omega(y)$.

The group G is said to be *p-filtered* if, in addition,

$$\omega(x^p) \geqslant \inf\{\omega(x) + 1, p\omega(x)\} \quad \text{for all } x \in G.$$

The group G is said to be *p-valued* if ω satisfies:

(iii) $\omega(x) < \infty$ for $x \neq e$;

(iv) $\omega(x) > (p-1)^{-1}$ for all $x \in G$;

(v) $\omega(x^p) = \omega(x) + 1$ for all $x \in G$.

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The group G is said to be *p-divisible* if it is p-valued and has the property that:

(vi) for all $x \in G$ with $\omega(x) > 1 + 1/(p-1)$ there exists $y \in G$ with $y^p = x$.

Finally, a p-divisible group G is said to be *saturated* if:

(vii) G is complete for the topology defined by the filtration.

Note that a filtration satisfying the conditions for being a p-valuation is automatically p-filtered.

Recall that a pro-p-group is the inverse limit of finite p-groups. This is the case that we are going to work with.

Proposition 2.1.2 [Laz65, ch. II, Proposition 2.1.3]. *A p-filtered group is a pro-p-group if and only if it is compact.*

We denote by $\mathbb{F}_p[\epsilon]$ the polynomial ring with generator ϵ in degree one.

Definition and Lemma 2.1.3 [Laz65, ch. II, § 1.1, ch. III, § 2.1.1 and Definition 2.1.3]. Suppose that (G, ω) is filtered.

(i) For every $\nu \in \mathbb{R}_+^*$,

$$G_\nu := \{x \in \mathbb{G} \mid \omega(x) \geqslant \nu\} \quad \text{and} \quad G_\nu^+ := \{x \in G \mid \omega(x) > \nu\} \subseteq G$$

are normal subgroups.

(ii) Let

$$\operatorname{gr}(G) := \bigoplus_{\nu \in \mathbb{R}_+^*} G_\nu / G_\nu^+.$$

Then $\operatorname{gr}(G)$ is a graded Lie algebra over $\mathbb{F}_p$, with the Lie bracket being induced by the commutator in G.

(iii) If (G, ω) is p-valued, then $\operatorname{gr}(G)$ is even a graded $\mathbb{F}_p[\epsilon]$-Lie algebra, with the action of ϵ being induced by $x \mapsto x^p$ (where $x \in G_\nu$ and $x^p \in G_{\nu+1}$).

(iv) In this case, $\operatorname{gr}(G)$ is free as a graded $\mathbb{F}_p[\epsilon]$-module. The *rank of* G is, by definition, the rank of the $\mathbb{F}_p[\epsilon]$-module $\operatorname{gr}(G)$.

Example 2.1.4 [Laz65, ch. V, § 2.2.1]. Let (G, ω) be a complete p-valued group of finite rank d. If $\{x_i\}_{i=1,\dots,d} \subseteq G$ are representatives of an ordered basis of the $\mathbb{F}_p[\epsilon]$-module $\operatorname{gr}(G)$, then every $y \in G$ can be expressed uniquely as an ordered product $y = \prod_{i=1}^d x_i^{\lambda_i}$ with $\lambda_i \in \mathbb{Z}_p$, and we have

$$\omega(y) = \inf_i (\omega(x_i) + v(\lambda_i))$$

where the valuation on $\mathbb{Z}_p$ is normalized by $v(p) = 1$; (G, ω) has rank d.

Definition 2.1.5 [Laz65, ch. V, §§ 2.2.1 and 2.2.7].

(i) The family $\{x_i\}_{i=1,\dots,d}$ in the above example is called an *ordered basis of* G.

(ii) The p-valued group (G, ω) is said to be *equi-p-valued* if there exists an ordered basis $\{x_i\}$ as above such that

$$\omega(x_i) = \omega(x_j) \quad \text{for all } 1 \leqslant i, j \leqslant d.$$

2.2 Serre's standard groups as examples

Let E be a finite extension of $\mathbb{Q}_p$ with ring of integers R. Let $\mathfrak{m}$ be the maximal ideal of R. Then E is a discretely valued field. We normalize its valuation v by $v(p) = 1$. Let e be the ramification index of $E/\mathbb{Q}_p$.

Any formal group law $F(X, Y)$ in n variables over R defines a group structure G on $\mathfrak{m}^n$. These are the *standard groups* as defined by Serre.

For $(x_1, \dots, x_n) \in \mathfrak{m}^n$, define

$$\omega(x_1, \dots, x_n) := \inf_i \{v(x_i)\}.$$

Then, for $\lambda \geqslant 0$, we have

$$G_\lambda := \{x \in G : \omega(x) \geqslant \lambda\}.$$

Proposition 2.2.1 [Ser65, Part II: Lie Groups, § 4.23, Theorem 1 and Corollary, § 4.25, Corollary 2 of Theorem 2]. *Let G be a standard group. Then G is a pro-p-group and for all $\lambda \geqslant 0$ the group G_λ is a normal subgroup of G. Moreover, the map ω defines a filtration on G in the sense of Definition 2.1.1(i)–(iii).*

One can show that a suitable open subgroup of the standard group G is saturated.

Let ρ denote the smallest integer that is larger than $e/(p-1)$.

Lemma 2.2.2. *Let E and G be as above. Then the subgroup (H, ω), where*

$$H := \left\{x \in G : \omega(x) > \frac{1}{p-1}\right\} = G_{\rho/e},$$

is saturated and of finite rank. It is equi-p-valued if and only if $e = 1$.

Proof. Note first that, according to [Ser65, Part II: Lie Groups, § 4.21, Corollary], the power series f_p which defines the p-power map is of the form

$$f_p(X) = p(X + \varphi(X)) + \psi(X)$$

with $\mathrm{ord}(\varphi) \geqslant 2$ and $\mathrm{ord}(\psi) \geqslant p$. It follows that

$$\omega(x^p) \geqslant \inf\{\omega(x) + 1, p\omega(x)\} \quad \text{for } x \in G,$$

because if x has coordinates x_i, then x^p has coordinates

$$f_{p,i}(x_1, \dots, x_n) = px_i + p\varphi_i(x) + \psi_i(x)$$

and the valuations of the summands are bounded below by $1 + \omega(x)$, $1 + 2\omega(x) > 1 + \omega(x)$ and $p\omega(x)$, respectively.

Since $\omega(x) > 1/(p-1)$ is equivalent to $\omega(x) + 1 < p\omega(x)$, this implies that on H,

$$\omega(x^p) \geqslant \omega(x) + 1.$$

On the other hand, let x_i be a coordinate of x such that $\omega(x) = v(x_i)$. Then

$$\omega(x^p) \leqslant v(px_i + p\phi_i(x) + \psi_i(x)) = 1 + \omega(x_i) = 1 + \omega(x),$$

and hence

$$\omega(x^p) = \omega(x) + 1$$

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for all $x \in H$. This shows that (H, ω) is p-valued. To see that H is saturated, we observe that by [Ser65, Part II: Lie Groups, § 4.26, Theorem 4] the p-power map induces an isomorphism

$$H_\lambda \xrightarrow{\cong} H_{\lambda+1}$$

for all $\lambda \in (1/(p-1), \infty) \cap v(\mathfrak{m})$. Since H is complete, this implies that the group is saturated.

The valuation on R takes values in $(1/e)\mathbb{Z}$, where e is the ramification index. By definition, $H = H_{\rho/e}$. We get

$$\operatorname{gr}(H) \cong \bigoplus_{\lambda \in (1/e)\mathbb{N},\, \lambda \geqslant \rho/e} k^n.$$

As an $\mathbb{F}_p[\epsilon]$-module, $\operatorname{gr}(H)$ is freely generated by an $\mathbb{F}_p$-basis of

$$\bigoplus_{\lambda \in (1/e)\mathbb{N},\, 1+\rho/e > \lambda \geqslant \rho/e} k^n.$$

This is finite because $[k : \mathbb{F}_p] < \infty$.

If $e = 1$, then only a single λ occurs in the sum, namely ρ. If $e > 1$, then $1 + \rho/e > (\rho + 1)/e$ and the sum has generators in more than one degree. □

An important example of the above construction arises from separated smooth group schemes $\mathbb{G}/R$. The formal completion $\widehat{\mathbb{G}}$ of $\mathbb{G}$ along its unit section is a formal group over R, and the associated standard group G is isomorphic, via $g \mapsto 1 + g$, to

$$G \cong \ker(\mathbb{G}(R) \to \mathbb{G}(k)).$$

Example 2.2.3. As an even more concrete example, consider $\mathbb{G} = \mathrm{GL}_n$ over R. Let π be the uniformizer. Then

$$\pi^\rho R = \left\{ x \in R \;\middle|\; v(x) > \frac{1}{p-1} \right\}.$$

It follows from Lemma 2.2.2 that

$$H := 1 + \pi^\rho M_n(R) \subset \mathrm{GL}_n(R)$$

is a saturated subgroup with respect to the filtration $\omega(1 + (x_{i,j})) = \inf_{i,j}(v(x_{i,j}))$. Since

$$\operatorname{gr}(H) \cong \bigoplus_{\lambda \geqslant \rho/e,\, \lambda \in (1/e)\mathbb{N}} M_n(k),$$

the rank of H is $n^2[R : \mathbb{Z}_p]$. Note that this is not equi-p-valued for $e > 1$.

However, we can view $\mathrm{GL}_n(R)$ as the group of $\mathbb{Z}_p$-valued points of the Weil restriction $\mathbb{G}' = \mathrm{Res}_{R/\mathbb{Z}_p}\mathrm{GL}_n$, which is a separated smooth group scheme over $\mathbb{Z}_p$ (see [BLR90, § 7.6, Proposition 5]). This point of view yields a different valuation on the corresponding standard group,

$$G' = \widehat{\mathbb{G}}'(\mathbb{Z}_p) = 1 + pM_n(R) \subseteq \mathrm{GL}_n(R).$$

By Lemma 2.2.2, (G', ω') is saturated and equi-p-valued if $p > 2$.

As an explicit example, choose $R = \mathbb{Z}_p[\pi]$ with $\pi^2 = p$ (hence $e = 2$) and $n = 1$. Take $p > 3$ for simplicity. Then $G = 1 + \pi R$ is of rank two with ordered basis $\{x_1 = 1 + \pi,\ x_2 = 1 + \pi^2\}$ such that

$$\omega(x_1) = \tfrac{1}{2} \quad \text{and} \quad \omega(x_2) = 1.$$

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On the other hand, $G' = 1 + pR$ also has rank two, with $\{x'_1 := 1 + p,\ x'_2 := 1 + \pi^3\}$ as an ordered basis such that

$$\omega'(x'_1) = \omega'(x'_2) = 1$$

Thus G' is saturated and equi-p-valued. Compare this with

$$\omega(x'_1) = 1 \quad \text{and} \quad \omega(x'_2) = \tfrac{3}{2}$$

under the inclusion $G' \subset G$.

2.3 Valued rings, modules and the functor Sat

Definition 2.3.1 [Laz65, ch. I, Definitions 2.1.1 and 2.2.1]. A *filtered ring* Ω is a ring together with a map

$$v : \Omega \to \mathbb{R}_+ \cup \{\infty\}$$

such that for any $\lambda, \mu \in \Omega$:

(i) $v(\lambda - \mu) \geqslant \min(v(\lambda), v(\mu))$;

(ii) $v(\lambda\mu) \geqslant v(\lambda) + v(\mu)$;

(iii) $v(1) = 0$.

Put

$$\Omega_\nu := \{\lambda \in \Omega \mid v(\lambda) \geqslant \nu\}.$$

A filtered ring Ω is said to be *valued* if, in addition to the above conditions, the following hold:

(ii′) $v(\lambda\mu) = v(\lambda) + v(\mu)$;

(iv) the topology defined on Ω by the filtration Ω_ν is separated.

Definition 2.3.2 [Laz65, ch. I, §§ 2.1.3 and 2.2.2]. A *filtered module* M over a filtered ring Ω is an Ω-module M together with a map

$$w : M \to \mathbb{R}_+ \cup \{\infty\}$$

such that for any $x, y \in M$ and $\lambda \in \Omega$:

(i) $w(x - y) \geqslant \min(w(x), w(y))$;

(ii) $w(\lambda x) \geqslant v(\lambda) + w(x)$.

Put

$$M_\nu := \{x \in M \mid w(x) \geqslant \nu\}.$$

A filtered module over a valued ring Ω is said to be *valued* if, in addition to the above conditions, the following hold:

(ii′) $w(\lambda x) = v(\lambda) + w(x)$;

(iii) the topology defined on M by the filtration M_ν is separated.

Let Ω be a commutative valued ring and let A be an Ω-algebra (for instance, a Lie algebra).

Definition 2.3.3 [Laz65, ch. I, § 2.2.4]. An Ω-algebra A over a commutative valued ring Ω is *valued* if it is valued as a ring and (with the same valuation map) valued as an Ω-module.

The following definition is an important technical tool in Lazard's work.

DEFINITION 2.3.4 [Laz65, ch. I, § 2.2.7]. A valued module M over a commutative valued ring Ω is said to be *divisible* if for all $\lambda \in \Omega$ and $x \in M$ with $v(\lambda) \leqslant w(x)$ there exists $y \in M$ such that $\lambda y = x$. The module M is *saturated* if it is divisible and complete.

Lazard showed in [Laz65, ch. I, § 2.2.10] that the completion of a divisible module is saturated.

DEFINITION 2.3.5 [Laz65, ch. I, § 2.2.11]. The *saturation* Sat M of a valued module M over a commutative valued ring Ω is the completion of

$$\operatorname{div} M := \{y \in K \otimes_\Omega M \mid \tilde{w}(y) \geqslant 0\}.$$

Here, K is the fraction field of Ω and the valuation w on M is extended to a map $\tilde{w}$ on $K \otimes_\Omega M$ by $\tilde{w}(\lambda^{-1} \otimes m) := w(m) - v(\lambda)$ (which is well-defined; see [Laz65, ch. I, § 2.2.8]).

The saturation Sat M satisfies the following universal property [Laz65, ch. I, § 2.2.11]: for any morphism $f : M \to N$ of M into a saturated Ω-module N, there is a unique extension to a map Sat $M \to N$.

2.4 Group rings

In this section we fix $\Omega = \mathbb{Z}_p$ with the standard valuation. All algebras are over $\mathbb{Z}_p$.

For any group G, let $\mathbb{Z}_p[G]$ be the group ring with coefficients in $\mathbb{Z}_p$.

DEFINITION 2.4.1 [Laz65, ch. II, Definition 2.2.1]. Let G be a pro-p-group. The *completed group ring* $\mathbb{Z}_p[[G]]$ is the projective limit

$$\mathbb{Z}_p[[G]] := \varprojlim \mathbb{Z}_p[G/U],$$

where U runs through all open normal subgroups of G and every $\mathbb{Z}_p[G/U]$ carries the p-adic topology.

In [Laz65] this ring is denoted by AlG.

DEFINITION 2.4.2 [Laz65, ch. III, Definition 2.3.1.2]. Let G be a p-filtered group. The *induced filtration* w on $\mathbb{Z}_p[G]$ is the lower bound for all filtrations (as a $\mathbb{Z}_p$-algebra) such that

$$w(x-1) \geqslant \omega(x) \quad \text{for all } x \in G.$$

PROPOSITION 2.4.3 [Laz65, ch. III, Theorem 2.3.3]. *Let G be p-valued. Then the induced filtration w on $\mathbb{Z}_p[G]$ is a valuation (as a $\mathbb{Z}_p$-module). If G is compact (or, equivalently, pro-p), then $\mathbb{Z}_p[[G]]$ is the completion of $\mathbb{Z}_p[G]$ with respect to the valuation topology.*

Example 2.4.4 [Laz65, ch. V, § 2.2.1]. Let G be p-valued, complete and of finite rank d. Let $\{x_i\}_{i=1,\dots,d} \subset G$ be an ordered basis of G. Then $\mathbb{Z}_p[[G]]$ admits $\{z^\alpha \mid \alpha \in \mathbb{N}^d\} \subseteq \mathbb{Z}_p[[G]]$,

$$z^\alpha := \prod_{i=1}^{d} (x_i - 1)^{\alpha_i},$$

as a topological $\mathbb{Z}_p$-basis satisfying

$$w(z^\alpha) = \sum_{i=1}^{d} \alpha_i \omega(x_i).$$

The associated graded is $U_{\mathbb{F}_p[\epsilon]}\mathrm{gr}(G)$, the universal enveloping algebra of the $\mathbb{F}_p[\epsilon]$-Lie algebra $\mathrm{gr}(G)$.

Remark 2.4.5. Note that if (G, ω) is saturated and non-trivial, then $\mathbb{Z}_p[[G]]$ is never saturated. Indeed, since $\mathrm{gr}(G) \neq 0$ is a free $\mathbb{F}_p[\epsilon]$-module, we have $\mathrm{gr}^\nu G \neq 0$ for arbitrarily large ν; in particular, there exists $g \in G$ with $\omega(g) \geqslant 1$. Then $x := g - 1 \in \mathbb{Z}_p[[G]]$ satisfies $w(x) \geqslant 1 = v(p)$, but x is not divisible by p in $\mathbb{Z}_p[[G]]$.

Lemma 2.4.6. *The inclusion* $\mathbb{Z}_p[G] \to \mathbb{Z}_p[[G]]$ *induces an isomorphism*

$$\mathrm{Sat}\, \mathbb{Z}_p[G] \cong \mathrm{Sat}\, \mathbb{Z}_p[[G]].$$

Proof. By [Laz65, ch. I, § 2.2.2], the natural map $\mathbb{Z}_p[G] \to \mathbb{Z}_p[[G]]$ is injective; it extends to

$$\mathrm{Sat}\, \mathbb{Z}_p[G] \to \mathrm{Sat}\, \mathbb{Z}_p[[G]].$$

On the other hand, $\mathrm{Sat}\, \mathbb{Z}_p[G]$ is complete, hence there is a natural map $\mathbb{Z}_p[[G]] \to \mathrm{Sat}\, \mathbb{Z}_p[G]$. As the right-hand side is saturated, it extends to

$$\mathrm{Sat}\, \mathbb{Z}_p[[G]] \to \mathrm{Sat}\, \mathbb{Z}_p[G].$$

The two maps are inverses of each other. □

2.5 Enveloping algebras

Let L be a valued $\mathbb{Z}_p$-Lie algebra and let UL be its enveloping algebra over $\mathbb{Z}_p$.

Definition 2.5.1 [Laz65, ch. IV, § 2.2.1]. The *canonical filtration*

$$w : UL \to \mathbb{R}_+ \cup \{\infty\}$$

is the lowest bound for all filtrations on UL, turning it into a valued $\mathbb{Z}_p$-algebra such that the canonical map $L \to UL$ is a morphism of valued modules.

Lemma 2.5.2 [Laz65, ch. IV, Corollary 2.2.5]. *The enveloping algebra UL equipped with the canonical filtration is a valued* $\mathbb{Z}_p$*-algebra, and the natural morphism*

$$U\mathrm{gr}(L) \to \mathrm{gr}(UL)$$

is an isomorphism.

2.6 Group-like and Lie-algebra-like elements

Everything in this section applies to $A = \mathrm{Sat}\, \mathbb{Z}_p[[G]]$ where G is a p-valued pro-p-group. We fix $\Omega = \mathbb{Z}_p$ with its standard valuation.

Definition 2.6.1 [Laz65, ch. IV, Definition 1.3.1]. Let A be a valued $\mathbb{Z}_p$-algebra with diagonal

$$\Delta : A \to \mathrm{Sat}(A \otimes_{\mathbb{Z}_p} A)$$

(see [Laz65, ch. IV, Definition 1.2.3]) and augmentation ϵ. Then $\mathcal{G}$, $\mathcal{L}$, $\mathcal{G}^*$ and $\mathcal{L}^*$ are defined by:

(i) $$\mathcal{G} = \{x \in A \mid \epsilon(x) = 1, \Delta(x) = x \otimes x\};$$

(ii) $$\mathcal{G}^* = \{x \in \mathcal{G} \mid w(x) > (p-1)^{-1}\};$$

(iii) $$\mathcal{L} = \{x \in A \mid \Delta(x) = x \otimes 1 + 1 \otimes x\};$$

(iv) $$\mathcal{L}^* = \{x \in \mathcal{L} \mid w(x) > (p-1)^{-1}\}.$$

These subsets have the following structures.

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Lemma 2.6.2 [Laz65, ch. IV, paragraphs 1.3.2.1 and 1.3.2.2]. *The subsets $\mathcal{G}$ and $\mathcal{G}^*$ are monoids with respect to the multiplication of A. If A is complete, then $\mathcal{G}^*$ is a group and $\mathcal{L}$ and $\mathcal{L}^*$ are Lie algebras. Moreover,* $\mathcal{L} = \operatorname{div} \mathcal{L}^*$.

When A is a saturated $\mathbb{Z}_p$-algebra, much more is known.

Theorem 2.6.3 [Laz65, ch. IV, Theorem 1.3.5]. *Let A be a saturated $\mathbb{Z}_p$-algebra with diagonal.*

- (i) *The exponential maps $\mathcal{G}^*$ to $\mathcal{L}^*$, and the logarithm maps $\mathcal{L}^*$ to $\mathcal{G}^*$; they are inverse homeomorphisms.*
- (ii) *The Lie algebra $\mathcal{L}$ is saturated; it is the saturation of $\mathcal{L}^*$.*
- (iii) *The subset $\mathcal{G}^*$ is a saturated group for the filtration $\omega(x) = w(x-1)$.*
- (iv) *The associated graded* $\operatorname{gr} \mathcal{L}^*$ *and* $\operatorname{gr} \mathcal{G}^*$ *are canonically isomorphic via the logarithm map.*
- (v) *The subsets $\mathcal{L}^*$ and $\mathcal{G}^*$ generate the same saturated associative subalgebra of A.*

This theorem has the following consequence for the universal enveloping algebra UL of a valued Lie algebra L.

Theorem 2.6.4 [Laz65, ch. IV, Lemma 3.1.2 and Theorem 3.1.3]. *Let L be a valued Lie algebra over $\mathbb{Z}_p$ and UL its universal enveloping algebra. Then*

$$\operatorname{Sat} UL = \operatorname{Sat} U \operatorname{Sat} L, \tag{1}$$

$$\mathcal{L} \operatorname{Sat} UL = \operatorname{Sat} L, \tag{2}$$

$$\mathcal{G} \operatorname{Sat} UL = \mathcal{G}^* \operatorname{Sat} UL. \tag{3}$$

The next result concerns the saturation of the group ring $\mathbb{Z}_p[G]$ (or, equivalently, of $\mathbb{Z}_p[[G]]$, owing to Lemma 2.4.6).

Theorem 2.6.5 [Laz65, ch. IV, Theorem 3.2.5]. *Let G be a saturated group and let* $A = \operatorname{Sat} \mathbb{Z}_p[G]$; *then*

$$\mathcal{G}^* = G.$$

Let $U\mathcal{L}$ be the universal enveloping algebra of $\mathcal{L}$; then the canonical map

$$\operatorname{Sat} U\mathcal{L} \to \operatorname{Sat} \mathbb{Z}_p[G]$$

is an isomorphism.

We introduce some new terminology.

Definition 2.6.6. Let G be a saturated group. We shall call

$$\mathcal{L}^*(G) = \mathcal{L}^* \subset \operatorname{Sat} \mathbb{Z}_p[G]$$

the *integral Lazard Lie algebra* of G.

The preceding theorem then reads

$$\operatorname{Sat} U\mathcal{L}^*(G) \cong \operatorname{Sat} \mathbb{Z}_p[G].$$

Example 2.6.7. Consider the saturated group $H = 1 + \pi^\rho M_n(R)$ from Example 2.2.3 and the algebra $\operatorname{Sat} \mathbb{Z}_p[H]$. We claim that the Lie algebra $\mathcal{L}^* = \mathcal{L}^*(H)$ is $\pi^\rho M_n(R)$ and that $\mathcal{L} = M_n(R)$. To see this, note that by Theorem 2.6.5 we have $H = \mathcal{G}^*$ and, by Theorem 2.6.3, $\mathcal{L}^*$ consists of the logarithms of $\mathcal{G}^*$. By [Laz65, ch. III, §§ 1.1.4 and 1.1.5], the maps

$$\mathrm{Log}: 1+\pi^\rho M_n(R) \to \pi^\rho M_n(R), \quad 1+x \mapsto \sum_{n\geqslant 1}(-1)^{n+1}\frac{x^n}{n} \tag{4}$$

and

$$\exp: \pi^\rho M_n(R) \to 1+\pi^\rho M_n(R), \quad x \mapsto \sum_{n\geqslant 0}\frac{x^n}{n!} \tag{5}$$

are both convergent and are inverses of each other. By Theorem 2.6.3, $\mathcal{L}$ is the saturation of $\mathcal{L}^*$ and is, by Definition 2.3.5,

$$\mathcal{L} = \{x \in K \otimes_R \pi^\rho M_n(R) \mid w(x) \geqslant 0\} = M_n(R).$$

Example 2.6.8. In general, the Lazard Lie algebra does not coincide with the algebraic Lie algebra. Let $\mathbb{G}$ be a separated smooth group scheme over $\mathbb{Z}_p$ and $\mathrm{Lie}(\mathbb{G})$ its $\mathbb{Z}_p$-Lie algebra. If $t_1, \ldots, t_n$ are formal coordinates of $\mathbb{G}$ around e, then $\partial/\partial t_1, \ldots, \partial/\partial t_n$ constitute a $\mathbb{Z}_p$-basis for $\mathrm{Lie}(\mathbb{G})$.

Let G be the associated standard group over $\mathbb{Z}_p$, as in § 2.2. Let H be the saturated subgroup of G; see Lemma 2.2.2. We have $H = G = (p\mathbb{Z}_p)^n$ if $p \neq 2$ and $H = (4\mathbb{Z}_2)^n$ if $p = 2$. Let $x_1, \ldots, x_n$ be the standard ordered basis of H. We put

$$\delta_i = \log x_i \in \mathrm{Sat}\, \mathbb{Z}_p[[H]].$$

By [Laz65, ch. IV, Lemma 3.3.6], the δ_i form a $\mathbb{Z}_p$-basis of $\mathcal{L}^*(H)$. As explained in [HK06, §§ 4.2 and 4.3], they can be viewed as derivations on $\mathbb{Z}_p[[t_1, \ldots, t_n]]$, the coordinate ring of $\hat{\mathbb{G}}$. Note, however, that the coordinate λ_i in [HK06] takes values in all of $\mathbb{Z}_p$ on G. Hence $\lambda_i = pt_i$ for $p \neq 2$. This implies

$$\delta_i = p\frac{\partial}{\partial t_i}\bigg|_{t=0}.$$

Hence, under the identification of [HK06, Proposition 4.3.1], we have

$$\mathcal{L}^*(H) = p\,\mathrm{Lie}(\mathbb{G}).$$

For $p = 2$, the argument gives

$$\mathcal{L}^*(H) = 4\,\mathrm{Lie}(\mathbb{G}).$$

2.7 Resolutions and cohomology

Definition 2.7.1 [Laz65, ch. I, §§ 2.1.16 and 2.1.17]. Let A be a filtered $\mathbb{Z}_p$-algebra and M a filtered A-module.

(i) A family of A-linearly independent elements $(x_i)_{i\in I}$ of M is said to be *filtered free* if for every family $(\lambda_i)_{i\in I}$ of elements of A that are almost all zero, we have

$$w\bigg(\sum_{i\in I}\lambda_i x_i\bigg) = \inf_i (w(x_i) + v(\lambda_i)).$$

We say that M is *filtered free* if it is generated by a filtered free family.

(ii) Suppose that A is complete. We say that M is *complete free* if it is the completion of the submodule generated by a filtered free family.

If A is complete and M is filtered free of finite rank, then M is also complete free.

Definition 2.7.2 [Laz65, ch. V, §§ 1.1.3, 1.1.4 and 1.1.7]. Let A be a filtered augmented $\mathbb{Z}_p$-algebra and M a filtered A-module.

(i) A *filtered acyclic resolution* $X_\bullet$ is a chain complex of filtered A-modules together with an augmentation $\epsilon : X_\bullet \to M$ such that for all $\nu \in \mathbb{R}_+$, the morphism $\epsilon_\nu : X_{\bullet\nu} \to M_\nu$ is a quasi-isomorphism.

(ii) A *split filtered resolution* $X_\bullet$ of M is a morphism $\epsilon : X_\bullet \to M$ of chain complexes of filtered A-modules together with filtered morphisms

$$\eta : M \to X_0, \quad s_n : X_n \to X_{n+1}$$

of $\mathbb{Z}_p$-modules (*sic*, not A-linear!) which define a homotopy between id and 0 on the extended complex $X_\bullet \xrightarrow{\epsilon} M$ and are such that $s_0\eta = 0$. Note that a split filtered resolution is a filtered acyclic resolution.

(iii) Let A be complete. A filtered acyclic resolution $X_\bullet$ by complete free modules will be called a *complete free acyclic resolution* of M.

(iv) Let $X_\bullet$ be a complete free acyclic resolution of the trivial A-module $\mathbb{Z}_p$, and let M be a complete A-module with linear topology. We call

$$H_c^n(A, M) = H^n(\mathrm{Hom}_c(X_\bullet, M))$$

(with Hom_c denoting continuous A-linear maps) the *nth continuous cohomology* of A with coefficients in M.

(v) Let A be an augmented $\mathbb{Z}_p$-algebra and M an A-module. We call

$$H^n(A, M) = \mathrm{Ext}_A^n(\mathbb{Z}_p, M)$$

the *nth cohomology* of A with coefficients in M.

3. An integral version of the Lazard isomorphism

The purpose of this section is to establish that continuous group cohomology and Lie algebra cohomology agree with integral coefficients, at least under certain technical assumptions. This generalizes Lazard's result for coefficients in $\mathbb{Q}_p$-vector spaces.

3.1 Results

We fix a saturated and compact group (G, ω) of finite rank d. In particular, G is a pro-p-group by Proposition 2.1.2. We assume that:

- (G, ω) is equi-p-valued;
- ω takes values in $(1/e)\mathbb{Z}$.

Recall that the integral Lazard Lie algebra

$$\mathcal{L}^*(G) = \mathcal{L}^* \operatorname{Sat} \mathbb{Z}_p[[G]]$$

is a finite free $\mathbb{Z}_p$-Lie algebra.

For technical reasons, we fix a totally ramified extension $\mathbb{Q}_p \subseteq K$ of degree e with ring of integers $\mathcal{O} \subseteq K$ and uniformizer $\pi \in \mathcal{O}$. The valuation on $\mathcal{O}$ is normalized by $v(p) = 1$.

Let M be a linearly topologized complete $\mathbb{Z}_p$-module with a continuous, $\mathbb{Z}_p$-linear action of G. Then M is a $\mathbb{Z}_p[[G]]$-module [Laz65, ch. II, Theorem 2.2.6]. We assume that:

- the $\mathbb{Z}_p[[G]]$-module structure on M extends to a $\operatorname{Sat} \mathbb{Z}_p[[G]]$-module structure.

Thus, M is canonically an $\mathcal{L}^*(G)$-module.

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In § 3.4 we will prove the following result.

Theorem 3.1.1. *Let (G, ω) and M be as above. Then the following hold.*

(i) *There is an isomorphism*

$$\phi_G(M) : H_c^*(G, M) \otimes_{\mathbb{Z}_p} \mathcal{O} \simeq H^*(\mathcal{L}^*(G), M) \otimes_{\mathbb{Z}_p} \mathcal{O}$$

of graded $\mathcal{O}$-modules; it is natural in M.

(ii) *If, in addition, M is a $\mathbb{Q}_p$-vector space, then the above isomorphism agrees with the one in [Laz65, ch. V, Theorem 2.4.9].*

(iii) *Let H be another group satisfying the assumptions of the theorem, and let $f : G \to H$ be a group homomorphism filtered for the chosen filtrations. In addition, assume that $\mathrm{gr}(H)$ is generated in degree $1/e$. Then the isomorphism is natural with respect to f.*

(iv) *If $\mathrm{gr}(G)$ has generators in degree $1/e$, then the isomorphism is compatible with cup-products as follows. Assume that M' and M'' satisfy the same assumptions as M does and that*

$$\alpha : M \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} M' \to M''$$

is $\mathrm{Sat}\, \mathbb{Z}_p[[G]]$-linear. Then the diagram

$$\begin{array}{ccc}
(H_c^*(G, M) \otimes_{\mathbb{Z}_p} H_c^*(G, M')) \otimes \mathcal{O} & \longrightarrow & H_c^*(G, M'') \otimes \mathcal{O} \\
\Big\downarrow{\scriptstyle \phi_G(M) \otimes \phi_G(M')} & & \Big\downarrow{\scriptstyle \phi_G(M'')} \\
(H^*(\mathcal{L}^*(G), M) \otimes_{\mathbb{Z}_p} H^*(\mathcal{L}^*(G), M')) \otimes \mathcal{O} & \longrightarrow & H_c^*(\mathcal{L}^*(G), M'') \otimes \mathcal{O}
\end{array}$$

commutes. Here, the horizontal maps are the $\mathcal{O}$-linear extensions of the cup-product defined by α.

Remark 3.1.2.

(i) If $H^*(\mathcal{L}^*(G), M)$ is a finitely generated $\mathbb{Z}_p$-module, e.g. when M is of finite type, then by the structure of finitely generated modules over principal ideal domains this implies the existence of an isomorphism of graded $\mathbb{Z}_p$-modules

$$H_c^*(G, M) \simeq H^*(\mathcal{L}^*(G), M).$$

However, it is not clear whether this isomorphism is natural or compatible with cup-products.

(ii) According to [Laz65, ch. V, §§ 2.2.6.3 and 2.2.7.2], the mod-p cohomology of an equi-p-valued group G is simply an exterior algebra

$$H_c^*(G, \mathbb{F}_p) = \Lambda_{\mathbb{F}_p}^*(H_c^1(G, \mathbb{F}_p)).$$

The cohomology with torsion-free coefficients is, however, more interesting; for example, if G is not abelian, then the $\mathbb{Q}_p$-Betti numbers of G are different from the $\mathbb{F}_p$-Betti numbers, showing that $H_c^*(G, \mathbb{Z}_p)$ contains non-trivial torsion.

It is not obvious which groups satisfy the assumptions of Theorem 3.1.1. We discuss in § 3.3 standard groups and uniform pro-p-groups which satisfy the assumptions of Theorem 3.1.1. Section 3.2 discusses the assumptions by means of some examples.

3.2 Some examples illustrating the assumptions of Theorem 3.1.1

In this section we illustrate the assumptions of Theorem 3.1.1 by a series of remarks and examples.

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The integral Lazard isomorphism may not hold for all topologically finitely generated pro-p-groups without p-torsion. However, the assumptions of the theorem are actually too restrictive.

Example 3.2.1. Assume $p \geqslant 5$, and let $D/\mathbb{Q}_p$ be the quaternion algebra, $\mathcal{O} \subseteq D$ its maximal order and $\Pi \in \mathcal{O}$ a prime element. Using [Laz65, ch. II, § 1.1.9], one can check that

$$G := 1 + \Pi\mathcal{O} \subseteq \mathcal{O}^*$$

is p-saturated. From [Rav86, Theorem 6.3.22] or [Hen07, Proposition 7], we know that

$$\dim_{\mathbb{F}_p} H_c^i(G, \mathbb{F}_p) = 1, 3, 4, 3, 1 \quad \text{for } i = 0, 1, 2, 3, 4, \text{ respectively.}$$

In particular, $H_c^*(G, \mathbb{F}_p) \neq \Lambda^* H_c^1(G, \mathbb{F}_p)$ and G does not admit an equi-p-valuation by [Laz65, ch. V, §§ 2.2.6.3 and 2.2.7.2]. However, by direct arguments one can establish an isomorphism

$$H_c^*(G, \mathbb{F}_p) \simeq H^*(\mathcal{L}^*(G), \mathbb{F}_p)$$

of graded $\mathbb{F}_p$-algebras; see Remark 3.4.10. The proof of [Hen07, Proposition 7] shows that the same result holds for coefficients in $\mathbb{Z}_p$.

In fact, not even saturatedness is necessary.

Example 3.2.2. Let $G = 1 + p^2\mathbb{Z}_p$ for $p \neq 2$. This group is not saturated for the obvious filtration; rather, we have

$$\mathrm{Sat}(G) = 1 + p\mathbb{Z}_p.$$

Put

$$\mathcal{L}^*(G) \subset \mathcal{L}^*(\mathrm{Sat}(G)),$$

the image of G under the logarithm map. We still get an isomorphism

$$H^*(G, \mathbb{Z}_p) \to H^*(\mathcal{L}^*(G), \mathbb{Z}_p)$$

induced by the logarithm; it is compatible with the one for $\mathrm{Sat}(G)$.

Remark 3.2.3.

(i) We are unaware of a group-theoretic characterization of those pro-p-groups that satisfy the assumptions of Theorem 3.1.1, but [ST03, Remark on p. 163] suggests that they are closely related to uniform pro-p-groups.

(ii) In general, it is difficult to decide whether a given $\mathbb{Z}_p[[G]]$-module structure extends over $\mathrm{Sat}\,\mathbb{Z}_p[[G]]$; we refer to [Tot99, p. 200], and especially to the proof of [Tot99, Corollary 9.3], for further discussion and useful sufficient conditions.

(iii) In Theorem 3.3.3 we establish a sufficient condition for both problems to be addressed here.

There are examples of groups that are saturated with respect to one filtration but not with respect to another. It can also happen that the group is saturated with respect to two filtrations but equi-p-valued for only one of them.

Example 3.2.4. Let $K/\mathbb{Q}_p$ be a finite extension with ramification index e. Let $\mathcal{O}$ be its ring of integers with uniformizer π. As discussed in Example 2.2.3, the group

$$1 + pM_n(\mathcal{O})$$

carries two natural filtrations ω and ω'. Recall that ρ is the smallest integer bigger than $e/(p-1)$.

(i) If $p = 5$ and $e = 2$, then $\rho = 1$ and hence $\pi^\rho \neq 5$. This implies that $1 + 5M_n(\mathcal{O})$ is saturated with respect to ω' but not with respect to ω.

(ii) If $p = 3$ and $e = 2$, then $\rho = 2$ and hence $\pi^\rho = 3$. The group $1 + 3M_n(\mathcal{O})$ is saturated with respect to both ω and ω' but equi-3-valued only with respect to ω'.

3.3 Standard groups and uniform pro-p-groups

We now discuss two examples in which the assumptions of Theorem 3.1.1 are satisfied. We consider first standard groups and then uniform pro-p-groups.

Example 3.3.1. Let $\mathbb{G}/\mathbb{Z}_p$ be a separated smooth group scheme and $G = \ker(\mathbb{G}(\mathbb{Z}_p) \to \mathbb{G}(\mathbb{F}_p))$ the associated standard group (see § 2.2); its filtration takes values in $\mathbb{Z}$. By Lemma 2.2.2, there is an open subgroup H of G which is saturated and equi-p-valued. If $p \neq 2$, then $H = G$ and the generators have degree one. If $p = 2$, then the generators have degree two. So H satisfies the assumptions of the theorem with $e = 1$.

Let $M = \mathbb{Z}_p$ with the trivial operation of H; then it also satisfies the assumptions of the theorem. Hence there is a natural isomorphism of graded $\mathbb{Z}_p$-modules,

$$H^*_c(H, \mathbb{Z}_p) \simeq H^*(\mathcal{L}^*(H), \mathbb{Z}_p).$$

For $p \neq 2$ this is compatible with cup-products.

This example generalizes to a larger class of groups. First, let us recall the notion of a *uniform* or *uniformly powerful* pro-p-group from [DdMS91, Definitions 3.1 and 4.1].

Definition 3.3.2. A pro-p-group G is *uniform* if:

(i) G is topologically finitely generated;

(ii) for $p \neq 2$ (respectively, $p = 2$), $G/\overline{G^p}$ (respectively, $G/\overline{G^4}$) is abelian;

(iii) writing $G = G_1 \supseteq G_2 \supseteq \cdots$ for the lower p-series of G, we have $[G_i : G_{i+1}] = [G_1 : G_2]$ for all $i \geqslant 2$.

To understand what is special about $p = 2$ here, note that the pro-2-group $\mathbb{Z}_2^* = 1 + 2\mathbb{Z}_2$ is not uniform while $1 + 4\mathbb{Z}_2$ is.

The Lie algebra $\mathfrak{g}$ of a uniform pro-p-group G was constructed in [DdMS91, § 8.2] and coincides with the integral Lazard Lie algebra $\mathcal{L}^*(G)$ by [DdMS91, Lemma 8.14].

Theorem 3.3.3. *Take a prime $p \neq 2$ (respectively, $p = 2$), and let G be a uniform pro-p-group and M a finite free $\mathbb{Z}_p$-module with a continuous action of G such that the resulting group homomorphism*

$$\varrho : G \longrightarrow \mathrm{Aut}_{\mathbb{Z}_p}(M)$$

has image in $1 + p\,\mathrm{End}_{\mathbb{Z}_p}(M)$ (respectively, in $1 + 4\,\mathrm{End}_{Z_2}(M)$). Then M is canonically a module for the Lie algebra $\mathfrak{g}$ of G and there is an isomorphism

$$H^*_c(G, M) \simeq H^*(\mathfrak{g}, M) \tag{6}$$

of graded $\mathbb{Z}_p$-modules which, in the case of $p \neq 2$, is compatible with cup-products whenever these are defined.

Remark 3.3.4. If G is an arbitrary $\mathbb{Q}_p$-analytic group acting continuously on the finite free $\mathbb{Z}_p$-module M, then there are arbitrarily small open subgroups $U \subseteq G$ such that the action of each U on M satisfies the assumptions of Theorem 3.3.3.

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Proof. We have the following two claims for $p \neq 2$ (respectively, for $p = 2$).

(i) The group G admits a valuation ω for which it is p-saturated of finite rank and equi-p-valued with an ordered basis consisting of elements of filtration 1 (respectively, of filtration 2).

(ii) The $\mathbb{Z}_p$-module M admits a valuation w for which it is saturated and such that for all $g \in G$ and $m \in M$, $w((g-1)m) \geqslant w(m) + \omega(g)$.

Granting these claims, we see as in [Tot99, pp. 200–201] that the $\mathbb{Z}_p[[G]]$-module structure of M extends over Sat $\mathbb{Z}_p[[G]]$; hence we obtain (6) by applying Theorem 3.1.1(i) with $\mathcal{O} = \mathbb{Z}_p$ and observing that $\mathfrak{g} = \mathcal{L}^*((G, \omega))$. The proof of claim (i) is essentially given in [ST03, Remark on p. 163], but we include the details here for the reader's convenience. The lower p-series

$$G = G_1 \supseteq G_2 \supseteq \cdots$$

(see [DdMS91, Definition 4.1]) consists of normal subgroups satisfying $(G_n, G_m) \subseteq G_{n+m}$ and $\bigcap_{n \geqslant 1} G_n = \{e\}$ (see [DdMS91, Proposition 1.16]); hence

$$\omega(x) := \sup\{n \in \mathbb{N} \mid x \in G_n\}, \quad x \in G,$$

defines a filtration of G by [Laz65, ch. II, Equation (1.1.2.4)]. Now, [DdMS91, Lemma 4.10] states that for all $n, k \geqslant 1$ the p^nth power map of G is a homeomorphism $G_k \xrightarrow{\cong} G_{k+n}$ and induces bijections $G_k/G_{k+l} \xrightarrow{\cong} G_{k+n}/G_{k+n+l}$ for all $l \geqslant 0$.

When $p \neq 2$, we get from this all the properties of ω in Definition 2.1.1 that we need: property (iv) is trivial since $1 > 1/(p-1)$; as regards property (v), if $x \in G$ has filtration $n = \omega(x)$, then $[x^p] \in G_{n+1}/G_{n+2}$ is non-trivial, i.e. $\omega(x^p) = \omega(x) + 1$; for property (vi), observe that if $x \in G$ satisfies $\omega(x) > 1 + 1/(p-1)$, then $\omega(x) \geqslant 2$ and hence $x \in G^p$.

As G is complete, we see that (G, ω) is p-saturated and, clearly, of finite rank. More precisely, from the above we get that $\operatorname{gr} G$ is $\mathbb{F}_p[\epsilon]$-free on $\operatorname{gr}^1 G$, and thus G is equi-p-valued with an ordered basis consisting of elements of filtration 1. This settles claim (i) in the case where $p \neq 2$.

In the $p = 2$ case, ω satisfies all the conditions in Definition 2.1.1 except (iv), so (G, ω) is, in particular, 2-filtered and $\operatorname{gr} G$ has the structure of a mixed Lie algebra over $\mathbb{F}_2$ (see [Laz65, ch. II, Definition 1.2.5]). Note that the only $\nu \in \mathbb{R}^+$ with $\nu \leqslant 1/(p-1) = 1$ and $\operatorname{gr}_\nu G \neq 0$ is $\operatorname{gr}_1 G$. From Definition 3.3.2(ii) we have $[\operatorname{gr}_1 G, \operatorname{gr}_1 G] = 0$, which easily implies that $\operatorname{gr} G$ is abelian. Since ω takes integer values, this means that

$$\omega([x, y]) \geqslant \omega(x) + \omega(y) + 1 \quad \text{for } x, y \in G.$$

Using this, it is easy to see that in the case where $p = 2$, $\omega' := \omega + 1$ is a filtration of G with the properties stated in claim (i).

As for claim (ii), with p being arbitrary now, we choose a $\mathbb{Z}_p$-basis $\{e_i\} \subseteq M$ and declare it to be a filtered basis with $w(e_i) = 0$; that is,

$$w\left(\sum_i \lambda_i e_i\right) = \inf_i \{v(\lambda_i)\} \quad \text{for } \lambda_i \in \mathbb{Z}_p.$$

Clearly, (M, w) is saturated. Assume $p \neq 2$.

We consider the continuous homomorphism of pro-p-groups

$$\varrho : G \longrightarrow 1 + p \operatorname{End}_{\mathbb{Z}_p}(M) =: G'$$

and claim that the lower-p-series of G' is given by $G'_n = 1 + p^n \operatorname{End}_{\mathbb{Z}_p}(M)$, $n \geqslant 1$. Since G' is powerful, [DdMS91, Lemma 2.4] gives $G'_{n+1} = \Phi(G'_n)$, the Frattini subgroup, for all $n \geqslant 1$;

therefore, arguing inductively, it suffices to show that

$$\Phi(1+p^n \operatorname{End}_{\mathbb{Z}_p}(M)) = 1+p^{n+1}\operatorname{End}_{\mathbb{Z}_p}(M).$$

Since the Frattini subgroup is generated by pth powers and commutators, we have the '$\supseteq$' part; [DdMS91, Proposition 1.16] then gives

$$\begin{aligned}&\Phi(1+p^n \operatorname{End}_{\mathbb{Z}_p}(M))/(1+p^{n+1}\operatorname{End}_{\mathbb{Z}_p}(M))\\ &\quad=\Phi((1+p^n \operatorname{End}_{\mathbb{Z}_p}(M)))/(1+p^{n+1}\operatorname{End}_{\mathbb{Z}_p}(M))\\ &\quad=\Phi((\mathbb{F}_p,+)^{n^2})=0.\end{aligned}$$

Since ϱ respects the lower p-series, we conclude that

$$\varrho(G_n)\subseteq 1+p^n\operatorname{End}_{\mathbb{Z}_p}(M)\quad \text{for } n\geqslant 1,$$

which implies that $w((g-1)m)\geqslant w(m)+\omega(g)$ for all $g\in G$ and $m\in M$.

This settles claim (ii) in the case where $p\neq 2$; the argument in the $p=2$ case is an obvious modification which we will leave to the reader.

Finally, to show compatibility with cup-products, we assume $p\neq 2$ and suppose that M' and M'' satisfy the same assumptions as M does and that

$$\alpha : M\otimes_{\mathbb{Z}_p} M'\to M''$$

is G-linear, defining cup-products in $H^*_c(G,-)$. Then both the source and the target of α are canonically $\operatorname{Sat}\mathbb{Z}_p[[G]]$-modules, as seen above, and α is $\operatorname{Sat}\mathbb{Z}_p[[G]]$-linear. Hence (6) is compatible with cup-products by Theorem 3.1.1(iv). □

3.4 Proof of Theorem 3.1.1

We now describe the set-up for the rest of the section.

Fix a saturated group (G,ω) of finite rank d, and let

$$\mathcal{L}^*(G)=\mathcal{L}^*\operatorname{Sat}\mathbb{Z}_p[[G]]$$

be its integral Lazard Lie algebra; this is a finite free $\mathbb{Z}_p$-module.

We fix an ordered basis $\{x_1,\dots,x_d\}\subseteq G$ and put $\omega_i:=\omega(x_i)$. For every $0\leqslant k\leqslant n$ let

$$\mathcal{I}_k:=\{(i_1,\dots,i_k)\mid 1\leqslant i_1<\cdots<i_k\leqslant n\},$$

and for $I\in\mathcal{I}_k$ write $|I|:=\sum_{s=1}^k\omega_s$. For $I\in\mathcal{I}_0=\emptyset$, we put $|I|=0$ with an abuse of notation.

By assumption there exists an integer $e\geqslant 1$ such that $\omega(G)\subseteq(1/e)\mathbb{Z}$, and fix a totally ramified extension $\mathbb{Q}_p\subseteq K$ of degree e with ring of integers $\mathcal{O}\subseteq K$ and uniformizer $\pi\in\mathcal{O}$. The valuation on $\mathcal{O}$ is normalized by $v(p)=1$. The artificial introduction of $\mathcal{O}$ is a trick invented by Totaro in [Tot99]. In this section, all valued modules and algebras are over $\mathcal{O}$. In particular, the saturation functor is taken in the category of valued $\mathcal{O}$-modules.

The inclusion $\mathbb{Z}_p\subseteq\mathcal{O}$ induces

$$\mathbb{F}_p[\epsilon]=\operatorname{gr}\mathbb{Z}_p\subseteq\operatorname{gr}\mathcal{O}=\mathbb{F}_p[\epsilon_K]$$

where ϵ (respectively, ϵ_K) is the leading term of $p\in\mathbb{Z}_p$ (respectively, of $\pi\in\mathcal{O}$). We have $\epsilon_K^e\in\mathbb{F}_p^*\cdot\epsilon$; in particular, the degree of ϵ_K is $1/e$.

If M is a valued $\mathcal{O}$-module, then $\operatorname{gr}(M)$ is canonically an $\mathbb{F}_p[\epsilon_K]$-module. As pointed out by Totaro in [Tot99, p. 201], it follows directly from the definitions that

$$\operatorname{gr}(\operatorname{Sat}(M))=\big(\operatorname{gr}(M)\otimes_{\mathbb{F}_p[\epsilon_K]}\mathbb{F}_p[\epsilon_K^{\pm 1}]\big)_{\text{degree}\geqslant 0}.$$

Let

$$A := \mathcal{O}[[G]] := \varprojlim_{U \subseteq G \text{ open normal}} \mathcal{O}[G/U]$$

and

$$B := U_{\mathcal{O}}(\mathcal{L}^*(G) \otimes_{\mathbb{Z}_p} \mathcal{O})^\wedge = U_{\mathbb{Z}_p}(\mathcal{L}^*(G))^\wedge \otimes_{\mathbb{Z}_p} \mathcal{O},$$

the completion of the universal enveloping algebra with respect to its canonical filtration. (This filtration is easily seen, by using the Poincaré–Birkhoff–Witt theorem, to be the p-adic filtration; the claimed equality follows because $\mathcal{O}$ is finite free as a $\mathbb{Z}_p$-module.) Finally, by virtue of Theorem 2.6.5, we introduce

$$C := \text{Sat } A \cong \text{Sat } B.$$

Lemma 3.4.1. *We have* $\text{gr}(A) = \text{gr}(B)$ *inside* $\text{gr}(C)$.

Proof. On the one hand,

$$\text{gr}(A) = \text{gr}(\mathbb{Z}_p[[G]] \otimes_{\mathbb{Z}_p} \mathcal{O}) = U_{\mathbb{F}_p[\epsilon]}(\text{gr } G) \otimes_{\mathbb{F}_p[\epsilon]} \mathbb{F}_p[\epsilon_K] = U_{\mathbb{F}_p[\epsilon_K]}(\text{gr } G \otimes_{\mathbb{F}_p[\epsilon]} \mathbb{F}_p[\epsilon_K]);$$

on the other hand,

$$\text{gr}(B) = \text{gr}(U_{\mathbb{Z}_p}(\mathcal{L}^*(G)) \otimes_{\mathbb{Z}_p} \mathcal{O}) = U_{\mathbb{F}_p[\epsilon_K]}(\text{gr } \mathcal{L}^*(G) \otimes_{\mathbb{F}_p[\epsilon]} \mathbb{F}_p[\epsilon_K]).$$

Since $\text{gr } \mathcal{L}^*(G) = \text{gr } G$ by Theorem 2.6.5 and Theorem 2.6.3(iv), the claim follows. □

Remark 3.4.2. Totaro showed in [Tot99, pp. 201–202] that, moreover, for

$$\mathfrak{t} := (\text{gr } G \otimes_{\mathbb{F}_p[\epsilon]} \mathbb{F}_p[\epsilon_K^{\pm 1}])_{\text{degree} \geqslant 0},$$

a finite graded free $\mathbb{F}_p[\epsilon_K]$-Lie algebra with generators in degree zero, we have $\text{gr}(C) = U_{\mathbb{F}_p[\epsilon_K]}(\mathfrak{t})$.

Lemma 3.4.3.

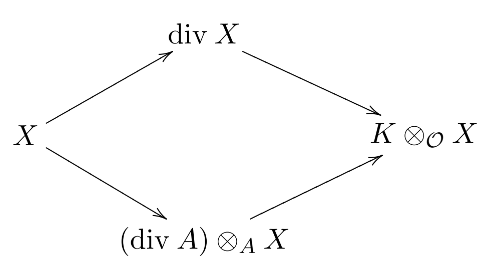
(i) *Let X be a filtered free A-module with A-basis $e_1, \ldots, e_r$. Then* $\text{Sat } X$ *is a filtered free C-module on generators*

$$e'_i = \pi^{-ew(e_i)} e_i, \quad i = 1, \ldots, r.$$

(ii) *Let Y be a filtered free B-module with B-basis $f_1, \ldots, f_s$. Then* $\text{Sat } Y$ *is a filtered free C-module on generators*

$$f'_j = \pi^{-ew(f_j)} f_j, \quad j = 1, \ldots, s.$$

Proof. It suffices to consider the case of the algebra A; the argument for B is the same. Without loss of generality, take $r = 1$. By construction (and because X is torsion-free), we have the following embeddings.



By assumption, any element x of $K \otimes X$ can be written in the form

$$x = \pi^v a e_1 = \pi^{v + ew(e_1)} a e'_1 \in \mathbb{Z} \quad \text{with } a \in A.$$

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It is in div X if and only if

$$w(x) = \frac{v}{e} + w(a) + w(e_1) \geqslant 0.$$

The above is equivalent to $\pi^{v+ew(e_1)}a \in \operatorname{div} A$ and hence to $x \in \operatorname{div}(A)e_1'$. Therefore $\operatorname{div} X = (\operatorname{div} A) \otimes_A X$.

Finally, apply the completion functor to the equality to finish the proof. □

Remark 3.4.4. This is the step where we make use of the coefficient extension to $\mathcal{O}$.

Both A and B are canonically subrings of C, and our first aim is to compare the cohomology of the (abstract) rings A and B with that of C. Both A and B are augmented $\mathcal{O}$-algebras, hence we have an A-module and a B-module structure on $\mathcal{O}$, both of which we will refer to as trivial.

Proposition 3.4.5.

(i) *The trivial A-module $\mathcal{O}$ admits a resolution $X_\bullet$ such that X_k is filtered free of rank $\binom{d}{k}$ over A on generators $\{e_I \mid I \in \mathcal{I}_k\}$ of filtration $w(e_I) = |I|$.*

(ii) *The trivial B-module $\mathcal{O}$ admits a resolution $Y_\bullet$ such that Y_k is filtered free of rank $\binom{d}{k}$ over B on generators $\{f_I \mid I \in \mathcal{I}_k\}$ of filtration $w(f_I) = |I|$.*

(iii) *Furthermore, $X_\bullet$ and $Y_\bullet$ can be chosen so that $\operatorname{gr} X_\bullet = \operatorname{gr} Y_\bullet$ as complexes of $\operatorname{gr}(A)$- or equivalently $\operatorname{gr}(B)$-modules.*

Proof.

(i) The base extension from $\mathbb{Z}_p$ to $\mathcal{O}$ of the quasi-minimal complex of G has the desired properties [Laz65, ch. V, Definition 2.2.2]. To see that the generators have the indicated filtration, remember that the quasi-minimal complex is obtained by lifting the standard complex $\overline{X}_\bullet$ of the $\mathbb{F}_p[\epsilon]$-Lie algebra $\operatorname{gr}(G)$ which has $\overline{X}_k = \Lambda^k_{\mathbb{F}_p[\epsilon]}(\operatorname{gr}(G))$ finite graded free on

$$\{x_{i_1} G^+_{\omega_1} \wedge \cdots \wedge x_{i_k} G^+_{\omega_k}\}.$$

(ii) The Lie algebra $\mathcal{L}^*(G)$ is $\mathbb{Z}_p$-free on generators $\operatorname{Log}(x_i)$ of filtration ω_i. Hence the standard complex of $\mathcal{L}^*(G) \otimes_{\mathbb{Z}_p} \mathcal{O}$ is as desired.

(iii) The equality $\operatorname{gr} X_\bullet = \operatorname{gr} Y_\bullet$ follows from $\operatorname{gr}(G) \cong \operatorname{gr} \mathcal{L}^*(G)$ by construction. □

Example 3.4.6. If G is equi-p-valued, i.e. if $\omega_i = \omega_j$ for all i and j, then $X_\bullet$ and $Y_\bullet$ are *minimal* in the sense of [Laz65, ch. V, § 2.2.5], i.e. $X_\bullet \otimes \mathbb{F}_p$ and $Y_\bullet \otimes \mathbb{F}_p$ have zero differentials.

In the following, we fix complexes $X_\bullet$ and $Y_\bullet$ that satisfy the conclusion of Proposition 3.4.5. Note that C is an augmented $\mathcal{O}$-algebra with augmentation, which extends both the one of A and the one of B.

Lemma 3.4.7. *Both* $\operatorname{Sat} X_\bullet$ *and* $\operatorname{Sat} Y_\bullet$ *are finite filtered resolutions of the trivial C-module $\mathcal{O}$, with the modules* $\operatorname{Sat} X_k$ *(respectively,* $\operatorname{Sat} Y_k$*) being filtered free on generators $\{\pi^{-e|I|} e_I \mid I \in \mathcal{I}_k\}$ (respectively, $\{\pi^{-e|I|} f_I \mid I \in \mathcal{I}_k\}$) of filtration zero over C.*

Proof. Clearly, $\operatorname{Sat} X_\bullet$ and $\operatorname{Sat} Y_\bullet$ are canonically complexes of C-modules. Since both $X_\bullet$ and $Y_\bullet$ admit the structure of a *split* resolution, and this structure is preserved by the additive functor Sat, both $\operatorname{Sat} X_\bullet$ and $\operatorname{Sat} Y_\bullet$ are resolutions of $\operatorname{Sat} \mathcal{O} = \mathcal{O}$.

The statement on generators follows directly from Lemma 3.4.3. □

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For $0 \leqslant k \leqslant n$ and $I \in \mathcal{I}_k$, denote by $e'_I \in \operatorname{Sat} X_k$ and $f'_I \in \operatorname{Sat} Y_k$ the C-generators found above; that is, $e'_I := \pi^{-e|I|} e_I$ and $f'_I := \pi^{-e|I|} f_I$.

We see that the canonical morphisms of complexes over C,

$$C \otimes_A X_\bullet \hookrightarrow \operatorname{Sat} X_\bullet$$

and

$$C \otimes_B Y_\bullet \hookrightarrow \operatorname{Sat} Y_\bullet,$$

are injective.

We pause to remark that, evidently, the above injections are isomorphisms rationally, a key input in Lazard's comparison isomorphism for rational coefficients. Similarly, an integral version of this comparison isomorphism is essentially equivalent to $C \otimes_A X_\bullet$ being isomorphic to $C \otimes_B Y_\bullet$, and we proceed to prove this in a special case as follows.

PROPOSITION 3.4.8. *There exists an isomorphism*

$$\phi : \operatorname{Sat} X_\bullet \to \operatorname{Sat} Y_\bullet$$

of filtered complexes over C such that $\operatorname{gr} \phi = \operatorname{id}$ and $H^0(\phi)$ is the identity of $\mathcal{O}$. Any two such ϕ are chain homotopic where the homotopy h can be chosen so that $\operatorname{gr}(h) = 0$.

Proof. In order to construct the isomorphism it suffices, upon invoking [Laz65, ch. V, Lemma 2.1.5] (which is applicable by Lemma 3.4.7), to canonically identify the complexes $\operatorname{gr} \operatorname{Sat} X_\bullet$ and $\operatorname{gr} \operatorname{Sat} Y_\bullet$ of $\operatorname{gr}(C)$-modules. Recall from Proposition 3.4.5 that

$$\operatorname{gr} X_\bullet = \operatorname{gr} Y_\bullet.$$

This implies

$$\begin{aligned}\operatorname{gr} \operatorname{Sat} X_\bullet &= (\operatorname{gr} X_\bullet \otimes_{\mathbb{F}_p[\epsilon_K]} \mathbb{F}_p[\epsilon_K^{\pm 1}])_{\text{degree} \geqslant 0} \\ &= (\operatorname{gr} Y_\bullet \otimes_{\mathbb{F}_p[\epsilon_K]} \mathbb{F}_p[\epsilon_K^{\pm 1}])_{\text{degree} \geqslant 0} = \operatorname{gr} \operatorname{Sat} Y_\bullet.\end{aligned}$$

Now, $H^0(\phi)$ is an $\mathcal{O}$-linear automorphism of $\mathcal{O}$ and hence given by multiplication with a unit $\alpha \in \mathcal{O}^*$. Using the fact that its associated graded map is the identity, one easily obtains $\alpha = 1$, as claimed.

We turn to the construction of the homotopy. Let ϕ and ϕ' be isomorphisms as above. Let $e'_I \in \operatorname{Sat}(X_0)$ be a basis element. We need to define $h_0(e_I) \in \operatorname{Sat}(Y_0)$ such that

$$dh_0(e'_I) = (\phi - \phi')(e'_I) =: y_I.$$

By assumption, $\operatorname{gr}(\phi - \phi') = 0$ and hence $y_I \in \operatorname{Sat}(Y_0)_{1/e}$. As ϕ and ϕ' are isomorphisms of resolution of $\mathcal{O}$, we have $\epsilon(y_I) = 0$. Recall that $\operatorname{Sat} X_\bullet$ and $\operatorname{Sat} Y_\bullet$ are *filtered* resolutions. Hence y_I has a preimage $\tilde{y}_I \in \operatorname{Sat}(Y_1)_{1/e}$. Put

$$h_0(e'_I) = \tilde{y}_I.$$

By C-linearity, this defines h_0, which then satisfies $\operatorname{gr}(h_0) = 0$. As usual, the same argument can be used inductively to define h_i for all $i \geqslant 0$. ☐

PROPOSITION 3.4.9. *If, in the situation of Proposition 3.4.8, (G, ω) is assumed to be equi-p-valued, then ϕ restricts to an isomorphism*

$$\psi : C \otimes_A X_\bullet \to C \otimes_B Y_\bullet$$

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of complexes over C. If, moreover, $\mathrm{gr}(G)$ is generated in degree $1/e$, then any two such isomorphisms are homotopic.

Proof. We have the following solid diagram of complexes over C.

$$\begin{array}{ccc} C \otimes_A X_\bullet & \overset{\iota_1}{\hookrightarrow} & \mathrm{Sat}\, X_\bullet \\ \Big\downarrow{\scriptstyle \psi} & & \simeq \Big\downarrow{\scriptstyle \phi} \\ C \otimes_B Y_\bullet & \overset{\iota_2}{\hookrightarrow} & \mathrm{Sat}\, Y_\bullet \end{array}$$

Since the horizontal maps are injective, ϕ factors as a chain-map if for every $0 \leqslant k \leqslant n$ we have

$$\phi_k(C \otimes_A X_k) \subseteq C \otimes_B Y_k. \tag{$*$}$$

If ψ exists, it is necessarily an isomorphism by completeness and the fact that its associated graded map is the identity. Alternatively, observe that the following argument applies likewise to ϕ^{-1} to produce an inverse of ψ.

To see what $(*)$ means, fix $0 \leqslant k \leqslant n$ and recall the C-generators $e_I \in C \otimes_A X_k$, $e'_I \in \mathrm{Sat}\, X_k$, $f_I \in C \otimes_B Y_k$ and $f'_I \in \mathrm{Sat}\, Y_k$, where $I \in \mathcal{I}_k$, which satisfy $\iota_1(e_I) = \pi^{e|I|} e'_I$ and $\iota_2(f_I) = \pi^{e|I|} f'_I$. Upon making the expansion

$$\phi_k(e'_I) = \sum_{J \in \mathcal{I}_k} c_{I,J} f'_J \quad \text{with } c_{I,J} \in C,$$

we see, by using the saturatedness of C, that $(*)$ for our fixed k is equivalent to the statement

$$w(c_{I,J}) \geqslant |J| - |I| \quad \text{for all } I, J \in \mathcal{I}_k, \tag{$**$}$$

where w denotes the filtration of C. If (G, ω) is equi-p-valued, the difference on the right-hand side of the inequality in $(**)$ is always zero, so that $(**)$ is trivially true.

By Proposition 3.4.8, any two such ϕ are chain homotopic via a homotopy $h : \mathrm{Sat}\, X_\bullet \to \mathrm{Sat}\, Y_\bullet$ such that $\mathrm{gr}(h) = 0$. It remains to check that this homotopy restricts to a homotopy $h : C \otimes_A X_\bullet \to C \otimes_B Y_\bullet$. We use the same generators as before. The additional assumption that $\mathrm{gr}(G)$ is generated in degree $1/e$ implies that $|I| = k/e$ for $I \in \mathcal{I}_k$.

Consider e'_I for $I \in \mathcal{I}_k$. Then $h_k(e'_I) \in \mathrm{Sat}\, Y_{k+1}$, and it expands as

$$h_k(e'_I) = \sum_{J \in \mathcal{I}_{k+1}} d_{I,J} f'_J \quad \text{with } d_{I,J} \in C.$$

Now $\mathrm{gr}(h) = 0$, hence $\pi | d_{I,J}$ for all I and J. Since $e'_I = \pi^{-k} e_I$ and $f'_J = \pi^{-(k+1)} f_J$, this implies

$$h_k(e_I) = \sum_{J \in \mathcal{I}_{k+1}} d_{I,J} \pi^{-1} f_J$$

with $d_{I,J}\pi^{-1} \in C$ as required. □

Remark 3.4.10. It seems difficult to directly relate the complexes $C \otimes_A X_\bullet$ and $C \otimes_B Y_\bullet$ using the filtration techniques successfully employed, for example, in [ST03, Tot99]; this is essentially because these complexes do not satisfy any reasonable exactness properties.

In fact, we have $H_*(C \otimes_A X_\bullet) = \mathrm{Tor}_*^A(C, \mathcal{O})$ and $H_*(C \otimes_B Y_\bullet) = \mathrm{Tor}_*^B(C, \mathcal{O})$, and one can check that unless $G = \{e\}$, the algebra C is not flat over either A or B.

We have examples of saturated but not equi-p-valued groups and an isomorphism ϕ as above which does not restrict as in Proposition 3.4.9; however, in all these examples it was possible, by inspection, to modify ϕ suitably.

It thus remains a tantalizing open problem to decide whether the assumption of 'equi-p-valued' is superfluous in Proposition 3.4.9. Of course, a positive answer would greatly extend the range of applicability of our integral Lazard comparison isomorphism.

Proof of Theorem 3.1.1. There is a filtration ω of G such that (G, ω) is p-saturated and equi-p-valued of finite rank, with $\omega(G) \subseteq (1/e)\mathbb{Z}$ for some integer $e \geqslant 1$. We are therefore in the situation studied in this subsection; in particular, recall $\mathcal{O}$, A, B, C, $X_\bullet$ and $Y_\bullet$ from above. The continuous group cohomology $H^*_c(G, M)$ is defined using continuous cochains and the Bar differential as in [Laz65, ch. V, § 2.3.1.]. By [Laz65, ch. V, Proposition 1.2.6 and Equation (2.2.3.1)], we have

$$H^*_c(G, M) \simeq H^*_c(\mathbb{Z}_p[[G]], M) \simeq \operatorname{Ext}^*_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p, M)$$

and, analogously,

$$H^*_c(G, M \otimes_{\mathbb{Z}_p} \mathcal{O}) \simeq \operatorname{Ext}^*_A(\mathcal{O}, M \otimes_{\mathbb{Z}_p} \mathcal{O})$$

by the flatness of $\mathcal{O}$ over $\mathbb{Z}_p$. Define $N := M \otimes_{Z_p} \mathcal{O}$. Since $X_\bullet$ is a finite free resolution of $\mathcal{O}$ over A, we obtain

$$\operatorname{Ext}^*_A(\mathcal{O}, N) = H^* \operatorname{Hom}_A(X_\bullet, N) = H^* \operatorname{Hom}_C(C \otimes_A X_\bullet, N),$$

by virtue of the fact that the A-module structure on N extends to a C-module structure, and then

$$\begin{aligned} \cdots &\overset{\text{Proposition 3.4.9}}{\simeq} H^* \operatorname{Hom}_C(C \otimes_B Y_\bullet, N) = H^* \operatorname{Hom}_B(Y_\bullet, N) \\ &\simeq H^*(\mathcal{L}^*(G) \otimes_{\mathbb{Z}_p} \mathcal{O}, N) \simeq H^*(\mathcal{L}^*(G), M) \otimes_{\mathbb{Z}_p} \mathcal{O}, \end{aligned}$$

where the penultimate isomorphism is due to [Tot99, Lemma 9.2]. Summing up, we have the isomorphism

$$H^*_c(G, M) \otimes_{\mathbb{Z}_p} \mathcal{O} \simeq H^*(\mathcal{L}^*(G), M) \otimes_{\mathbb{Z}_p} \mathcal{O} \tag{7}$$

of $\mathcal{O}$-modules.

We now turn to functoriality. Let $f : G \to H$ be a filtered group homomorphism. We write $A(G)$, $C(G)$ and $X_\bullet(G)$, $Y_\bullet(G)$ (respectively, $A(H)$, $C(H)$ and $X_\bullet(H)$, $Y_\bullet(H)$) for the rings A, C and complexes $X_\bullet$, $Y_\bullet$ corresponding to the group G (respectively, H). The group homomorphism induces a commutative diagram as follows.

$$\begin{array}{ccc} \operatorname{gr}(G) & \xrightarrow{\operatorname{gr}(f)} & \operatorname{gr}(H) \\ \downarrow{\cong} & & \downarrow{\cong} \\ \operatorname{gr}(\mathcal{L}^*(H)) & \longrightarrow & \operatorname{gr}(\mathcal{L}^*(H)) \end{array}$$

As in Proposition 3.4.8, this lifts to a diagram of filtered complexes of $\operatorname{Sat} A(G)$-modules

$$\begin{array}{ccc} \operatorname{Sat} X_\bullet(G) & \longrightarrow & \operatorname{Sat} X_\bullet(H) \\ \downarrow{\cong} & & \downarrow{\cong} \\ \operatorname{Sat} Y_\bullet(G) & \longrightarrow & \operatorname{Sat} Y_\bullet(H) \end{array}$$

which commutes up to homotopy and is such that taking gradeds gives back the previous diagram, while taking gradeds of the homotopy yields 0. As in Proposition 3.4.9, the preceding diagram

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restricts to a diagram of filtered complexes of $C(G)$-modules

$$\begin{array}{ccc} C(G)\otimes X_\bullet(G) & \longrightarrow & C(G)\otimes X_\bullet(H) \\ \downarrow \cong & & \downarrow \cong \\ C(G)\otimes Y_\bullet(G) & \longrightarrow & C(G)\otimes Y_\bullet(H) \end{array}$$

which commutes up to homotopy.

Compatibility with cup-products is the case of $\Delta : G \to G \times G$. Note that the generators of $G \times G$ are of the form $(x, 1)$ and $(1, y)$ for generators x and y of G. Their filtration is the same as that of x and y. □

4. The Lazard isomorphism for algebraic group schemes

In this section we give, in the case of p-adic Lie groups arising from algebraic groups, a direct description of a map from analytic group cohomology to Lie algebra cohomology by differentiating cochains. In Proposition 4.2.4 we recall that this map coincides with Lazard's isomorphism. This analytic description directly generalizes to K-Lie groups where K is a finite extension of $\mathbb{Q}_p$. In Theorem 4.3.1 we show that the map is also an isomorphism in the case of K-Lie groups defined by algebraic groups.

4.1 Group schemes

Let p be a prime number, let K be a finite extension of $\mathbb{Q}_p$ and let R be its ring of integers with prime element π. Throughout this section, $\mathbb{G}$ will be a separated smooth group scheme over R and $\mathfrak{g}$ its R-Lie algebra in the following sense:

$$\mathfrak{g} = \mathrm{Lie}(\mathbb{G}) = \mathrm{Der}_R(\mathcal{O}_{\mathbb{G},e}, R).$$

Then $\mathfrak{g}_K := \mathfrak{g} \otimes_R K = \mathrm{Lie}(\mathbb{G}_K)$ is its Lie algebra as a K-manifold.

Note that this category is stable under base change and Weil restriction for finite flat ring extensions $R \to S$. If $A \to B$ is a ring extension with B finite and locally free over A and X is an A-scheme, we write $X_B = X \times_A \mathrm{Spec}\, B$. If Y is a B-scheme, we write $\mathrm{Res}_{B/A} X$ for the Weil restriction, i.e. $\mathrm{Res}_{B/A} X(T) = X(T_B)$ for all A-schemes T. See [BLR90, § 7.6] for properties of the Weil restriction. In particular, if $\mathbb{G}$ is a group scheme over a discrete valuation ring R, then $\mathbb{G}$ is quasi-projective by [BLR90, § 6.4, Theorem 1]. This suffices to guarantee that $\mathrm{Res}_{S/R}(\mathbb{G})$ exists for finite extensions S/R.

The following bit of algebraic geometry will be needed in the proofs.

Lemma 4.1.1. *Let L/K be a finite extension and S the ring of integers of L. Consider a separated smooth group scheme $\mathbb{G}$ over S. Then $\mathbb{G}$ is a direct factor of $\mathrm{Res}_{S/R}(\mathbb{G})_S$.*

Proof. Let X be an S-scheme. For all S-schemes T, we describe T-valued points of $\mathrm{Res}_{S/R}(X)_S$ as follows:

$$\begin{aligned}\mathrm{Mor}_S(T, \mathrm{Res}_{S/R}(X)_S) &= \mathrm{Mor}_R(T, \mathrm{Res}_{S/R}(X)) = \mathrm{Mor}_S(T \times_R \mathrm{Spec}\, S, X) \\ &= \mathrm{Mor}_S(T \times_S \mathrm{Spec}(S \otimes_R S), X).\end{aligned}$$

The natural map $\iota : S \to S \otimes_R S$, $s \mapsto s \otimes 1$, induces the transformation of functors

$$\mathrm{Mor}_S(T \times_S \mathrm{Spec}\, S, X) \xrightarrow{\iota} \mathrm{Mor}_S(T \times_S \mathrm{Spec}(S \otimes_R S), X) = \mathrm{Mor}_S(T, \mathrm{Res}_{S/R}(X)_S)$$

and hence a morphism

$$\iota : X \to \mathrm{Res}_{S/R}(X)_S.$$

This is none other than the adjunction morphism.

The multiplication $\mu_S : S \otimes_R S \to S$ is a section of ι. This again induces a transformation of functors

$$\mathrm{Mor}_S(T, \mathrm{Res}_{S/R}(X)_S) = \mathrm{Mor}_S(T \times_S \mathrm{Spec}(S \otimes_R S), X) \xrightarrow{\mu_S} \mathrm{Mor}_S(T \times_S S, X)$$

and hence a morphism

$$\mu_S : \mathrm{Res}_{S/R}(X)_S \to X.$$

(Put $T = \mathrm{Res}_{S/R}(X)_S$ and the identity on the left.) By construction, μ_S is a section of ι. Both are natural in X; hence, as group schemes, $\mathbb{G}$ is a direct factor of $\mathrm{Res}_{S/R}(\mathbb{G})_S$. □

Remark 4.1.2. If L/K is Galois of degree d, then $\mathrm{Res}_{L/K}(\mathbb{G})_L \cong \mathbb{G}^d$. This carries over to the integral case if the extension is unramified. The assertion becomes false for ramified covers. Note, however, that the weaker statement of the lemma remains true.

4.2 Analytic description of the Lazard morphism

Let $\mathbb{G}$ be a smooth connected group scheme over R with Lie algebra $\mathfrak{g}$. Let $\mathcal{G} \subset \mathbb{G}(R)$ be an open sub-Lie-group.

We denote by $\mathcal{O}_{\mathrm{la}}(\mathcal{G})$ (locally) analytic functions on $\mathcal{G}$, i.e. those that can be locally written as a convergent power series with coefficients in K. Let $H^i_{\mathrm{la}}(\mathcal{G}, K)$ denote (locally) analytic group cohomology, i.e. cohomology of the bar complex $\mathcal{O}_{\mathrm{la}}(\mathcal{G}^n)_{n \geqslant 0}$ with the usual differential. We denote by $H^i(\mathfrak{g}, K)$ Lie algebra cohomology, i.e. cohomology of the complex $\Lambda^*(\mathfrak{g}_K^\vee)$ with differential induced by the dual of the Lie bracket.

Definition 4.2.1. The *Lazard morphism* is the map

$$\Phi : H^i_{\mathrm{la}}(\mathcal{G}, K) \to H^i(\mathfrak{g}, K)$$

induced by the morphism of complexes

$$\mathcal{O}_{\mathrm{la}}(\mathcal{G}^n) \to (\mathfrak{g}_K^n)^\vee \to \Lambda^n \mathfrak{g}_K^\vee,$$
$$f \mapsto df_e.$$

Remark 4.2.2. It is not completely obvious that Φ is a morphism of complexes. See [HK06, §§ 4.6 and 4.7].

Remark 4.2.3. The map Φ is compatible with the multiplicative structure.

Recall from Lemma 2.2.2 that in the case where $K = \mathbb{Q}_p$, the kernel G of $\mathbb{G}(\mathbb{Z}_p) \to \mathbb{G}(\mathbb{F}_p)$ is filtered and has a subgroup $\mathcal{G}$ of finite index which is saturated and equi-p-valued. Indeed, for $p \neq 2$ we have $G = \mathcal{G}$.

Let $\mathcal{L}^* = \mathcal{L}^*(\mathcal{G})$ be its integral Lazard Lie algebra (see Definition 2.6.6). As reviewed in Example 2.6.8, there is a natural isomorphism

$$\mathfrak{g} \otimes \mathbb{Q}_p \cong \mathcal{L}^* \otimes \mathbb{Q}_p.$$

Proposition 4.2.4 [HK06, Theorem 4.7.1]. *For $K = \mathbb{Q}_p$ and $\mathcal{G}$ saturated, the Lazard morphism Φ (see Definition 4.2.1) agrees, under the identification of Lie algebras given in Example 2.6.8, with the isomorphism defined by Lazard [Laz65, ch. V, Theorems 2.4.9 and 2.4.10].*

In particular, Φ is an isomorphism in this case.

Remark 4.2.5. This is a case to which our integral version of the result (Theorem 3.1.1) can be applied. As shown there, this is again the same isomorphism.

4.3 The isomorphism over a general base

Theorem 4.3.1. *Let $\mathbb{G}$ be a smooth group scheme over R with connected generic fiber and let $\mathcal{G} \subset \mathbb{G}(R)$ be an open subgroup. Then the Lazard morphism Φ (see Definition 4.2.1) is an isomorphism.*

The rest of the article will be devoted to the proof of this theorem.

Remark 4.3.2. Let us sketch the argument. We shall first show injectivity. For this, we can restrict to smaller and smaller subgroups $\mathcal{G}$ and even to their limit. In the limit, the statement follows by base change from Lazard's result for $R = \mathbb{Z}_p$. We then show surjectivity. Finite dimensionality of Lie algebra cohomology implies that the morphism is surjective for sufficiently small $\mathcal{G}$. Algebraicity then implies surjectivity also for the maximal $\mathcal{G}$.

By construction, the Lazard morphism Φ depends only on an infinitesimal neighborhood of e in $\mathcal{G}$. Hence it factors through the Lazard morphism for all open sub-Lie-groups of $\mathcal{G}$ and even through its limit

$$\Phi_\infty : \varinjlim_{\mathcal{G}' \subset \mathcal{G}} H^i_{\mathrm{la}}(\mathcal{G}', K) \to H^i(\mathfrak{g}, K).$$

Lemma 4.3.3. *The limit morphism Φ_∞ is an isomorphism.*

Proof. For $K = \mathbb{Q}_p$, this holds by Proposition 4.2.4 and the work of Lazard [Laz65, ch. V, Theorems 2.4.9 and 2.4.10].

The system of open sub-Lie-groups of $\mathcal{G}$ is filtered; hence

$$\varinjlim_{\mathcal{G}' \subset \mathcal{G}} H^i_{\mathrm{la}}(\mathcal{G}', K) = H^i(\mathcal{O}_{\mathrm{la}}(\mathcal{G}^\bullet)_e),$$

where $\mathcal{O}_{\mathrm{la}}(\mathcal{G}^n)_e$ is the ring of germs of locally analytic functions in e. Note that $\mathbb{G}(\mathbb{Z}_p)$ also carries the structure of a rigid analytic variety, and germs of locally analytic functions are none other than germs of rigid analytic functions, therefore they can be identified with a limit of Tate algebras.

First, suppose that $\mathbb{G} = \mathbb{H}_R$ for a smooth group scheme $\mathbb{H}$ over $\mathbb{Z}_p$. Then

$$\mathcal{O}_{\mathrm{la}}(\mathcal{G}^n)_e \cong \mathcal{O}_{\mathrm{la}}(\mathbb{H}(\mathbb{Z}_p)^n) \otimes K$$

because Tate algebras are well-behaved under base change (see [BGR84, ch. 6.1, Corollary 8]). Moreover, Φ_∞ is compatible with base change. Since it is an isomorphism for $\mathbb{H}$, it is also an isomorphism for $\mathbb{G}$.

Now consider general $\mathbb{G}$. By Lemma 4.1.1, $\mathbb{G}$ is a direct factor of some group of the form $\mathbb{H}_R$ where $\mathbb{H}$ is a group over $\mathbb{Z}_p$. Indeed, $\mathbb{H} = \mathrm{Res}_{R/\mathbb{Z}_p}(\mathbb{G})$. By naturality, $\Phi_{\infty,\mathbb{G}}$ is a direct factor of $\Phi_{\infty,\mathbb{H}_R}$ and hence, by the special case, an isomorphism. □

Corollary 4.3.4. *The Lazard morphism Φ is injective.*

Proof. Since $\mathbb{G}(R)$ is compact, all open sub-Lie-groups are of finite index. If $\mathcal{G}' \subset \mathcal{G}$ is an open normal subgroup, we have

$$H^i_{\mathrm{la}}(\mathcal{G}, K) \cong H^i_{\mathrm{la}}(\mathcal{G}', K)^{\mathcal{G}/\mathcal{G}'}.$$

Hence the restriction maps

$$H^i_{\mathrm{la}}(\mathcal{G}, K) \to H^i_{\mathrm{la}}(\mathcal{G}', K)$$

are injective. Because the system of open normal subgroups is filtered, this also implies that

$$H^i_{\mathrm{la}}(\mathcal{G}, K) \to \varinjlim_{\mathcal{G}'} H^i_{\mathrm{la}}(\mathcal{G}', K)$$

is injective. The injectivity of Φ then follows from the injectivity of Φ_∞. □

Lemma 4.3.5. *Let $\mathcal{G} \subset \mathbb{G}(R)$ be an open subgroup. Then there is an open subgroup $\mathcal{H} \subset \mathcal{G}$ such that the Lazard morphism for $\mathcal{H}$ is bijective.*

Remark 4.3.6. Note that this is precisely what Lazard proved over $\mathbb{Q}_p$ with $\mathcal{H}$ being the saturated subgroup of $\mathcal{G} = \mathbb{G}(\mathbb{Z}_p)$.

Proof. As Φ_∞ is bijective and Φ injective, it suffices to show that there is $\mathcal{H}$ such that the restriction map

$$H^i_{\mathrm{la}}(\mathcal{H}, K) \to \varinjlim_{\mathcal{G}'} H^i_{\mathrm{la}}(\mathcal{G}', K)$$

is surjective. Let α be a cocycle with class $[\alpha] \in \varinjlim_{\mathcal{G}'} H^i_{\mathrm{la}}(\mathcal{G}', K)$. By definition, it is represented by a cochain on some $\mathcal{G}'$. It is a cocycle (possibly on some smaller $\mathcal{G}'$). Hence $[\alpha]$ is in the image of the restriction map for $\mathcal{G}'$.

Lie algebra cohomology is finite-dimensional by definition, hence this is also true of $\varinjlim_{\mathcal{G}'} H^i_{\mathrm{la}}(\mathcal{G}', K)$. By intersecting the $\mathcal{G}'$ for a basis we get the group $\mathcal{H}$ that we wanted to construct. □

Proof of Theorem 4.3.1. Injectivity has already been proved in Corollary 4.3.4. We use an argument of Casselman and Wigner [CW74, § 3] to conclude the proof. The operation of $\mathbb{G}_K$ on $H^i(\mathfrak{g}, K)$ is algebraic. Hence the stabilizer $\mathbb{S}_K$ is a closed subgroup of $\mathbb{G}_K$. On the other hand, it contains some open subgroup of $\mathbb{G}(R)$. This implies that $\mathbb{S}_K = \mathbb{G}_K$ because $\mathbb{G}_K$ is connected. Hence $\mathcal{G} \subset \mathbb{G}(R)$ operates trivially, and thus

$$\Phi : H^i_{\mathrm{la}}(\mathcal{G}, K) \to H^i(\mathfrak{g}, K)$$

is surjective. □

Remark 4.3.7. The argument also works for cohomology with coefficients in a finite-dimensional algebraic representation of the group.

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Appendix A. A remark of Weigel on equi-p-valued groups

In this short appendix we explain an interesting remark that T. Weigel communicated to us during the proofreading of the present paper.

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We fix an *odd* prime p and mention, without going into any detail, that at the prime 2 similar but strictly weaker results are available.

Proposition A.1 (Weigel). *Let (G, ω) be a p-saturated group of finite rank which is equi-p-valued.*

(i) *The filtration of G defined by ω is the lower p-series.*

(ii) *There is a valuation ω' on G which has all the properties of ω and is such that, in addition, there is an ordered basis $g_i \in G$ satisfying $\omega'(g_i) = 1$ for all i.*

(iii) *The group G is uniformly powerful.*

Proof. (i) Denote by t the valuation of an ordered basis of G. Since (G, ω) is equi-p-valued, we have

$$\mathrm{gr}(G) = \mathbb{F}_p[\epsilon] \cdot \mathrm{gr}^t(G);$$

that is, the filtration of G defined by ω has jumps exactly as follows:

$$G = G^t \supseteq G^{t+1} \supseteq G^{t+2} \cdots .$$

For every $n \geqslant 0$ we see that

$$G^{t+n}/G^{t+n+1} \simeq \mathrm{gr}^{t+n}(G) \simeq \epsilon^n \cdot \mathrm{gr}^t(G)$$

is elementary p-abelian; hence the Frattini subgroup $\Phi(G^{t+n})$ of G^{t+n} satisfies

$$\Phi(G^{t+n}) \subseteq G^{t+n+1}.$$

On the other hand, we have inclusions

$$G^{t+n+1} \subseteq (G^{t+n})^p \subseteq \Phi(G^{t+n}),$$

the first one by divisibility and the second one by a general property of Frattini subgroups. This proves (i).

(ii) Since (G, ω) is p-saturated, we have

$$\frac{1}{p-1} < t \leqslant \frac{p}{p-1}.$$

We now check that $\omega' := \omega + 1 - t$ has the desired properties.

Write $c := 1 - t$ and first assume that $t \geqslant 1$, i.e. $c \leqslant 0$. Then for all $x, y \in G$,

$$\omega'([x, y]) = \omega([x, y]) + c \geqslant \omega(x) + \omega(y) + c = \omega'(x) + \omega'(y) - c \geqslant \omega'(x) + \omega'(y).$$

This shows that ω' has properties (i)–(iii) of Definition 2.1.1. Since (recall that $p \neq 2$)

$$\omega' \geqslant 1 > \frac{1}{p-1},$$

ω' has property (iv) of Definition 2.1.1, and property (v) is trivially satisfied. Take $x \in G$ with

$$\omega(x) + c = \omega'(x) > \frac{p}{p-1}.$$

Then, again because $c \leqslant 0$, we have

$$\omega(x) > \frac{p}{p-1} - c \geqslant \frac{p}{p-1},$$

and $x \in G^p$ since ω is saturated. Hence ω' has property (vi) of Definition 2.1.1, and property (vii) is trivially satisfied.

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Now assume that $t < 1$, i.e. $c > 0$. For all x and y in an ordered basis of G we have

$$\omega([x,y]) \geqslant \omega(x) + \omega(y) = 2t > t,$$

which, using the structure of $\mathrm{gr}(G)$, implies the first inequality in

$$\omega([x,y]) \geqslant t + 1 = \omega(x) + \omega(y) + 1 - t = \omega(x) + \omega(y) + c.$$

By uniformity, this inequality holds for all $x, y \in G$, and thus ω' has property (ii); hence properties (i)–(iii) of Definition 2.1.1 are satisfied. Take $x \in G$ with

$$\omega(x) + c = \omega'(x) > \frac{p}{p-1}.$$

Then

$$\omega(x) > \frac{p}{p-1} - c = \frac{p}{p-1} + t - 1 = t + \frac{1}{p-1} > t.$$

By the structure of $\mathrm{gr}(G)$, this implies

$$\omega(x) \geqslant t + 1.$$

Using the fact that (G, ω) is p-saturated, we have $t + 1 \geqslant p/(p-1)$ and $x \in G^p$. This implies that ω' has property (vi) of Definition 2.1.1. Since $\omega' \geqslant 1$, ω' has property (iv) and the rest of assertion (ii) in the proposition is clear.

(iii) We first show that G is uniform. Choose ω' as in part (ii) above; then

$$G = G^1 \supseteq G^2 \supseteq \cdots$$

is the lower p-series and

$$[G^n : G^{n+1}] = |\epsilon^n \cdot \mathrm{gr}^1(G)| = [G^1 : G^2].$$

Furthermore, G/G^p is abelian because

$$[G, G] \subseteq G^2 \subseteq (G^1)^p = G^p.$$

□

We conclude by explaining the simplifications implied by this result for our integral Lazard isomorphism, i.e. Theorem 3.1.1.

THEOREM A.2. *Let p be an odd prime and (G, ω) a compact, saturated, equi-p-valued group. Let M be as in Theorem 3.1.1. Then there is an isomorphism of graded $\mathbb{Z}_p$-modules*

$$\phi_G(M) : H_c^*(G, M) \simeq H^*(\mathcal{L}^*(G), M),$$

which is natural in M and G and compatible with cup-products.

Proof. Replace ω by the filtration constructed above. Then we are in the $e = 1$ case of Theorem 3.1.1 with generators in degree one. For naturality in G, note that every group homomorphism f (is continuous and) respects the lower p-series; hence the isomorphism is natural with respect to f. In particular, then, the isomorphism is always compatible with cup-products. □

REFERENCES

BGR84 S. Bosch, U. Güntzer and R. Remmert, *Non-Archimedean analysis: a systematic approach to rigid analytic geometry*, Grundlehren der Mathematischen Wissenschaften, vol. 261 (Springer, Berlin, 1984).

https://doi.org/10.1112/S0010437X10004884 Published online by Cambridge University Press

BLR90 S. Bosch, W. Lütkebohmert and R. Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 21 (Springer, Berlin, 1990).

CW74 W. Casselman and D. Wigner, *Continuous cohomology and a conjecture of Serre's*, Invent. Math. **25** (1974), 199–211.

DdMS91 J. D. Dixon, M. P. F. du Sautoy, A. Mann and D. Segal, *Analytic pro-p groups*, London Mathematical Society Lecture Note Series, vol. 157 (Cambridge University Press, Cambridge, 1991).

Hen07 H.-W. Henn, *On finite resolutions of $K(n)$-local spheres*, in *Elliptic cohomology*, London Mathematical Society Lecture Note Series, vol. 342 (Cambridge University Press, Cambridge, 2007), 122–169.

HK06 A. Huber and G. Kings, *A p-adic analogue of the Borel regulator and the Bloch–Kato exponential map*, Preprint (2006), arXiv:math/0612611v1, J. Inst. Math. Jussieu, to appear.

Laz65 M. Lazard, *Groupes analytiques p-adiques*, Publ. Math. Inst. Hautes Études Sci. **26** (1965), 389–603.

Rav86 D. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics, vol. 121 (Academic Press, Orlando, FL, 1986).

ST03 P. Schneider and J. Teitelbaum, *Algebras of p-adic distributions and admissible representations*, Invent. Math. **153** (2003), 145–196.

Ser64 J.-P. Serre, *Groupes analytiques p-adiques, d'après Michel Lazard*, Séminaire Nicolas Bourbaki, no. 270 (1963–64).

Ser65 J.-P. Serre, *Lie algebras and Lie groups* (W. A. Benjamin, New York, 1965), 1964 lectures given at Harvard University.

Tot99 B. Totaro, *Euler characteristics for p-adic Lie groups*, Publ. Math. Inst. Hautes Études Sci. **90** (1999), 169–225.

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