Conjugation 2: Conjugate lines in a triangle

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1. Introduction

In a previous note [1] the present author has given an account of the conjugations known as isogonal conjugation and isotomic conjugation with respect to a given triangle and has shown how to generalise the idea of conjugation involving a pair of points so that $P(l, m, n)$ is $q$-conjugate, by means of a construction involving a transversal $q$, to the point $P'(p^2/l, q^2/m, r^2/n)$ and he has also demonstrated how to perform the geometrical constructions involved. When there is a self-conjugate point at the incentre $I$ one has isogonal conjugation, and when there is a self-conjugate point at the centroid $G$ one has isotomic conjugation. This type of conjugation includes cases in which no self-conjugate point $Q$ exists, this being the case when the conjugation is generated by a transversal $q$ that intersects two sides of the fundamental triangle internally. The purpose of [1] was first to show how the idea of isogonal and isotomic conjugation can be extended into a whole family of point conjugations of a particular type and secondly to serve as a preparation for extending those ideas to give a consistent definition of conjugation involving pairs of lines. It is not to be expected that we would be able to announce new results at this stage about the conjugation of points, as this topic has been studied extensively for over a hundred years. The excuse, if one is needed, for introducing the idea of line conjugations is that by doing so we are able to announce some new results of significance.

The notion that there might be a construct of pairs of conjugate lines with respect to a triangle seems to have no mention in the literature. However, as we hope to show, pairs of conjugate lines may indeed be constructed, which possess some intriguing properties. In this article we consider mainly pairs of isogonally conjugate lines and pairs of isotomically conjugate lines, but the concept may be generalised to produce pairs of $q$-conjugate lines, and these are used in Section 6 to enhance our understanding of the set of non-degenerate conics that pass through three non-collinear points. It should be noted that in this paper when we talk of a pair of $q$-conjugate lines this is a reciprocal relationship between two lines and is not to be confused with the locus of the points that are the $q$-conjugate points of a line, which is a line through a triangle vertex or a conic through all three triangle vertices.

The plan of the paper is as follows. First we provide a construction for producing pairs of $q$-conjugate lines and then we describe how this may be applied to construct pairs of isogonally conjugate lines and isotomically conjugate lines. In each case the self-conjugate lines are identified. We then produce the result with pairs of conjugate lines that corresponds to the result for pairs of conjugate points that if you take a fixed line (not passing through a vertex of $ABC$), then the isogonal (isotomic) conjugates of the points on the line trace out a conic passing through $A$, $B$, $C$ and furthermore
the tangents to this circumconic at the vertices meet the opposite sides at three collinear points.

We then prove a theorem about the isogonally (isotomically) conjugate lines of the sides of the pedal triangle of the Cevians of a point. Next we show how a connection arises between conjugation, Carnot's theorem [2] and the intersections of a conic with the sides of a triangle. Finally we prove an intriguing theorem about isotomic conjugate lines of pairs of parallel lines, which has application to the Droz-Farny line theorem [3, 4, 5]. In fact we claim that it provides the final resolution of the Droz-Farny problem.

2. The construction of $q$-conjugate, isogonally conjugate and isotomically conjugate pairs of lines

Let $ABC$ be a triangle and $LMN$ a transversal, with $L$ on $BC$, $M$ on $CA$, $N$ on $AB$. In order to obtain the $q$-conjugate line to $LMN$, it is best to use the concept of harmonic pole and polar, as defined, for example, in Semple and Kneebone [6]. If we take $LMN$ to have equation $lx + my + nz = 0$, then its harmonic pole is the point with coordinates $(1/1, 1/m, 1/n)$. The $q$-conjugate of this point is given in [1] and is the point with coordinates $(p^2l, q^2m, r^2n)$. The harmonic polar of this point is the line with equation

$$\frac{x}{p^2l} + \frac{y}{q^2m} + \frac{z}{r^2n} = 0. \quad (1)$$

This we take to be the definition of the line $L'M'N'$, the $q$-conjugate line to $LMN$. The definition is reciprocal so that $LMN$ is the $q$-conjugate line to $L'M'N'$. Thus $LMN$ and $L'M'N'$ are a pair of $q$-conjugate lines. Here $L'$ is the point on $BC$ with coordinates $(0, -q^2m, r^2n)$, with similar expressions by cyclic change for the coordinates of the points $M'$ on $CA$ and $N'$ on $AB$.

It is now evident that the isogonal conjugate to $lx + my + nz = 0$ has equation

$$\frac{x}{a^2l} + \frac{y}{b^2m} + \frac{z}{c^2n} = 0, \quad (2)$$

and the isotomic conjugate has equation

$$\frac{x}{l} + \frac{y}{m} + \frac{z}{n} = 0. \quad (3)$$

Geometrical constructions in the case of the isogonal and isotomic cases are as follows:

For the isogonal conjugate of $LMN$ take the lines $AL$, $BM$, $CN$ and reflect these lines in $AI$, $BI$, $CI$ respectively. Let these lines meet $BC$, $CA$, $AB$ respectively at $L'$, $M'$, $N'$ respectively. Then as shown below $L'M'N'$ is a straight line satisfying (2) and is hence the isogonal conjugate of $LMN$. For the isotomic conjugate of $LMN$ we take $L'$, $M'$, $N'$ to be the images of $L$, $M$, $N$ in $180^\circ$ rotations about the midpoints of $BC$, $CA$, $AB$ respectively. Then again as we show below $L'M'N'$ is a straight line satisfying (3) and is hence the isotomic conjugate of $LMN$. 

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The construction of the isogonal conjugate of the transversal $LMN$ is illustrated in Figure 1. Given that $LMN$ has equation $lx + my + nz = 0$, we now find the equations of the conjugate lines $L'M'N'$ in the two cases. In both cases the points $L$, $M$, $N$ have coordinates $(0, n, -m), (-n, 0, l), (m, -l, 0)$ respectively. For the isogonal conjugate line $L'$, $M'$, $N'$ have coordinates $(0, b^2m, -c^2n), (-a^2l, 0, c^2n), (a^2l, -b^2m, 0)$ respectively. For the isotomic conjugate line they have coordinates $(0, m, -n), (-l, 0, n), (l, -m, 0)$ respectively. It may now be checked that these points lie respectively on the lines satisfying (2) and (3).

3. Self-conjugate lines

It can be seen from (2) that the equations of the self-conjugate lines for isogonal conjugation are $x/a + y/b + z/c = 0, -x/a + y/b + +z/c = 0, x/a - y/b + z/c = 0$ and $x/a + y/b - z/c = 0$. Geometrically these arise as follows: let $AI, BI, CI$ meet $BC, CA, AB$ respectively at $U, V, W$. Let $U'$ be the harmonic conjugate of $U$ with respect to $B$ and $C$. Let $V'$ and $W'$ be defined similarly. Then the four self-conjugate lines are $U'V'W', U'VW, UV'W, UVW'$.

Similarly the self-conjugate lines for isotomic conjugation are the line at infinity and the sides of the medial triangle.

The self-conjugate lines for $q$-conjugation have equations $x/p + y/q + z/r = 0, -x/p + y/q + +z/r = 0, x/p - y/q + z/r = 0$ and
If a transversal q is involved that cuts two sides of a triangle internally then there are no self-conjugate points and no self-conjugate lines (unless one transfers the problem to the complex projective plane).

It is important to emphasise that a self-conjugate line in the context of this paper occurs when the two lines of a pair of q-conjugate lines coincide. It is not a line (such as AI in isogonal conjugation) in which the conjugates of the points lying on it also lie on that line.

4. Dual theorems

**Theorem 1**: Let m be a transversal of triangle ABC and P a variable point on the line. Let Pg be the isogonal conjugate of P. Then the locus of Pg as P varies on m is a circumconic S(m) of ABC. Furthermore, if the tangents to S(m) at A, B, C meet the sides BC, CA, AB respectively at U, V, W, then UVW coincides with m', the isogonal conjugate line of m.

This theorem is illustrated in Figure 2. Apart from the statement that UVW and m' coincide this is a known theorem.

**Proof**: Let m have equation lx + my + nz = 0. Then it is well known that the equation of S(m) is

\[ a^2lyz + b^2mxz + c^2nxy = 0. \]  (4)
For the second part, start with a transversal $LMN$, with $L, M, N$ on $BC$, $CA, AB$ respectively; the point-conjugate of $AL$ is $AU$, with $U$ on $BC$ etc., and $UVW$ is the line-conjugate of $LMN$, by definition. But the point-conjugate of $LMN$ is the conic $S(m)$ through $A, B, C$ and since $AL, LMN$ meet at $L$ on $BC$, their point-conjugates will touch at $A$, that is, $AU$ is the tangent to $S(m)$ at $A$. Here we have used [6, p. 234, theorem 12].

As an example, if $m$ is the line at infinity, then $l = m = n = 1$ and $S(m)$ is the circumcircle. In this case the line $m'$ is the polar line of the symmedian point. Theorem 1 is true if isogonal conjugation is replaced by isotomic or indeed any other $q$-conjugation.

**Theorem 2:** Let $ABC$ be a triangle and let $P$ be a point not on the sides of the triangle or on the circumcircle. Let $l$ be a variable line through $P$ and let $l'$ be its isogonal conjugate line. Then the envelope of $l'$ as $l$ varies is an inconic $s(P)$ of $ABC$. Furthermore, the lines from the vertices of $ABC$ to the points of contact of $s(P)$ with opposite sides are concurrent at $P_g$, the isogonal conjugate of $P$.

![Figure 3](https://www.cambridge.org/core/fig/)

**FIGURE 3**

This theorem is illustrated in Figure 3. This is the dual of theorem 1. However, as we have not used the concept of duality in defining line-conjugation we give a separate proof. Again the theorem is true if isogonal conjugation is replaced by isotomic or indeed any other $q$-conjugation.

**Proof:** Let $P$ have coordinates $(l, m, n)$. Use a parameter $t$ for lines through the point, so one such line $l$ has line coordinates $(1/l, t/m, -(1+t)/n)$. The isogonal line-conjugate is the line with coordinates $$(1/l', t/m, -(1+t)/n').$$ The point-conjugate of $l'$ is then $l'' = (1/l', t/m, (1+t)/n')$. The point $P$ is the pole of $l''$ with respect to $S(m)$. Therefore, the point $P$ is the pole of $l''$ with respect to $S(m)$. Hence, $P$ is the isogonal conjugate of $P_g$, the isogonal conjugate of $P$. The theorem is proved.
\((l \parallel a^2, m \parallel b^2t, -n \parallel c^2(1 + t))\). Using \((u, v, w)\) for the general line, we have 
\(u/v = b^2l/t/a^2m, u/w = -c^2l(1 + t)/a^2n\). Eliminating \(t\) we find the line equation of the envelope is the inscribed conic envelope with equation 
\(b^2c^2lvw + c^2a^2muv + a^2b^2nuv = 0\). Transferring back to point coordinates the envelope of \(l'\) is

\[
\begin{align*}
&b^4c^4x^2 + c^4a^4y^2 + a^4b^4z^2 \\
-2ab^4c^2mnyz - 2ab^4c^2nlzx - 2ab^4c^4lmyx = 0.
\end{align*}
\]

This is a conic \((P)\) touching the sides of \(ABC\). For example it touches \(x = 0\) at the point with coordinates \((0, b^2/m, c^2/n)\). The three lines from the vertices to the point of contact on the opposite side are therefore Cevians with the Cevian point (sometimes referred to as the Brianchon point of the inconic) with coordinates \((a^2/l, b^2/m, c^2/n)\). This point is \(P_g\), the isogonal conjugate of \(P\).

This result provides an opportunity to apply it to the theory of triangle porisms. We refer to Poncelet's porism, one version of which says that when two conics are so situated that there is a triangle inscribed in one and circumscribed to the second, then there are infinitely many such triangles. The best known case is when a triangle is given together with its circumcircle and incircle and then an infinite number of triangles can be drawn having the same circumcircle and incircle. All one has to do is to choose any point \(X\) on the circumcircle and draw a pair of tangents from it to the incircle. If these tangents meet the circumcircle again at points \(Y\) and \(Z\), then the chord \(YZ\) always touches the incircle, so \(XYZ\) is a triangle with the required property. Another less well known case is Brocard's porism in which the triangle \(ABC\) inscribed in a circle has Brocard's ellipse as inconic, the Brianchon point for this inconic being the symmedian point \(K\) of \(ABC\). Theorem 2 provides a method of construction of this inconic as the envelope of the isogonal line-conjugates of lines through the centroid of \(ABC\), the centroid being the isogonal conjugate of the symmedian point. The particularly interesting thing about this porism concerns the other triangles \(XYZ\) circumscribing this inconic and inscribed in the same circle; all have Brianchon point at the same point -- that is the locus of the symmedian point of the triangles in the porism is a stationary point. See [7]. In other porisms in which the circumconic is the circumcircle, the locus is a conic. Theorem 2 also tells us that given a circle and a fixed triangle \(ABC\) inscribed in it, then by varying \(P\) one can construct any inconic of \(ABC\), and thereby a porism is generated, one of the triangles in the porism being \(ABC\).

When the lines \(l\) are parallel so that they all pass through the same point at infinity, and isotomic conjugation is involved, then the inconic is a parabola, and all parabolic inconics follow in this way. Since the isotomic conjugate of points on the line at infinity lie on the outer Steiner ellipse we have a neat way of seeing that the Brianchon point of a parabolic inconic must lie on this ellipse.

Again, when the conjugation is isotomic, if you take two parallel lines, then their isotomic conjugates meet the triangle sides at pairs of points. It turns out that the three midpoints of these pairs are collinear, forming a
generalised Droz-Farny line. This Droz-Farny line is the isotomic conjugate of the line parallel and midway between the initial pair of parallel lines. This application is dealt with in detail in Section 7.

5. Pedal triangles of Cevians

If Cevians through a point \( P \) meet the sides of triangle \( ABC \) at \( L, M, N \) then we define the pedal triangle of the Cevians to be the triangle \( LMN \). (This is not to be confused with the pedal triangle of a point, which is the triangle joining the feet of the perpendiculars from the point to the sides of the triangle.) The following theorem is merely an illustrative example of line-conjugation and has no immediate consequences as far as we can see.

**Theorem 3:** Let \( ABC \) be a triangle and \( P \) a general point. Let \( AP, BP, CP \) meet the sides \( BC, CA, AB \) in points \( L, M, N \). Take the isogonal conjugates of \( MN, NL, LM \) to be lines \( l, m, n \) respectively. Then the triangle formed by \( l, m, n \) is inscribed in triangle \( ABC \) and if its vertices are denoted by \( L' = m \wedge n, M' = n \wedge l, N' = l \wedge m \), the lines \( AL', BM', CN' \) meet at \( P_g \), the isotogonal conjugate of \( P \).

**Proof:** Let \( P \) have coordinates \((u, v, w)\). The coordinates of \( M, N \) are \((u, 0, w), (u, v, 0)\) respectively. The equation of \( MN \) is thus \( x/u - y/v - z/w = 0 \). The line \( l \), which is the isogonal conjugate of \( MN \) thus has equation \( ux/a^2 - vy/b^2 - wz/c^2 = 0 \). Similarly the isogonal conjugates \( m, n \) have equations \(-ux/a^2 + vy/b^2 - wz/c^2 = 0\), \(-ux/a^2 - vy/b^2 + wz/c^2 = 0\) respectively. The point \( L' = m \wedge n \) thus has coordinates \((0, b^2/v, c^2/w)\), which lies on \( BC \) and is the foot of the Cevian through the point \( P_g \) with coordinates \((a^2/u, b^2/v, c^2/w)\). The theorem follows by symmetry.

The theorem is also true if the word isogonal is replaced by isotomic or any other \( q \)-conjugation.

6. The intersections of a conic with the sides of a triangle

In this section we describe how a conic that does not pass through a vertex of a given triangle determines a conjugation, and conversely how a triangle \( ABC \) with an inscribed triangle \( UVW \) and any conjugation enables one to determine points \( P, Q, R \) on the sides so that a conic may be drawn through all six points \( U, V, W, P, Q, R \). The construction for doing this is justified by Carnot's theorem, see [2, p. 327]. This theorem is the relationship Carnot derived for this to happen, which is

\[
\begin{vmatrix}
BU & CV & AW & BP & CQ & AR \\
UC & VA & WB & PC & QA & RB
\end{vmatrix} = 1.
\]

The procedure is illustrated in Figure 4. We start with a triangle, its circumcircle and three points \( U, V, W \) on \( BC, CA, AB \) respectively and a
transversal $LMN$ that may or may not intersect two of the sides of $ABC$ internally. The transversal is going to produce the conjugation, as first described in [1], but we outline the construction again for convenience and for the purpose of identifying the points on the figure. It should be made clear that the circumcircle may be replaced by any other circumconic. All that happens if this replacement is made is that the points $P, Q, R$ that result from the conjugation change position (unless the transversal $LMN$ is also changed appropriately).

The construction is as follows. $AU$ is extended to meet the circumcircle at $U'$, and then $U'L$ is drawn to meet the circumcircle again at $P'$. Finally $AP'$ is drawn to meet $BC$ at $P$. If $U'$ and $P'$ coincide this is no problem, it just means that $U$ and $P$ coincide and the resulting conic touches $BC$ at $U$. Points $V', Q', Q$ are obtained similarly using $B$ and $M$ and points $W', R', R$ by using $C$ and $N$. The theorem then, as we prove below, is that the six points $U, V, W, P, Q, R$ lie on a conic.

The converse is also true that if one starts with the conic $UVPQR$ and if $U'P'$ meets $BC$ at $L$, $V'Q'$ meets $CA$ at $M$ and $W'R'$ meets $AB$ at $N$, then $L$,
M, N are collinear, thereby establishing the conjugation. It should be noted that AU, BV, CW do not need to meet in a point, but if they do then AP, BQ, CR also meet in a point. This follows from Carnot’s theorem and Ceva’s theorem. It should also be noted that the lines UV, VW, WU are conjugate lines to PQ, QR, RP respectively, in the sense described in Section 2.

Theorem 4: Using the notation just described the points U, V, W, P, Q, R lie on a conic.

Proof: Areal coordinates are again used. Let U, V, W have coordinates U (0, v, w), V (t, 0, s), W (p, q, 0) respectively. Let LMN be the transversal with equation lx + my + nz = 0 meeting the sides at L(0, n, -m), M(-n, 0, l), N(m, -l, 0).

The equation of AU is wy = vz and the circumcircle has equation $a^2yz + b^2zx + c^2xy = 0$. These meet again at the point $U'$ with coordinates $(-a^2, (b^2w + c^2v)/w, (b^2w + c^2v)/v)$. The equation of $LU'$ is

$$ (b^2w + c^2v)(mv + nw)x + a^2vw(my + nz) = 0. \quad (6) $$

This meets the circumcircle again at the point $P''$ with coordinates $(-a^2/(mv + nw), b^2/mv, c^2/nw)$. AP' meets BC again at $P(0, b^2/mv, c^2/nw)$.

We now have

$$ \left( \frac{BU}{UC} \right) \left( \frac{BP}{PC} \right) = \left( \frac{w}{v} \right) \left( \frac{c^2}{mv} \right) \left( \frac{b^2}{nv} \right) = \left( \frac{m}{n} \right) \left( \frac{c^2}{b^2} \right). $$

With similar analysis for points Q, R by cyclic change of letters we have

$$ \left( \frac{BU}{UC} \right) \left( \frac{CV}{VA} \right) \left( \frac{AW}{WB} \right) \left( \frac{BP}{PC} \right) \left( \frac{CQ}{QA} \right) \left( \frac{AR}{RB} \right) = \left( \frac{m}{n} \right) \left( \frac{n}{l} \right) \left( \frac{l}{m} \right) \left( \frac{c^2}{b^2} \right) \left( \frac{b^2}{c^2} \right) = 1. $$

And hence the theorem holds by Carnot’s theorem. The working is reversible, so the converse is also true.

If one studies the coordinates of the points P, Q, R it is clear that the conjugation is isogonal conjugation when l = m = n = 1 and the transversal is the line at infinity. In the construction this means that $U'P'$ is parallel to BC and so on. Also the conjugation is isotomic conjugation when $l : m : n = a^2 : b^2 : c^2$.

So far in this section we have looked at the construction in terms of a point conjugation, but since UV, VW, WU are three transversals of ABC it is possible to consider their images under the corresponding line conjugation. From Section 2 it is clear that their line conjugates are the lines PQ, QR, RP. We therefore have the following theorem.

Theorem 5: Let ABC be a triangle and let UVW be an inscribed triangle. Take the conjugate lines with respect to any line conjugation (of the type we have described in this paper), then these lines will also form an inscribed triangle PQR and a conic may be drawn through the six points U, V, W, P, Q, R.
7. The Droz-Farny problem

Given triangle $ABC$ and its orthocentre $H$, if you draw any pair of perpendicular lines $m_1, m_2$ through $H$ to meet $BC$ at $L_1, L_2$ respectively, $CA$ at $M_1, M_2$ respectively and $AB$ at $N_1, N_2$ respectively, and if $L, M, N$ are the midpoints of $L_1L_2, M_1M_2, N_1N_2$ then $LMN$ is a straight line. It is called a Droz-Farny [2] line, because it was he who first posed the collinearity involved as a problem. For many years only analytic proofs of the Droz-Farny theorem were available, but recently Ayme [4] has given a delightful and instructive synthetic proof. The construction above is remarkable in that any pair of perpendicular lines through $H$ will serve to create a Droz-Farny line. It is also the case that $L, M, N$ need not be the midpoints, but can be points dividing the three segments $L_1L_2, M_1M_2, N_1N_2$ in any fixed ratio and yet $L, M, N$ are still collinear. Even more recently, in [5] Thas has shown that if $H$ is replaced by any point $K$, then given any axis $m_1$ through $K$, there is always another axis $m_2$ through $K$ with the Droz-Farny property, the difference being that when $K$ is not the orthocentre the axes $m_1$ and $m_2$ are not necessarily at right angles. This observation was made independently not long afterwards in a paper by Bradley, Monk and Smith [8]. Their paper was concerned with the Euler porism in which the circumconic is the circumcircle, and the inconic is the envelope of the Droz-Farny lines generated from $H$. This inconic has centre at the nine-point centre and its foci are at $O$, the circumcentre and $H$. They noticed that the porism generalises when $H$ is replaced by another point $K$. In a more recent paper [9] the present author has shown how every general point $K$ (that is, other than the vertices $A, B, C$ and the orthocentre $H$) is associated with one pair of axes at right angles to each other.

The methods outlined above require elaborate constructions when the axes producing the Droz-Farny lines do not intersect at $H$. For example for the pair of axes through a general point $K$, one has to draw the rectangular hyperbola through $A, B, C, H, K$ and the unique pair of axes through $K$ that are at right angles and which lead to a generalised Droz-Farny line have directions parallel to the asymptotes of this hyperbola. It is not surprising to discover that pairs of axes that are not at right angles arise from other hyperbolas through $A, B, C, K$ and are parallel to their asymptotes. There are an infinite number of such hyperbolas, but only one which is a rectangular hyperbola, so it follows that there is only one pair of axes, leading to a generalised Droz-Farny line, that are at right angles. For this approach to generalised Droz-Farny lines see [9]. Thas [5] follows a different route to this result, making use of a projective generalization of the original construction.

Since [9] was written it has emerged that there is a method of obtaining all Droz-Farny lines and generalised Droz-Farny lines and the pairs of axes that produce them, using isotomic line-conjugation. We now describe this as our last application in this article of line-conjugation. It remains to be seen whether line-conjugation throws further light on other properties in the Euclidean plane. A sceptic might argue that line-conjugation has been
implicit in twentieth century geometry all along, even if its theory has not previously been given any emphasis. However we would argue that any method, such as the one that follows, which throws further light on known results is worth following up. The application in fact disposes of the Droz-Farny problem.

Once again we use areal coordinates. Let there be two axes $L_1, L_2$ with equations $lx + my + nz = 0$ and $px + qy + rz = 0$. These meet the line $BC$ at the points $L, P$ with coordinates $(0, n, -m)$ and $(0, r, -q)$. The points $M, Q$ and $N, R$ on $CA$ and $AB$ respectively can be written down similarly. The midpoints $U, V, W$ of $LP, MQ, NR$ can now be found and the determinantal condition for their collinearity can be obtained. This condition turns out wonderfully to factorise (of course, it had to) and the two possibilities are

$$lq + mr + np = lr + mp + nq, \quad (1)$$

$$\frac{1}{lq} + \frac{1}{mr} + \frac{1}{np} = \frac{1}{lr} + \frac{1}{mp} + \frac{1}{nq}. \quad (2)$$

Similar analysis can be worked out when the lines are each divided in the same ratio, and this is left as an exercise for the reader.

Condition (1) is that the two lines should be parallel. This is an obvious solution, the line through the midpoints being the line parallel to $LMN$ and $PQR$ and midway between them, but it is not one that provides a generalised Droz-Farny line, since the axes involved have no finite common point. Condition (2) is that the lines with equations $x/l + y/m + z/n = 0$ and $x/p + y/q + z/r = 0$ are parallel. These lines are the isotomics of the original lines and pass through points $L', M', N'$ for the first and $P', Q', R'$ for the second, which points are respectively the rotations by 180° of the points $L, M, N, P, Q, R$ respectively about the midpoints of the lines on which they lie.

Clearly the argument is reversible so that if we start with a pair of parallel lines $L_1, L_2$ meeting $BC, CA, AB$ respectively in $L', M', N'$ and $P', Q', R'$ and then we find the isotomics $L_1$ and $L_2$ meeting $BC, CA, AB$ respectively in $L, M, N$ and $P, Q, R$ it follows that if $U, V, W$ are the midpoints of the pairs $L, P$ and $M, Q$ and $N, R$ respectively, then $U, V, W$ are collinear and form a generalised Droz-Farny line, since in this case $L_1$ and $L_2$ are not parallel. Furthermore $UVW$ is the isotomic of the parallel line midway between the lines $L_1, L_2$. Figure 5 illustrates this and the configuration has been arranged so that $L_1$ and $L_2$ are lines at right angles meeting at the orthocentre, producing one of the classic original Droz-Farny lines.

Also shown in the diagram is the Euler inconic, which is the ellipse with foci $O$ and $H$ touching the sides of $ABC$, and which is the envelope of the Droz-Farny lines as the lines $L_1$ and $L_2$ rotate. See [8].
Conditions (1) and (2) are exhaustive, which means to say that all generalised Droz-Farny lines arise from this construction. In summary, any pair of parallel lines produces a generalised Droz-Farny line, which is the isotomic of the parallel line midway between them. From this it emerges that any line in the plane qualifies in many ways as a generalised Droz-Farny line.

We now give an areal analysis of this situation. We suppose the parallel lines have equations $lx + my + nz = 0$ and $(l + t)x + (m + t)y + (n + t)z = 0$ for some value of $t$. The isotomics have equations $x/l + y/m + z/n = 0$ and $x/(l + t) + y/(m + t) + z/(n + t) = 0$ and these axes meet at a point $P(t)$ with coordinates $(l(m - n)(l + t), m(n - l)(m + t), n(l - m)(n + t))$. As one alters the variable $t$ to increase or decrease the separation of the parallel lines the point $P(t)$ moves on the line $x/l + y/m + z/n = 0$, the isotomic of the first line. The associated line midway between the two parallel lines has equation...
\[(l + t/2)x + (m + t/2)y + (n + t/2)z = 0\] and the associated generalised Droz-Farny line has equation \(x/(2l + t) + y/(2m + t) + z/(2n + t) = 0\).

As \(t\) varies, the envelope of this line is the inconic with equation
\[
(m - n)^2 x^2 + (l - m)^2 y^2 + (n - l)^2 z^2 + 2(l - m)(l - n)yz + 2(m - n)(m - l)zx + 2(n - l)(n - m)xy = 0,
\]
which is a parabola that touches the sides of \(ABC\). This parabola is uniquely associated with the choice of the first line \(lx + my + nz = 0\), which has the direction \((m - n, n - l, l - m)\) in the sense that lines parallel to it all pass through the point at infinity with these coordinates. In other words, every direction in the plane is associated with a set of generalised Droz-Farny lines that envelope an inscribed parabola. Conversely given any inscribed parabola there exists a unique direction and associated set of generalised Droz-Farny lines. The circumcircle of \(ABC\) and any one of these inscribed parabolas together with the given triangle create a porism, showing that there are a host of other triangles all with the same property.

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References