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PRO-*C* COMPLETIONS OF CROSSED MODULES

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Crossed modules occur in the theory of group presentations, in group cohomology and in providing algebraic models for certain homotopy types. There are profinite analogues of each of these contexts. In this paper, we examine the problem of extending the profinite completion functor on groups to one on crossed modules thus providing a method for comparing the information contained in profinite and abstract crossed modules in each of these situations.

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Although it originated in algebraic topology, the theory of crossed modules has recently become a useful tool in combinatorial and cohomological group theory. Almost all of the algebraic results would seem to generalise, with suitable modification, to the case where the groups involved are profinite groups and the homomorphism and action are continuous. This raises the possibility of finding new methods for use in Galois cohomology, in the general study of Galois groups and the related fundamental groups of schemes.

In this paper we study one of the basic problems of this new area, namely the interrelationship between crossed modules and pro-C crossed modules for C a full class of finite groups (see below). Explicitly we show that there is a pro-C completion functor defined on the category of crossed modules, taking values in the category of pro-C crossed modules. Having shown that such a pro-C completion exists, we ask the obvious question: since a crossed module consists of two groups C and G and a group homomorphism ∂ between them (satisfying certain axioms), what is the relationship between the pro-C completion of (C,G,∂), which we will denote by $\widehat{(C,G,\partial)}$, and the pro-C completions of C,G as groups? Finally we study the pro-C completion of a free crossed module and relate it to a free pro-C crossed module on associated data. Applications are considered in [**11**] and [**12**].

Terminology

In this paper C will denote a class of finite groups which is closed under the formation of subgroups, homomorphic images, finite products and which contains at least one non-trivial group. Pro-C groups are profinite groups whose finite quotients are in C.

The class C will be assumed to be full in the sense that C must also be closed under extension of groups. Of particular interest for future applications are the cases when C is the class of finite p-groups or finite solvable groups.

1. Crossed modules and pro-C crossed modules

We recall the definition of a crossed module, (see for instance Brown and Huebschmann [2] for a more detailed treatment), and introduce the pro-C analogue.

1.1. *A crossed module* (C, G, ∂) consists of groups C and G, a left action of G on C, which will be written $(g, c) \to {}^{g}c$ for $g \in G$, $c \in C$, and a group homomorphism ∂: $C \to G$ satisfying the following conditions:

(CM1) for all $c \in C$, and $g \in G$

$$\partial({}^{g}c) = g(\partial c)g^{-1};$$

(CM2) for all $c_1, c_2 \in C$

$$c_2 c_1 c_2^{-1} = {}^{\partial(c_2)}c_1.$$

((CM2) is often called the *Peiffer relation* or *Peiffer identity*).

Examples. (1) For H a normal subgroup of G, the inclusion homomorphism $i: H \to G$ makes (H, G, i) into a crossed modules where G acts on H by conjugation.

(2) If M is a left G-module and $0: M \to G$ is the zero homomorphism, then $(M, G, 0)$ is a crossed module.

Definition. Let (C, G, ∂), and (C', G', ∂') be crossed modules. A morphism

$$(\mu, \eta): (C, G, \partial) \to (C', G', \partial')$$

of crossed modules consists of group homomorphisms $\mu: C \to C'$, $\eta: G \to G'$ such that

(i) $\partial' \mu = \eta \partial$ and

(ii) $\mu({}^{g}c) = {}^{\eta(g)}\mu(c)$ for all $c \in C, g \in G$.

This notion of morphism easily gives us a category Cmod of crossed modules and crossed module morphisms. There are special classes of morphisms that we will need later, namely those in which $G = G'$ and η is the identity morphism. For fixed G, such a morphism (μ, Id_G): $(C, G, \partial) \to (C', G, \partial')$ will be called a morphism of crossed modules over G. These give a subcategory Cmod/G of Cmod.

1.2. The pro-C analogues of these concepts are now easy to give.

A *pro-C crossed module* (C, G, ∂) is a crossed module in which C and G are pro-C topological groups, ∂ is a continuous homomorphism and the left G-action on C is a continuous G-action. Closed normal subgroups give examples of such as do zero

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morphisms from pseudocompact left G-modules to G, (see Brumer [5] for the theory of pseudocompact modules).

A morphism

$$(\mu,\eta)\colon (C,G,\partial)\to(C',G',\partial')$$

of pro-C crossed modules is a morphism of the underlying crossed modules in which both μ and η are continuous morphisms of pro-C groups. This gives us categories "Pro-CCmod" and "Pro-CCmod/G" for G a pro-C group and also a functor

$$U_{C\mathrm{mod}}\colon \text{Pro-}CC\mathrm{mod}\to C\mathrm{mod}\,.$$

which forgets the topology.

Recall the corresponding situation for groups; the forgetful functor

$$U_{\mathrm{Grps}}\colon \text{Pro-}C.\mathrm{Grps}\to\mathrm{Grps}$$

(in the hopefully obvious notation) has a left adjoint, known as the *pro-C completion functor*, which we will denote by a "$\hat{\ }$". This is defined as follows:

If G is a group, let $\Omega(G)$ be the directed set of normal finite index subgroups W of G with $G/W\in C$, then

$$\hat{G}\cong\varprojlim_{W\in\Omega(G)} G/W$$

We will sometimes write $W\lhd_{\mathrm{fin}}G$ as indicating that $W\in\Omega(G)$. This notation is useful in as much as it is more suggestive of the actual concept involved, but can also become somewhat cumbersome so we will use both notations.

We wish to see if the crossed module forgetful functor

$$U_{C\mathrm{mod}}\colon \text{Pro-}CC\mathrm{mod}\to C\mathrm{mod}$$

also has a left adjoint. The obvious approach using some idea of "normal" subcrossed module of finite index is technically messy so we use an equivalent formulation involving Loday's notion of cat^1-groups.

2. Cat^1-groups, their pro-C analogue and the completion process

The equivalence between crossed modules and internal categories in the category of groups has been known for some time (see the comments on this in Brown–Spencer [4]). A neat reformulation of the latter type of object was given by Loday in [13] (see also Brown–Loday [3]). There one also finds the introduction of the convenient term "cat^1-group".

2.1. A *cat^1-group* is a triple (G,s,t) consisting of a group G and endomorphisms s, the source map, and t, the target map of G satisfying the following axioms:

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(i) $st=t$ and $ts=s$,

(ii) $[\operatorname{Ker} s, \operatorname{Ker} t]=1$.

Here, of course, $[\operatorname{Ker} s, \operatorname{Ker} t]$ indicates the subgroup of G generated by the commutators $[g,h]=ghg^{-1}h^{-1}$ with $g \in \operatorname{Ker} s, h \in \operatorname{Ker} t$.

There is an obvious notion of a morphism between cat^1-groups: if (G,s,t), and (G',s',t') are cat^1-groups, a morphism

$$\phi:(G,s,t)\to(G',s',t')$$

is a group homomorphism $\phi:G\to G'$ such that

$$s'\phi=\phi s$$

and

$$t'\phi=\phi t.$$

This gives a category, which we will denote $\mathrm{Cat}^1(\mathrm{Grps})$, of cat^1-groups and morphisms between them.

2.2. In [**13**], Loday shows that there is an equivalence between the categories $C\mathrm{Mod}$ and $\mathrm{Cat}^1(\mathrm{Grps})$. This equivalence is constructed as follows:

Given $\partial: C\to B$, a crossed module, we form the semi-direct product, $G=C\times B$, using the action of B on C. The structural maps s,t are given by

$$s(c,b)=(1,b)$$

and

$$t(c,b)=(1,\partial(c)b)$$

for $c\in C$, $b\in B$. This clearly satisfies the axioms for a cat^1-group. On the other hand, given a cat^1-group (G,s,t), we set $C=\operatorname{Ker} s$, $B=\operatorname{Im} s$, and $\partial=t|C$, the restriction of t to C. The action of B on C is by conjugation within G. Again the axioms are easily checked.

2.3. We next introduce the pro-C analogue of the above. A *cat^1-pro-C-group* is a cat^1-group (G,s,t) in which G is a pro-C group and s and t are continuous endomorphisms of G. A morphism of cat^1-pro-C groups is a morphism ϕ: $(G,s,t)\to(G',s',t')$ of the underlying cat^1-groups such that ϕ is a continuous morphism of pro-C groups. This gives a category of cat^1-pro-C groups that we will denote $\mathrm{Cat}^1(\text{Pro-}C.\mathrm{Grps})$. There is a forgetful functor from $\mathrm{Cat}^1(\text{Pro-}C.\mathrm{Grps})$ to $\mathrm{Cat}^1(\mathrm{Grps})$ which will be denoted by $U_{C\mathrm{Grps}}$.

Lemma. *There is an equivalence of categories*

$$\text{Pro-}C\text{Cmod} \xrightarrow{\simeq} \text{Cat}^1(\text{Pro-}C.\text{Grps})$$

compatible, via the forgetful functors, with the equivalence between Cmod *and Cat*^{1}*(Grps), i.e. the diagram*

$$\begin{array}{ccc} \text{Pro-}C\text{Cmod} & \xrightarrow{\simeq} & \text{Cat}^1(\text{Pro-}C.\text{Grps}) \\ {\scriptstyle U_{\text{Cmod}}}\downarrow & & \downarrow{\scriptstyle U_{\text{CGrps}}} \\ \text{Cmod} & \xrightarrow{\simeq} & \text{Cat}^1(\text{Grps}) \end{array}$$

commutes.

Proof. In fact, if (C,B,∂) is a pro-C crossed module then $G=C\times B$ is a pro-C group and the endomorphisms s and t given earlier are continuous so the resulting (G,s,t) is a cat^1-pro-C group. Similarly, if (G,s,t) is a cat^1-pro-C groups then $(\text{Ker}\, s,\ \text{Im}\, s, t|_{\text{Ker}\, s})$ is a pro-C crossed module.

This lemma will enable us to prove the existence of a left adjoint for U_{Cmod}: Pro-CCmod$\rightarrow$Cmod by constructing one for U_{CGrps}: Cat^1(Pro-C.Grps)$\rightarrow$Cat^1(Grps). This latter construction will need projective limits within Cat^1(Pro-C.Grps) and so we will briefly look at their construction as it sheds more light on the pro-C completion functor that will result from their use.

Given a projective system $F: I\rightarrow \text{Cat}^1(\text{Pro-}C.\text{Grps})$, one notes that F is a projective system of groups together with two endomorphisms of projective systems, $s,t: F\rightarrow F$ satisfying $st=t$ and $ts=s$, plus a commutator condition. We form $\varprojlim F$ by taking the limit of this underlying system of pro-C groups together with the induced endomorphisms $\varprojlim s$ and $\varprojlim t$. Writing the result as $(\bar{F},\bar{s},\bar{t})$, we have merely to check the commutator condition $[\text{Ker}\,\bar{s}, \text{Ker}\,\bar{t}]=1$. However $\bar{F}$ can be realised as a subgroup of the product $\prod_{i\in I} F(i)$, and $\bar{t}((x_i))=(t(i)x_i)$, similarly for $\bar{s}$, so as the commutator subgroup $[\text{Ker}\, s(i), \text{Ker}\, t(i)]$ is trivial for each i in I, it is so for the limit as it can be calculated "pointwise".

Proposition 2.4. *A pro-C completion functor from Cat*^{1}*(Grps) to Cat*^{1}*(Pro-C.Grps) exists, (i.e. the forgetful functor* U_{CGrps} *has a left adjoint).*

Proof. An exact sequence

$$1\rightarrow(K,s',t')\xrightarrow{u}(G,s,t)\xrightarrow{v}(H,s'',t'')\rightarrow 1$$

of cat^1-groups is an exact sequence

$$1\rightarrow K\rightarrow G\rightarrow H\rightarrow 1$$

of the underlying groups and continuous maps compatible with the source and target maps. In this situation, we say that the cat^1-group (H, s'', t'') is the *quotient* of (G, s, t) by the normal sub-cat^1-group (K, s', t'). The latter is of *finite index* in (G, s, t) if H is finite.

Given any cat^1-group (G, s, t) the set of its normal sub-cat^1-groups (N, s', t') of finite index with $G/N \in C$ is directed by inclusion so we can form an inverse system of finite quotients of (G, s, t) and can take its limit within the category of cat^1-pro-C groups. (As usual one considers each finite cat^1-group as a pro-C one having the discrete topology.)

Thus we define a functor

$$\widetilde{\quad} : \mathrm{Cat}^1(\mathrm{Grps}) \to \mathrm{Cat}^1(\text{Pro-}C.\,\mathrm{Grps})$$

by

$$\widetilde{(G, s, t)} = \varprojlim \{\text{finite } C\text{-quotients of } (G, s, t)\}$$

General considerations of category theory then imply that this functor is left adjoint to the forgetful functor from Cat^1(Pro-C.Grps) to Cat^1(Grps).

Corollary 2.5. *A pro-C completion functor from* Cmod *to Pro-C*Cmod *exists* (*i.e. the forgetful functor* U_{Cmod} *has a left adjoint*)

Proof. In the diagram

$$\begin{array}{ccc} \text{Pro-}C\mathrm{Cmod} & \xrightarrow{\simeq} & \mathrm{Cat}^1(\text{Pro-}C.\mathrm{Grps}) \\ U_{\mathrm{Cmod}} \downarrow & & \downarrow U_{C\mathrm{Grps}} \\ \mathrm{Cmod} & \xrightarrow{\simeq} & \mathrm{Cat}^1(\mathrm{Grps}) \end{array}$$

we have found a left adjoint to the (vertical) functor on the right. This induces via the equivalence of categories a left adjoint for the left hand (vertical) functor.

Remark. One can attempt to use the functors defining the two equivalences to give an "explicit" description of this pro-C completion functor, but in what follows we shall merely use its existence and the universal property that it satisfies to compare it with the pro-C completion of the individual groups involved.

Notation. We will denote by $\widetilde{(C, G, \partial)}$ or, less accurately, $(\tilde{C}, \tilde{G}, \tilde{\partial})$, the pro-$C$ completion of the crossed module (C, G, ∂).

3. Relations between the pro-C completions of groups and of crossed modules

It is natural to want to compare this pro-C completion $(\tilde{C}, \tilde{G}, \tilde{\partial})$ with the pro-C

completions $\hat{C}, \hat{G}$ and $\hat{\partial}$ of the individual pieces of data involved. One may even wonder why $(\hat{C}, \hat{G}, \hat{\partial})$ is not itself always the same as $(\tilde{C}, \tilde{G}, \tilde{\partial})$. To start the study of this problem we first look at $\tilde{G}$.

Proposition 3.1. *For any crossed module* (C, G, ∂), $\tilde{G} \cong \hat{G}$.

Proof. This follows from an adjoint functor argument as follows: There is a forgetful functor

$$R\colon \mathrm{CMod} \to \mathrm{Grps}$$

given by $R(C, G, \partial) = G$ and also an analogous one R_{pC}: Pro-CCMod$\to$Pro-CGrps. These have left adjoints L and L_{pC} defined by $L(G) = (G, G, id_G)$ and similarly for L_{pC}.

We have a diagram of left and right adjoints

$$\begin{array}{ccc}
\text{Pro-}C\text{CMod} & \underset{L_{pC}}{\overset{R_{pC}}{\rightleftarrows}} & \text{Pro-}C\text{Grps} \\
(\tilde{\ }) \uparrow\downarrow U_{\mathrm{CMod}} & & (\hat{\ }) \uparrow\downarrow U_{\mathrm{Grps}} \\
\mathrm{CMod} & \underset{L}{\overset{R}{\rightleftarrows}} & \mathrm{Grps}
\end{array}$$

The right adjoint diagram commutes, so the left adjoint diagram commutes up to isomorphism, i.e.

$$\widetilde{(G, G, id_G)} \cong (\hat{G}, \hat{G}, id_{\hat{G}})$$

but better we have a sequence of isomorphisms: for a pro-C group H,

$$\begin{aligned}
&\text{Pro-}C\text{Grps}\,(R_{pc}(\widetilde{C, G, \partial}), H) \\
&\quad \cong \text{Pro-}C\text{CMod}((\widetilde{C, G, \partial}), L_{pc}(H)) \\
&\quad \cong \mathrm{CMod}((C, G, \partial), U_{\mathrm{CMod}} L_{pc}(H)) \\
&\quad \cong \mathrm{CMod}((C, G, \partial), L U_{\mathrm{Grps}}(H)) \qquad \text{by observation} \\
&\quad \cong \mathrm{Grps}(R(C, G, \partial), U_{\mathrm{Grps}}(H)) \\
&\quad \cong \mathrm{Grps}(G, U_{\mathrm{Grps}}(H)) \\
&\quad \cong \text{Pro-}C\text{Grps}(\hat{G}, H)
\end{aligned}$$

as required; hence $\hat{G} \cong \tilde{G}$, independently of what C is.

In order to study conditions which imply that $\tilde{C}$ and $\hat{C}$ are isomorphic it is convenient to introduce a condition that we will call the "cofinality condition".

Let (C, G, ∂) be a crossed module and write $\Omega_G(C)$ for the directed subset of $\Omega(C)$, the set of finite index normal subgroups of C, consisting of those $W \in \Omega(C)$, $C/W \in C$, which

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are G-invariant. We will say that (C, G, ∂) satisfies the *cofinality condition* if $\Omega_G(C)$ is cofinal in $\Omega(C)$.

Proposition 3.2. *If $G \in C$, then any crossed G-module, (C, G, ∂), satisfies the cofinality condition.*

Proof. Given any $W \in \Omega(C)$, let

$$W' = \bigcap_{g \in G} {}^{g}W,$$

be the intersection of all translates of W under the G-action. Then W' is G-invariant and as G in C, W' is of finite index $C/W' \in C$. As W' is contained in W, this completes the proof.

Theorem 3.3. *If (C, G, ∂) satisfies the cofinality condition, then $\tilde{C} \cong \hat{C}$.*

Proof. Recall that one has an isomorphism

$$\hat{C} \cong \varprojlim_{W \in \Omega(C)} C/W.$$

As $\Omega_G(C)$ is cofinal in $\Omega(C)$, we have that this is isomorphic to $\varprojlim_{W \in \Omega_G(C)} C/W$, so when considering an element of $\hat{C}$ we can represent it as a compatible family $(c_w W)_{W \in \Omega_G(C)}$ of elements with $c_w \in C$. Of course there is a natural map

$$\begin{array}{ccc} C & \xrightarrow{\phi_2} & \tilde{C} \\ {\scriptstyle \partial}\downarrow & & \downarrow{\scriptstyle \tilde{\partial}} \\ G & \xrightarrow{\phi_1} & \hat{G} \end{array}$$

coming from the adjointness. As $\hat{G}$ is pro-C, we have a factorisation via

$$\begin{array}{ccc} \hat{C} & \longrightarrow & \tilde{C} \\ & {\scriptstyle \hat{\partial}}\searrow \quad \swarrow{\scriptstyle \tilde{\partial}} & \\ & \hat{G} & \end{array}$$

and the various universality properties imply that it suffices to prove that $(\hat{C}, \hat{G}, \hat{\partial})$ is a pro-C crossed module in order to prove that $\tilde{C} \cong \hat{C}$. Thus we need to show that the G-action on C extends to a $\hat{G}$-action on $\hat{C}$ such that $\hat{\partial}$ is $\hat{G}$-equivariant and the Peiffer relation holds.

We need to define therefore a map

$$\hat{G} \times \hat{C} \to \hat{C}.$$

This we can attempt to do either topologically or using the identification of the category Pro-C.Grps with the category pro(Fin.Grps) of projective systems in the category of finite groups. For this we need for each $W \in \Omega_G(C)$ to pick a $(V, W) \in \Omega(G) \times \Omega_G(C)$ such that there is a map

$$\psi_W : G/V \times C/W' \to C/W$$

and that these maps are compatible with the bonding maps of the systems $\{G/V\}$ and $\{C/W\}$.

We pick $W' = W$ and $V = St_G(C/W)$. To see the reason for the latter choice, we note that since W is G-equivariant, there is a G-action on C/W; a finite group. This gives a homomorphism

$$G \to \mathrm{Aut}_C(C/W)$$

giving $V = St_G(C/W)$ as its kernel and we note that $V \lhd_{\mathrm{fin}} G$, since $\mathrm{Aut}_C(C/W)$ is finite.

We define ψ_W by the obvious rule

$$\psi_W(gV, cW) = {}^g cW.$$

Now assume $W' \subset W$, $W' \in \Omega_G(C)$, then we get a G-equivariant epimorphism

$$p_W^{W'} : C/W' \to C/W$$

and since if $v \in St_G(C/W')$, ${}^v cc^{-1} \in W'$ we have $V' = St_G(C/W') \subset V$ and an epimorphism in $q_V^{V'} : G/V' \to G/V$. Thus we have a commutative diagram

$$\begin{array}{ccc} G/V \times C/W & \longrightarrow & C/W \\ {\scriptstyle q_V^{V'} \times p_W^{W'}} \uparrow & & \uparrow {\scriptstyle p_W^{W'}} \\ G/V' \times C/W' & \longrightarrow & C/W' \end{array}$$

i.e. $\{\psi_W : W \in \Omega_G(C)\}$ is a map of projective systems. That it is an action is then clear.

To check the axioms we need an explicit description of $\hat{\partial}: \hat{C} \to \hat{G}$. Given $U \lhd_{\text{fin}} G$, there is a composed homomorphism $C \to G \to G/U$. Take N to be its kernel then since ∂ is G-equivariant and G/U is finite, it follows that N is in $\Omega_G(C)$ and that $U \subset St_G(C/N)$. These observations readily imply that $\hat{\partial}$, defined by

$$\hat{\partial}_u(cN_U) = \partial c_U U,$$

is not only well defined, but is $\hat{G}$-equivariant.

The proof that the Peiffer relation holds now follows from the Peiffer identity in (C, G, ∂) and the descriptions of $\hat{\partial}$ and the $\hat{G}$-action.

Corollary 3.4. *If G is in C, and (C, G, ∂) a crossed module then $(\hat{C}, \hat{G}, \hat{\partial})$ is a crossed module, which is the pro-C completion of (C, G, ∂).*

4. Pro-C completions of free crossed modules

Free crossed modules arise naturally in the study of presentations of groups (cf. Brown–Huebschmann [**2**]); their existence is also a crucial factor in many of the homological and cohomological applications of crossed modules (cf. Huebschmann [**10**]). Elsewhere, [**11**], we have discussed more fully the existence and basic properties of free pro-C crossed modules, here it will suffice to recall the definition of freeness in both the abstract and the pro-C cases.

Definition 4.1. Let $\partial: C \to B$ be a crossed B-module. Let S be a set and $g: S \to B$ some function. Then we say that $\partial: C \to B$ is a *free crossed module* on the function g if the following properties are satisfied:

(i) g lifts to a function $f: S \to C$ so $g = \partial f$

and

(ii) given any crossed module $\partial': C' \to B$ and function v: $S \to C'$ such that $\partial' v = \partial f$, there is a unique morphism $\phi: (C, \partial) \to (C', \partial')$ of crossed B-modules such that $\phi f = v$.

4.2. The pro-C analogue of this concept is obtained by insisting that all crossed modules be pro-C, all maps continuous and that one replaces the set S by a profinite (i.e. compact Hausdorff totally disconnected) space. Explicitly we have:

Given a pro-C crossed module (M, G, ∂) and a continuous map $v: X \to G$ where X is a profinite space, we say (M, G, ∂) is a *free* pro-C *crossed module* on v if

(i) v factors as ∂u with $u: X \to M$ continuous

and

(ii) given any pro-C crossed module (N, G, ∂') over G and a continuous function $h: X \to N$ satisfying $\partial' h = \partial u = v$ there is a unique morphism ψ: $(M, G, \partial) \to (N, G, \partial')$ of pro-C crossed modules over G such that $h = \psi u$.

Proposition 4.3. *If $\partial: C \to G$ is a free crossed module on a function $f: S \to G$, then $(\tilde{C}, \hat{G}, \tilde{\partial})$ is the free pro-C crossed module on the profinite completion $\hat{f}: \hat{S} \to \hat{G}$ of f.*

Remark. We should remark that $\hat{f}$ is obtained from the composite continuous map,

$$S \to G \to \hat{G}$$

where S is given the discrete topology, via the factorisation

$$S \to \hat{S} \xrightarrow{\hat{f}} \hat{G}$$

The profinite completion of a space can be obtained by taking the Boolean algebra of clopen (i.e. closed-open) subsets of the space and then forming the maximal ideal space of that Boolean algebra.

Proof of 4.3. We start by introducing some useful notation. We have already introduced $C\text{Mod}/G$ and Pro-$CC\text{Mod}/\hat{G}$ for the categories of crossed modules over G and pro-C crossed modules over $\hat{G}$ respectively. We also introduce categories: Sets/G and Spaces/$\hat{G}$ to denote the category of functions with codomain G (resp. continuous functions with codomain $\hat{G}$ and domain a profinite space). There are forgetful functors from $C\text{Mod}/G$ (resp. Pro-$CC\text{Mod}/\hat{G}$) to Sets/G (resp. Spaces/$\hat{G}$) and the existence of free crossed modules in the two instances correspond to the existence of left adjoints for these functors; thus

$$C\text{Mod}/G((C(S), G, \partial_f), (D, G, \partial')) \cong \text{Sets}/G((S, G, f), U(D, G, \partial'))$$

where $(C(S), G, \partial_f)$ is the free crossed module on (S, G, f) and

$$\text{Pro-}CC\text{Mod}/_{\hat{G}}(\bar{C}(X), \hat{G}, \bar{\partial}_f), (E, \hat{G}, \partial')) \cong \text{Spaces}/_{\hat{G}}((X, \hat{G}, f), U(E, \hat{G}, \partial'))$$

where $(\bar{C}(X), \hat{G}, \bar{\partial}_f)$ denotes the free pro-C module on $(X, \hat{G}, f)$.

Then we have

$$\begin{aligned}\text{Pro-}CC\text{Mod}/_{\hat{G}}(\widetilde{(C(S), G, \partial_f)}, (\tilde{D}, \hat{G}, \partial)) &\cong C\text{Mod}/_{\hat{G}}((C(S), G, \partial_f), U_\phi(\tilde{D}, \hat{G}, \partial)) \\ &\cong \text{Sets}/_G((S, G, f), UU_\phi(\tilde{D}, \hat{G}, \partial)) \\ &\cong \text{Spaces}/_{\hat{G}}((\hat{S}, \hat{G}, \hat{f}), U(\tilde{D}, \hat{G}, \partial))\end{aligned}$$

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$$\cong \text{Pro-}CC\text{Mod}/_{\hat{G}}((\bar{C}(\hat{S}),\hat{G},\bar{\partial}_f),(\tilde{D},\hat{G},\partial))$$

(Here we have used the base restriction functor U_ϕ along the homomorphism $\phi: G \to \hat{G}$. This functor is given by pullback along ϕ (see [11] for more on this construction).)

Thus $(\bar{C}(\hat{S}),\hat{G},\bar{\partial}_f) \cong \widehat{(C(S),G,\partial_f)}$.

5. Remarks on the non-exactness of C-completions

If G is a group and $N \lhd G$, it does not follow that $\hat{N}_C \lhd \hat{G}_C$, i.e. pro-C completion is not exact. This causes difficulties in algebraic geometry. Friedlander [7] gives an example in which the pro-L completion of a covering morphism does not yield an exact sequence under π_1 as expected. The base of the covering is a surface and the characteristic is 0, so otherwise the situation is extremely well behaved.

Anderson [1] points out that things can go wrong even for finite groups. Let $Sl(2,5)$ be, as usual, the group of 2×2 matrices of determinant 1 over the field $\mathbf{Z}_5$. The centre of $Sl(2,5)$ is of order 2. Completing at the prime 2 kills $Sl(2,5)$ but leaves the centre alone. Thus from

$$Z(Sl(2,5)) \lhd Sl(2,5),$$

one obtains

$$Z(Sl(2,5)) \to \{1\}.$$

Considering this second example from the viewpoint of this article, we note that as $Sl(2,5)$ is finite, the 2-completion of any normal pair $N \lhd Sl(2,5)$ should be a pro-2 crossed module (by 3.2) and of course since $Z(Sl(2,5))$ is cyclic of order 2, this is indeed so.

Friedlander's example is somewhat deeper. One expects fibrations to yield crossed modules under π_1 at least in topological cases, cf. [13]. Thus the fact that on completing away from the prime 2, a fibration sequence associated with a covering should yield a crossed module in π_1 and not a normal inclusion should not be cause for surprise. Friedlander's results from [6] may perhaps need considering from this viewpoint.

A similar phenomenon occurs in the theory of group presentations. For abstract groups, one knows that for any one relator group, G, in which the relator is not a proper power, one has that $\operatorname{cd} G = 2$. However for pro-p groups, Gildenhuys [8] has given an example of a pro-p presentation with two generators and one relation which is not a proper power and yet is such that the group thus presented has infinite cohomological dimension. This in crossed module terms can be explained as follows. The free crossed module

$$C(P) \xrightarrow{\partial} F(x,y)$$

of Gildenhuy's presentation has ∂ an inclusion. If we pro-p complete this crossed module, we get the corresponding construction in pro-p and the map in this free pro-p crossed module is no longer an inclusion, it has a kernel cyclic $\hat{\mathbf{Z}}_p(G)$-module with a periodic resolution (see [12] for a detailed treatment of this).

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