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On functoriality of Zelevinski involutions

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Abstract

In this paper, we present a conjecture on a relation between the Zelevinski involutions for reductive groups over a p-adic field and the conjectural A-packets. As evidence for this conjecture, we prove that the Zelevinski involutions, which are regarded as operators on the space of virtual characters, commute with the endoscopic transfers under the assumption of the fundamental lemma for groups and for Lie algebras.

1. Introduction

Let F be a p-adic field and G a connected reductive algebraic group defined over F. We denote by W_F the Weil group of F. Let ${}^LG = \hat{G} \rtimes W_F$ be the L-group of G. We denote by $\mathcal{L}^G$ the set of standard Levi subgroups of G. For $M \in \mathcal{L}^G$, we denote by $r(M)$ the semisimple split F-rank of M. Let $\Pi(G)$ be the set of equivalence classes of irreducible admissible representations of $G(F)$ and let $\mathbb{C}[\Pi(G)]$ be the space of virtual characters of $G(F)$. The parabolic induction defines a homomorphism $i_M^G : \mathbb{C}[\Pi(M)] \longrightarrow \mathbb{C}[\Pi(G)]$ and the (normalized) Jacquet functor defines a homomorphism $r_M^G : \mathbb{C}[\Pi(G)] \longrightarrow \mathbb{C}[\Pi(M)]$. Following Kato [Kat93], we define the Zelevinski involution $\mathbf{D}_G$ by

$$\mathbf{D}_G = \sum_{M \in \mathcal{L}^G} (-1)^{r(M)} i_M^G \circ r_M^G.$$

Let $\{M\}$ be the set of associate standard Levi subgroups of M. We say that $\pi \in \Pi(G)$ is of type $\{M\}$ if $r_M^G(\pi)$ is a non-zero linear combination of supercuspidal representations of $M(F)$. If π is of type $\{M\}$, then we put $r_\pi = r(M)$. For $\pi \in \Pi(G)$, we define

$$\mathbf{d}_G(\pi) = (-1)^{r_\pi} \mathbf{D}_G(\pi).$$

Aubert [Aub95, Aub96] proved that $\mathbf{d}_G(\pi)$ is irreducible. Thus the Zelevinski involution preserves the irreducibility. It seems natural to consider the relation between the Zelevinski involution and the conjectural Langlands functoriality. Nevertheless, the Zelevinski involution does not preserve the L-packets. We consider the A-packets conjectured by Arthur [Art89, Conjecture 6.1]. (In this paper, we follow the formulation of [Art89, Conjecture 6.1], although we can find a modified conjecture due to Vogan in [Vog93]). For a Langlands parameter $\phi : W_F \times SU_2(\mathbb{C}) \longrightarrow {}^LG$, we denote by $\Pi_\phi(G)$ the corresponding conjectural L-packet. Although $SU_2(\mathbb{C})$ is isomorphic to $SL_2(\mathbb{C})$, we denote the second factor of this group by $SU_2(\mathbb{C})$ in order to distinguish it from the factor $SL_2(\mathbb{C})$ used to define the Arthur parameters in [Art89]. Let

$$\psi : W_F \times SU_2(\mathbb{C}) \times SL_2(\mathbb{C}) \longrightarrow {}^LG$$

be an Arthur parameter. We put

$$S_\psi = \mathrm{Cent}(\psi, \hat{G}),$$
$$\mathbb{S}_\psi = S_\psi / S_\psi^0,$$

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where S_ψ^0 is the identity component of S_ψ. Let $\Pi_\psi(G)$ be the conjectural A-packet of ψ and $\rho = \rho_\psi : S_\psi \longrightarrow \mathbb{C}^\times$ the conjectural normalizing function. For $\overline{s} \in \mathbb{S}_\psi$ and $\pi \in \Pi_\psi(G)$, we define $\langle \overline{s}, \pi | \rho \rangle$ as in [Art89, Conjecture 6.1]. Then it is conjectured that

$$\overline{s} \longrightarrow \langle \overline{s}, \pi | \rho \rangle$$

is a virtual character of $\mathbb{S}_\psi$. We say that a virtual character $\theta \in \mathbb{C}[\Pi(G)]$ is stable if θ is stable as a distribution on $G(F)$. Let $\mathbb{C}[\Pi(G)]^{\mathrm{st}}$ be the space of stable virtual characters of $G(F)$ and $\mathbb{C}[\Pi_\psi(G)]$ the subspace of $\mathbb{C}[\Pi(G)]$ generated by $\Pi_\psi(G)$. We put $\mathbb{C}[\Pi_\psi(G)]^{\mathrm{st}} = \mathbb{C}[\Pi(G)]^{\mathrm{st}} \cap \mathbb{C}[\Pi_\psi(G)]$.

In the case that G is quasi-split, we put

$$\mathbb{S}_\psi^* = S_\psi / S_\psi^0 \cdot Z_{\hat{G}}^\Gamma,$$

where $Z_{\hat{G}}^\Gamma$ is the subgroup of the center $Z_{\hat{G}}$ of $\hat{G}$ consisting of the elements fixed by $\Gamma = \mathrm{Gal}(\overline{F}/F)$. We fix Whittaker data χ for G (see [KS99, § 5.3]). We determine the base point $\pi_\chi \in \Pi_{\phi_\psi}(G)$ as in [Art89, § 6], where $\Pi_{\phi_\psi}(G)$ is the L-packet corresponding to ψ. We define $\langle \overline{s}, \pi | \pi_\chi \rangle$ as in [Art89, Conjecture 6.1]. Then it is conjectured that $\langle \cdot, \pi | \pi_\chi \rangle$ is an irreducible character of $\mathbb{S}_\psi$. As F is a p-adic field, the following hypothesis is believed.

Hypothesis 1.1. *We have*

$$\dim \mathbb{C}[\Pi_\psi(G)]^{\mathrm{st}} = 1.$$

In the following, we assume the Arthur conjecture [Art89, Conjecture 6.1] and Hypothesis 1.1.

Now we turn to the Zelevinski involution. We identify $SU_2(\mathbb{C})$ with $SL_2(\mathbb{C})$ and define $d(\psi)$ by

$$d(\psi)(w \times t \times u) = \psi(w \times u \times t), \quad w \times t \times u \in W_F \times SU_2(\mathbb{C}) \times SL_2(\mathbb{C}).$$

Then $d(\psi)$ is the Arthur parameter constructed from ψ by interchanging the role of $SU_2(\mathbb{C})$ and $SL_2(\mathbb{C})$.

Conjecture 1.2. *We have*

$$\mathbf{d}_G(\Pi_\psi(G)) = \Pi_{d(\psi)}(G).$$

Since $S_\psi = S_{d(\psi)}$, we may identify $\mathbb{S}_\psi$ with $\mathbb{S}_{d(\psi)}$. Let ρ_d be the conjectural normalizing function of $d(\psi)$. In the case that G is quasi-split, we denote the base point in $\Pi_{\phi_{d(\psi)}}(G)$ by $\pi_{d,\chi}$.

Conjecture 1.3. *There exists a one-dimensional character μ of $\mathbb{S}_\psi$ such that*

$$\langle \overline{s}, \mathbf{d}_G(\pi) | \rho_d \rangle = \mu(\overline{s}) \langle \overline{s}, \pi | \rho \rangle$$

for all $\overline{s} \in \mathbb{S}_\psi$.

If G is quasi-split, then the above formula is equal to

$$\langle \overline{s}, \mathbf{d}_G(\pi) | \pi_{d,\chi} \rangle = \mu(\overline{s}) \langle \overline{s}, \pi | \pi_\chi \rangle.$$

(In the general case, the character μ may not be determined by the above relation.) The following conjecture is a special case of Conjecture 1.2.

Conjecture 1.4. *If G is quasi-split and if $\mathbb{S}_\psi^* = \{1\}$, then*

$$\mathbf{d}_G(\Pi_\psi(G)) = \Pi_{d(\psi)}(G).$$

As F is a p-adic field, it is believed that the condition $G = G^*$ and $\mathbb{S}_\psi^* = \{1\}$ implies that $\Pi_\psi(G) = \{\pi_\chi\}$ and $\Pi_{d(\psi)}(G) = \{\pi_{d,\chi}\}$. If we assume this, then Conjecture 1.4 asserts that $\mathbf{d}_G(\pi_\chi) = \pi_{d,\chi}$. In general, nevertheless, $\mathbf{d}_G(\pi_\chi)$ may not be equivalent to $\pi_{d,\chi}$. In fact, even in the case that $G = SL_2$, there exists ψ such that $\mathbb{S}_\psi^* \neq \{1\}$ and that $\mathbf{d}_G(\pi_\chi) \neq \pi_{d,\chi}$ (see [LL79]).

In the case that $G = GL_n$, Conjecture 1.2 follows from the results of Moeglin and Waldspurger [MW86].

Recently, Takuya Konno and Kazuko Konno checked that Conjecture 1.2 is compatible with their candidates for the A-packets on the quasi-split unitary group in four variables in [Kon03].

Conjecture 1.3 implies that the Zelevinski involutions behave well under the endoscopic transfers. In this paper, we discuss the relation between the Zelevinski involutions and the endoscopic transfers. By Corollary 3.4, we have

$$\mathbf{D}_G(\mathbb{C}[\Pi(G)]^{\mathrm{st}}) = \mathbb{C}[\Pi(G)]^{\mathrm{st}}.$$

Let $(H, \mathcal{H}, s, \xi)$ be a set of (standard) endoscopic data. For the sake of brevity, we assume that $\mathcal{H} \cong {}^L H$. Unfortunately the existence of the endoscopic transfer is still hypothetical. In this paper, we assume the fundamental lemma for groups [Art96, Hypothesis 3.1] and for Lie algebras [Wal97, Conjecture 1.3] to define the endoscopic transfer

$$\mathrm{Tran}_H^G : \mathbb{C}[\Pi(H)]^{\mathrm{st}} \longrightarrow \mathbb{C}[\Pi(G)]$$

of virtual characters (see Proposition 4.6). Let A_0 be a maximal split torus of G and let $A_{H,0}$ be a maximal split torus of H. We put $a(G) = \dim(A_0)$ and $a(H) = \dim(A_{H,0})$. Then we have the following theorem (see Theorem 6.6).

Theorem 1.5. *Assume the fundamental lemma for groups and for Lie algebras. Then we have*

$$\mathbf{D}_G \circ \mathrm{Tran}_H^G = (-1)^{a(G)-a(H)} \mathrm{Tran}_H^G \circ \mathbf{D}_H.$$

(In the case that $\mathcal{H} \ncong {}^L H$, we take a z-pair (H_1, ξ_{H_1}) as in [KS99, § 2.2]; see the formula in Theorem 6.7.)

By using this theorem, we can reduce Conjecture 1.2 to Conjecture 1.4 (see Lemma 7.2). Moreover, if G is quasi-split, then by using Theorem 1.5, we can show that Conjecture 1.4 implies the following formula (see Proposition 7.4):

$$\langle \overline{s}, \mathbf{d}_G(\pi) | \pi_{d,\chi} \rangle = \langle \overline{s}, \mathbf{d}_G(\pi_\chi) | \pi_{d,\chi} \rangle \langle \overline{s}, \pi | \pi_\chi \rangle,$$

where $\langle \cdot, \mathbf{d}_G(\pi_\chi) | \pi_{d,\chi} \rangle$ is a one-dimensional character of $\mathbb{S}_\psi$. This is Conjecture 1.3. In the case that G is not quasi-split, Conjecture 1.4 implies the following formula:

$$\langle \overline{1}, \pi | \rho \rangle = \langle \overline{1}, \mathbf{d}_G(\pi) | \rho_d \rangle.$$

In the theory of endoscopy, some relations are defined modulo inner automorphisms. To avoid this ambiguity, we fix endoscopic data, an inner twisting and splittings in the following way. Let $\varphi : G \longrightarrow G^*$ be a quasi-split inner twisting of G and A_0^* a maximal split torus of G^*. We fix an F-splitting $(B_0^*, T_0^*, \{X_\alpha\})$ of G^*, an F-splitting $(B_{H,0}, T_{H,0}, \{Y_\alpha\})$ of H, a Γ-splitting $(\mathcal{B}, \mathcal{T}, \{\mathcal{X}_{\check{\alpha}}\})$ of $\hat{G}$ and a Γ-splitting $(\mathcal{B}_H, \mathcal{T}_H, \{\mathcal{Y}_{\check{\alpha}}\})$ of $\hat{H}$. Then we may identify $\hat{T}_0^*$ with $\mathcal{T}$ and $\hat{T}_{H,0}$ with $\mathcal{T}_H$. We may assume that $A_0^* \subset T_0^*$ and that $A_{H,0} \subset T_{H,0}$. We say that a subtorus of A_0^* is standard if it is equal to the split component of the center of a standard Levi subgroup of G^*. We assume that $s \in \mathcal{T}$, $\xi(\mathcal{T}_H) = \mathcal{T}$ and $\xi(\mathcal{B}_H) \subset \mathcal{B}$. Let $i_0^* : T_{H,0} \longrightarrow T_0^*$ be the dual homomorphism of $\xi^{-1} : \mathcal{T} \longrightarrow \mathcal{T}_H$. We may assume that $i_0^*(A_{H,0})$ is a standard subtorus of A_0^* (see § 4). We choose an inner twisting φ such that $\varphi(A_0)$ is a standard subtorus of A_0^*. We define a positive root system of G by the pullback of the positive root system of G^*. For $M \in \mathcal{L}^G$, we have $\varphi(M) \in \mathcal{L}^{G^*}$.

In § 2, we collect the properties of double cosets of Weyl groups with respect to endoscopic groups and standard Levi subgroups, which is a generalization of [Car93, § 2.7]. The proofs of the results in § 2 are contained in the Appendix. Assume that $G = G^*$. Let

$$\Omega(G) = \mathrm{Norm}(A_0, G) / \mathrm{Cent}(A_0, G),$$
$$\Omega(H) = \mathrm{Norm}(A_{H,0}, H) / \mathrm{Cent}(A_{H,0}, H),$$

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be the Weyl groups. We denote the set of roots of (G, A_0) by $R(G) = R(G, A_0)$ and the set of roots $(H, A_{H,0})$ by $R(H) = R(H, A_{H,0})$. We identify $A_{H,0}$ with the image $i_0^*(A_{H,0})$ in $A_0 = A_0^*$. By Lemma 4.3, we may regard $\Omega(H)$ as a subgroup of $\Omega(G)$. For $M \in \mathcal{L}^G$, we put

$$\Omega(G)_{M,H} = \{\omega \in \Omega(G) \mid \omega(A_{H,0}) \supset A_M\},$$

where A_M is the split component of the center of M. Let

$$\tilde{D}_M = \{\omega \in \Omega(G)_{M,H}^{-1} \mid \omega(R^+(M)) > 0\}.$$

Then by Lemma 2.1, we can define $D_{M,H}$ by

$$D_{M,H} = \{\omega \in \tilde{D}_M^{-1} \mid \omega(R^+(H)) > 0\}.$$

Proposition 2.2 asserts that $D_{M,H}$ is a system of representatives for

$$\Omega(M)\backslash\Omega(G)_{M,H}/\Omega(H).$$

For $\omega \in D_{M,H}$, let $R = R(H) \cap \operatorname{res}_{A_{H,0}}(\omega^{-1}(R(M)))$, then by Lemma 2.3, R is a root system of a standard Levi subgroup M_ω of H. For $L \in \mathcal{L}^H$, we put

$$D_{M,H,L} = \{\omega \in D_{M,H} \mid M_\omega = L\}$$

and

$$a_{M,H,L} = \sharp D_{M,H,L}.$$

Then we have the following formula, which is a generalization of [Car93, Proposition 2.7.7]:

$$\sum_{M \in \mathcal{L}^G} (-1)^{r(M)} a_{M,H,L} = (-1)^{a(G)-a(H)} \cdot (-1)^{r(L)}. \tag{1.1}$$

We turn to the general G. For $M \in \mathcal{L}^G$, let $M^* = \varphi(M)$ and $D_{M,H} = D_{M^*,H}$. For $\omega \in D_{M,H}$, we put $M_\omega = (M^*)_\omega$. Let ${}^L M_\omega$ be the L-group of M_ω. Then we may regard ${}^L M_\omega$ as a standard Levi subgroup of ${}^L H$. We choose a representative $\hat{n}_\omega \in \operatorname{Norm}(\mathcal{T}, \hat{G})$ for

$$\omega \in \Omega(G^*) \subset \Omega(G^*, T_0^*) \cong \Omega(\hat{G}, \mathcal{T}).$$

We put $s'_\omega = \operatorname{Int} \hat{n}_\omega(s)$ and $\xi_\omega = \operatorname{Int} \hat{n}_\omega \circ \xi$. Then Lemma 5.1 asserts that $(M_\omega, {}^L M_\omega, s'_\omega, \xi_\omega)$ is a set of endoscopic data for M. The following formula in Theorem 5.6 is an analogue of the formula of Bernstein and Zelevinski [BZ77, Lemma 2.12]:

$$r_M^G \circ \operatorname{Tran}_H^G = \sum_{\omega \in D_{M,H}} \operatorname{Tran}_{M_\omega}^M \circ r_{M_\omega}^H. \tag{1.2}$$

Note that $D_{M,H}$ depends on the choice of a set of endoscopic data, an inner twisting and splittings.

In §§ 4 and 5, we prove this formula. In § 6, we prove Theorem 1.5 by using (1.1) and (1.2). In § 7, we discuss the conjectural relation between the Zelevinski involutions and the Arthur conjecture.

2. Cosets of Weyl groups

Properties of the double cosets of Weyl groups with respect to standard Levi subgroups are well known [Car93, § 2.7]. In this section, we collect results on the double cosets of Weyl groups with respect to endoscopic groups and standard Levi subgroups. Proofs of the results in this section are contained in the Appendix.

Let $\mathfrak{a}^G$ be a finite-dimensional vector space over the real number field $\mathbb{R}$ and let $(\mathfrak{a}^G)'$ be the vector space of linear forms of $\mathfrak{a}^G$. In this paper, we allow a root system to have a center. Therefore, a subset $R \subset (\mathfrak{a}^G)'$ is called a root system on $\mathfrak{a}^G$ if R is a (non-reduced) root system in the vector space generated by R. Let $R(G)$ be a root system on $\mathfrak{a}^G$. We denote by $\Omega(G)$ the Weyl group of $R(G)$.

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We fix a positive definite $\Omega(G)$-invariant symmetric bilinear form $(\ ,\ )$ on $(\mathfrak{a}^G)'$. This defines an isomorphism from $(\mathfrak{a}^G)'$ to $\mathfrak{a}^G$ and a positive definite symmetric bilinear form on $\mathfrak{a}^G$. For $\tilde{\alpha} \in R(G)$, we denote the corresponding reflection on $\mathfrak{a}^G$ by $s_{\tilde{\alpha}}$. We fix a positive root system. We denote the set of positive roots by $R^+(G)$ and the set of negative roots by $R^-(G)$. The set of simple roots is denoted by $S(G)$. For a subset $S(M)$ of $S(G)$, we denote the corresponding subroot system by $R(M)$. We put $R^+(M) = R(M) \cap R^+(G)$ and

$$\mathfrak{a}_M = \{a \in \mathfrak{a}^G \mid \tilde{\alpha}(a) = 0 \text{ for all } \tilde{\alpha} \in S(M)\}.$$

Let $\Omega(M)$ be the Weyl group of $R(M)$. We have

$$R(M) = \{\tilde{\alpha} \in R(G) \mid \tilde{\alpha}(a) = 0 \text{ for all } a \in \mathfrak{a}_M\},$$
$$\Omega(M) = \{\omega \in \Omega(G) \mid \omega(a) = a \text{ for all } a \in \mathfrak{a}_M\}.$$

We say that a subroot system $R \subset R(G)$ is *standard* if there exists $S(M) \subset S(G)$ such that $R = R(M)$ and a subspace $\mathfrak{a} \subset \mathfrak{a}^G$ is *standard* if there exists $S(M) \subset S(G)$ such that $\mathfrak{a} = \mathfrak{a}_M$. For a subspace $\mathfrak{a} \subset \mathfrak{a}^G$, we put

$$\Omega(G)_{\mathfrak{a}} = \{\omega \in \Omega(G) \mid \omega(\mathfrak{a}) = \mathfrak{a}\}.$$

We denote the restriction of $\tilde{\alpha} \in R(G)$ to $\mathfrak{a}$ by $\mathrm{res}_{\mathfrak{a}}(\tilde{\alpha})$. If $\omega \in \Omega(G)_{\mathfrak{a}}$, then we denote the restriction of ω to $\mathfrak{a}$ by $\mathrm{res}_{\mathfrak{a}}(\omega)$. For a standard subspace $\mathfrak{a}_M \subset \mathfrak{a}^G$, we put

$$R(G; \mathfrak{a}_M) = \{\mathrm{res}_{\mathfrak{a}_M}(\tilde{\alpha}) \mid \tilde{\alpha} \in R(G)\} - \{0\}.$$

For $\alpha \in R(G; \mathfrak{a}_M)$, we denote by s_α the reflection on $\mathfrak{a}_M$ corresponding to α with respect to the restriction of the bilinear form $(\ ,\ )$ to $\mathfrak{a}_M$. In general, $R(G; \mathfrak{a}_M)$ may not be a root system on $\mathfrak{a}_M$. In this paper, we say that $\alpha \in R(G; \mathfrak{a}_M)$ is $R(G)$-*symmetric* if there exists $\omega \in \Omega(G)_{\mathfrak{a}_M}$ such that $\mathrm{res}_{\mathfrak{a}_M}(\omega) = s_\alpha$. If there exists a simple root $\tilde{\alpha}$ such that $\alpha = \mathrm{res}_{\mathfrak{a}_M} \tilde{\alpha}$, then Lemma 4 asserts that α is $R(G)$-symmetric if and only if we have $\omega_-^\alpha \tilde{\alpha} = -\tilde{\alpha}$, where ω_-^α is the longest element in the Weyl group of $S(M) \cup \{\tilde{\alpha}\}$. We denote by $R(G; \mathfrak{a}_M)_{\mathrm{sym}}$ the subset consisting of the $R(G)$-symmetric roots on $\mathfrak{a}_M$. We put $R^+(G; \mathfrak{a}_M) = \{\mathrm{res}_{\mathfrak{a}_M}(\tilde{\alpha}) \mid \tilde{\alpha} \in R^+(G)\} - \{0\}$ and $R^-(G; \mathfrak{a}_M) = -R^+(G; \mathfrak{a}_M)$. Since $\mathfrak{a}_M$ is a standard subspace, the intersection of $R^+(G; \mathfrak{a}_M)$ and $R^-(G; \mathfrak{a}_M)$ are empty. We put $R^+(G; \mathfrak{a}_M)_{\mathrm{sym}} = R^+(G; \mathfrak{a}_M) \cap R(G; \mathfrak{a}_M)_{\mathrm{sym}}$. We write $\tilde{\alpha} > 0$ if $\tilde{\alpha}$ is a positive root and $\tilde{\alpha} < 0$ if $\tilde{\alpha}$ is a negative root. Moreover, for a subset $R \subset R(G)$, we write $R > 0$ if $R \subset R^+(G)$ and $R < 0$ if $R \subset R^-(G)$.

Until the end of this section, we fix $S(M^H) \subset S(G)$. We put $\mathfrak{a}^H = \mathfrak{a}_{M^H}$. We abbreviate $\mathrm{res}_{\mathfrak{a}^H}$ to res_H and $\Omega(G)_{\mathfrak{a}^H}$ to $\Omega(G)_H$. We also fix a root system $R(H) \subset R(G; \mathfrak{a}^H)_{\mathrm{sym}}$ on $\mathfrak{a}^H$. Let $R^+(H) = R(H) \cap R^+(G; \mathfrak{a}^H)_{\mathrm{sym}}$, then $R^+(H)$ is a positive system of $R(H)$. For $S(M) \subset S(G)$ and $\omega \in \Omega(G)$, we put

$$l_M(\omega) = \sharp\{\tilde{\alpha} \in R^+(M) \mid \omega\tilde{\alpha} < 0\}.$$

We denote by $\Omega(H)$ the Weyl group of $R(H)$ acting on $\mathfrak{a}^H$. Since $\alpha \in R(H)$ is $R(G)$-symmetric, there exists a unique $\tilde{s}_\alpha \in \Omega(G)_H$ such that $\mathrm{res}_H \tilde{s}_\alpha = s_\alpha$ and $l_{M^H}(\tilde{s}_\alpha) = 0$. Therefore, for each $\omega \in \Omega(H)$, there exists a unique $\tilde{\omega} \in \Omega(G)_H$ such that $\mathrm{res}_H(\tilde{\omega}) = \omega$ and $l_{M^H}(\tilde{\omega}) = 0$. The homomorphism $\omega \longrightarrow \tilde{\omega}$ allows us to regard $\Omega(H)$ as a subgroup of $\Omega(G)$. We put

$$\tilde{D}_M = \{\omega \in \Omega(G) \mid l_M(\omega) = 0 \text{ and } \omega(\mathfrak{a}_M) \subset \mathfrak{a}^H\}.$$

We write $\mathbb{R}_+^\times$ for $\{x \in \mathbb{R} \mid x > 0\}$.

Lemma 2.1. *Let $\omega \in \tilde{D}_M^{-1}$ and $\omega' \in \Omega(G)_H$. Let $\tilde{\alpha} \in R^+(G)$ be a positive root satisfying $\mathrm{res}_H(\tilde{\alpha}) \neq 0$. If $\omega\omega'\tilde{\alpha} > 0$, then for any $\tilde{\alpha}' \in R^+(G)$ satisfying $\mathrm{res}_H(\tilde{\alpha}') \in \mathbb{R}_+^\times \mathrm{res}_H(\tilde{\alpha})$, we have $\omega\omega'\tilde{\alpha}' > 0$ and if $\omega\omega'\tilde{\alpha} < 0$, then for any $\tilde{\alpha}' \in R^+(G)$ satisfying $\mathrm{res}_H(\tilde{\alpha}') \in \mathbb{R}_+^\times \mathrm{res}_H(\tilde{\alpha})$, we have $\omega\omega'\tilde{\alpha}' < 0$.*

Let $\omega \in \tilde{D}_M^{-1}$ and $\alpha \in R^+(G;\mathfrak{a}^H)$, then we say that $\omega\alpha$ is *positive* if we have $\omega\tilde{\alpha} > 0$ for all $\tilde{\alpha} \in R^+(G)$ satisfying $\mathrm{res}_H(\tilde{\alpha}) \in \mathbb{R}_+^\times \alpha$ and $\omega\alpha$ is *negative* if we have $\omega\tilde{\alpha} < 0$ for all $\tilde{\alpha} \in R^+(G)$ satisfying $\mathrm{res}_H(\tilde{\alpha}) \in \mathbb{R}_+^\times \alpha$. We write $\omega\alpha > 0$ if $\omega\alpha$ is positive and $\omega\alpha < 0$ if $\omega\alpha$ is negative. For $\omega \in \tilde{D}_M^{-1}$, we define $l_H(\omega)$ by

$$l_H(\omega) = \sharp\{\alpha \in R^+(H) \mid \omega\alpha < 0\}.$$

We put

$$D_{M,H} = \{\omega \in \tilde{D}_M^{-1} \mid l_H(\omega) = 0\}.$$

We also put

$$\Omega(G)_{M,H} = \{\omega \in \Omega(G) \mid \omega(\mathfrak{a}^H) \supset \mathfrak{a}_M\},$$

then $\Omega(G)_{M,H}$ is invariant under the left and right action of $\Omega(M)$, $\Omega(H)$, respectively.

Proposition 2.2. *The subset $D_{M,H} \subset \Omega(G)_{M,H}$ is a system of representatives for the set of double cosets $\Omega(M)\backslash\Omega(G)_{M,H}/\Omega(H)$.*

The set of standard subroot systems $\{R(M) \mid S(M) \subset S(G)\}$ of $R(G)$ is denoted by $\mathcal{L}^G$. We write $\mathcal{L}^H$ for the set of standard subroot systems of $R(H)$.

Lemma 2.3. *Let $\omega \in D_{M,H}$, then*

$$R(H) \cap \mathrm{res}_H(\omega^{-1}(R(M))) \in \mathcal{L}^H.$$

For $R(M) \in \mathcal{L}^G$ and $R(L) \in \mathcal{L}^H$, we put

$$\begin{aligned} D_{M,H,L} &= \{\omega \in D_{M,H} \mid R(H) \cap \mathrm{res}_H(\omega^{-1}(R(M))) = R(L)\},\\ a_{M,H,L} &= \sharp D_{M,H,L}. \end{aligned}$$

We write $r(M)$ for $\sharp S(M)$ and $r(L)$ for $\sharp S(L)$. The following theorem is the main result of this section.

Theorem 2.4. *We have*

$$\sum_{R(M)\in\mathcal{L}^G} (-1)^{r(M)} a_{M,H,L} = (-1)^{r(M^H)} \cdot (-1)^{r(L)}.$$

The following lemma will be used in § 6.

Lemma 2.5. *Let $R(M), R(M_0) \in \mathcal{L}^G$. Assume that $R(M_0)$ has no other associate standard subroot system than $R(M_0)$ itself. If $R(M)$ satisfies $\omega(\mathfrak{a}_M) \subset \mathfrak{a}_{M_0}$ for some $\omega \in \Omega(G)$, then we have*

$$R(M) \supset R(M_0).$$

3. Zelevinski involutions

Let G be a connected reductive linear algebraic group defined over a p-adic field F. We denote by Γ the Galois group $\mathrm{Gal}(\overline{F}/F)$ of F. We fix a minimal parabolic subgroup P_0 of G defined over F. We also fix a Levi subgroup $M_0 \subset P_0$ defined over F and a maximal torus $T_0 \subset M_0$ defined over F. We denote by A_0 the split component of T_0. Let G^* be a quasi-split inner form of G and let $\varphi : G \longrightarrow G^*$ be an inner twisting. We fix a Borel subgroup B_0^* of G^* defined over F and a maximal torus $T_0^* \subset B_0^*$ defined over F. Let $R(G^*, T_0^*)$ be the set of roots of (G^*, T_0^*), $R^+(G^*, T_0^*)$ the set of positive roots corresponding to B_0^* and $S(G^*, T_0^*)$ the set of simple roots. We write $\mathcal{L}^{G^*}$ for the set of standard Levi subgroups of G^*. Let A_0^* be the split component of T_0^*.

For $M^* \in \mathcal{L}^{G^*}$, we denote by A_{M^*} the split component of the center Z_{M^*} of M^*. We say that a subtorus $A \subset A_0^*$ is *standard* if there exists a standard Levi subgroup $M^* \in \mathcal{L}^{G^*}$ such that $A = A_{M^*}$. For a subtorus $A \subset A_0^*$, we put $M_A^* = \mathrm{Cent}(A, G^*)$. We may assume that φ satisfies $\varphi(T_0) = T_0^*$ and that $\varphi(A_0)$ is a standard subtorus of A_0^*. Moreover, we assume that $\varphi(P_0) \supset B_0^*$. Therefore, we can define a set of positive roots $R^+(G, T_0)$ by the pullback of $R^+(G^*, T_0^*)$. Let $\mathcal{L}^G$ be the set of standard Levi subgroups of G, then for any $M \in \mathcal{L}^G$, we have $\varphi(M) \in \mathcal{L}^{G^*}$. Moreover, it is easy to see that the restriction of φ to M is an inner twisting of M to the quasi-split inner form $\varphi(M)$. We put $r(M) = \sharp S(M, A_0)$. We denote by $\Omega(G, T_0)$ the Weyl group of (G, T_0) and by $\Omega(G, T_0)_F$ the subgroup consisting of the elements defined over F. We write $R(G, A_0)$ for the set of relative roots of (G, A_0). We define the set of positive roots $R^+(G, A_0)$ by the restriction of $R^+(G, T_0)$. Let $S(G, A_0)$ be the set of simple roots and $\Omega(G, A_0)$ the Weyl group of (G, A_0). We define $R(G^*, A_0^*)$, $R^+(G^*, A_0^*)$, $S(G^*, A_0^*)$, $\Omega(G^*, A_0^*)$ and $r(M^*)$ similarly. For a maximal torus $T \subset G$ defined over F, we denote by A_T the split component of T. Therefore, A_T is conjugate to a standard subtorus of A_0. Let $\Omega(G(F), T) = \mathrm{Norm}(T, G(F))/T(F)$, then $\Omega(G(F), T)$ is a subgroup of $\Omega(G, T)_F$. Since G^* is quasi-split, we have

$$\Omega(G^*(F), T_0^*) = \Omega(G^*, T_0^*)_F \cong \Omega(G^*, A_0^*).$$

We identify $\Omega(G^*, A_0^*)$ with $\Omega(G^*, T_0^*)_F$. We determine a Haar measure on $T(F)$ by the condition that the volume of the maximal compact subgroup of $T(F)$ is 1. Let G_{reg} be the set of strongly regular semisimple elements in $G(F)$ and G_{ell} the set of elliptic elements in G_{reg}. We denote by $\Pi(G)$ the set of equivalence classes of irreducible admissible representations of $G(F)$. We write $\mathbb{C}[\Pi(G)]$ for the space of virtual characters. Then $\mathbb{C}[\Pi(G)]$ consists of the finite linear combinations of $\Pi(G)$. For $M \in \mathcal{L}^G$, let

$$i_M^G : \mathbb{C}[\Pi(M)] \longrightarrow \mathbb{C}[\Pi(G)]$$

be the homomorphism corresponding to the (normalized) induction and

$$r_M^G : \mathbb{C}[\Pi(G)] \longrightarrow \mathbb{C}[\Pi(M)]$$

the homomorphism corresponding to the (normalized) Jacquet functor. Let $\pi \in \Pi(G)$, then by the theorem of Harish-Chandra [Har78], the distribution character of π can be represented by a locally constant function ch_π on G_{reg}. We define a function $I^G(\pi)$ on G_{reg} by

$$I^G(\pi, \gamma) = \Delta_G(\gamma)\, \mathrm{ch}_\pi(\gamma),$$

where $\gamma \in G_{\mathrm{reg}}$ and $\Delta_G(\gamma) = |\prod_\alpha(\alpha(\gamma) - 1)|_F^{1/2}$ is the Weyl denominator. We extend this definition to $\theta \in \mathbb{C}[\Pi(G)]$. We denote by $C_c^\infty(G)$ the space of locally constant compactly supported functions on $G(F)$. For $f \in C_c^\infty(G)$, we define

$$I(\gamma, f) = \Delta_G(\gamma) \int_{G(F)/G_\gamma(F)} f(g\gamma g^{-1})\, dg,$$

where $\gamma \in G_{\mathrm{reg}}$ and $G_\gamma = \mathrm{Cent}(\gamma, G)$. Let $\{\gamma\}_{\mathrm{st}}$ be the set of conjugacy classes in the stable conjugacy class of γ. We define

$$I^{\mathrm{st}}(\gamma, f) = \sum_{\gamma' \in \{\gamma\}_{\mathrm{st}}} I(\gamma', f).$$

Put $C_c^\infty(G)^- = \{f \in C_c^\infty(G) \mid I^{\mathrm{st}}(\gamma, f) = 0 \text{ for all } \gamma \in G_{\mathrm{reg}}\}$. We say that a distribution D on $G(F)$ is *stable* if D satisfies $D(f) = 0$ for all $f \in C_c^\infty(G)^-$. We also say that a virtual character $\theta \in \mathbb{C}[\Pi(G)]$ is *stable* if θ is stable as a distribution on $G(F)$. Then θ is stable if and only if $I^G(\theta, \gamma) = I^G(\theta, \gamma')$ holds for all $\gamma \in G_{\mathrm{reg}}$ and $\gamma' \in \{\gamma\}_{\mathrm{st}}$. We denote by $\mathbb{C}[\Pi(G)]^{\mathrm{st}}$ the subspace of stable virtual characters. For a maximal torus T^M of M and a maximal torus T^G of G, we put

$$I_{\mathrm{conj}}^{G,M}(T^M, T^G) = \{i : T^M \longrightarrow T^G \mid i = \mathrm{Int}\, g \text{ for some } g \in G(F)\}.$$

Then $\Omega(G(F),T^G)$ and $\Omega(M(F),T^M)$ act on $I^{G,M}_{\mathrm{conj}}(T^M,T^G)$ from the left and right, respectively. Let $\mathcal{T}^G_{\mathrm{conj}}$ be a system of representatives for the conjugacy classes of maximal tori of G. We choose a Haar measure on $M(F)$ in such a way that the following formula holds.

$$I^G(i^G_M(\theta),\gamma) = \sum_{T^M\in\mathcal{T}^M_{\mathrm{conj}}} \sum_{i\in I^{G,M}_{\mathrm{conj}}(T^M,G_\gamma)/\Omega(M(F),T^M)} I^M(\theta,i^{-1}(\gamma)), \quad \gamma\in G_{\mathrm{reg}}. \tag{3.1}$$

We put $M_{G\text{-reg}} = M(F)\cap G_{\mathrm{reg}}$. We denote by A^-_M the set of $a\in A_M(F)$ satisfying $|\alpha(a)|_F<1$ for all $\alpha\in R^+(G,A_0)-R^+(M,A_0)$. In this paper, we use Casselman's character formula in the following form.

Lemma 3.1. *Let $M\in\mathcal{L}^G$ and $a\in A^-_M$, then for each $m\in M_{G\text{-reg}}$, we can choose a positive number n_0 such that*

$$I^G(\theta,a^nm) = I^M(r^G_M(\theta),a^nm)$$

holds for any $\theta\in\mathbb{C}[\Pi(G)]$ and $n\geqslant n_0$.

Proof. Let P be the standard parabolic subgroup with Levi factor M. For each $m\in M_{G\text{-reg}}$ we can choose an M-conjugate $m'\in M(F)$ and a positive number n_0 so that for all $n\geqslant n_0$ the parabolic subgroup $P_{a^nm'}$ in [Cas77] is a standard parabolic subgroup contained in P and $a^nm'\in G_{\mathrm{reg}}$. Let $L\in\mathcal{L}^G$ be the standard Levi subgroup of $P_{a^nm'}$. Then by [Cas77, Theorem 5.2], we have $I^G(\theta,a^nm') = I^L(r^G_L(\theta),a^nm')$ for all $n\geqslant n_0$ and $\theta\in\mathbb{C}[\Pi(G)]$. On the other hand, by applying [Cas77, Theorem 5.2] to $r^G_M(\theta)$, we have $I^M(r^G_M(\theta),a^nm') = I^L(r^M_L\circ r^G_M(\theta),a^nm')$. Thus $I^G(\theta,a^nm') = I^M(r^G_M(\theta),a^nm')$. Since a^nm and a^nm' are M-conjugate, this completes the proof of the lemma. □

Let $\theta\in\mathbb{C}[\Pi(M)]$, $m\in M_{\mathrm{reg}}$ and $a\in A_M$, then

$$n\in\mathbb{Z}\longrightarrow I^M(\theta,a^nm)$$

is a finite linear combination of quasi-characters of $\mathbb{Z}$. Hence, we have the following lemma.

Lemma 3.2. *Let $\theta,\theta'\in\mathbb{C}[\Pi(M)]$, $m,m'\in M_{\mathrm{reg}}$ and $a,a'\in A_M$. If there exists a positive number n_0 such that*

$$I^M(\theta,a^nm) = I^M(\theta',a'^nm')$$

holds for all $n\geqslant n_0$, then we have

$$I^M(\theta,m) = I^M(\theta',m').$$

Lemma 3.3. *We have*

$$i^G_M(\mathbb{C}[\Pi(M)]^{\mathrm{st}})\subset\mathbb{C}[\Pi(G)]^{\mathrm{st}},$$
$$r^G_M(\mathbb{C}[\Pi(G)]^{\mathrm{st}})\subset\mathbb{C}[\Pi(M)]^{\mathrm{st}}.$$

Proof. Let $m,m'\in M_{G\text{-reg}}$, $\theta\in\mathbb{C}[\Pi(G)]^{\mathrm{st}}$ and $a\in A^-_M$. Suppose that m and m' are stably M-conjugate. Then a^nm and a^nm' are stably G-conjugate. Therefore, by using Lemma 3.1, we have $I^M(r^G_M(\theta),a^nm) = I^M(r^G_M(\theta),a^nm')$ for sufficiently large n. Hence, by Lemma 3.2, we have the required relation for r^G_M. The relation for i^G_M is well known. □

We define the Zelevinski involution $\mathbf{D}_G$ by

$$\mathbf{D}_G = \sum_{M\in\mathcal{L}^G}(-1)^{r(M)} i^G_M\circ r^G_M.$$

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(See [Kat93] and [Aub95].) It is known that $\mathbf{D}_G \circ \mathbf{D}_G = \mathrm{id}$ and that $\mathbf{D}_G \circ i_M^G = i_M^G \circ \mathbf{D}_M$. If $\pi \in \Pi(G)$, then Aubert [Aub95, Aub96] has shown that either $\mathbf{D}_G(\pi)$ or $-\mathbf{D}_G(\pi)$ is irreducible. Now, Lemma 3.3 implies the following corollary.

Corollary 3.4. *We have*

$$\mathbf{D}_G(\mathbb{C}[\Pi(G)]^{\mathrm{st}}) = \mathbb{C}[\Pi(G)]^{\mathrm{st}}.$$

4. Endoscopy

We denote by W_F the Weil group of F. We write $\hat{G}$ for the dual group of G and write ${}^LG$ for the L-group $\hat{G} \rtimes W_F$ of G. We fix a Γ-splitting $(\mathcal{B}, \mathcal{T}, \{\mathcal{X}_{\check{\alpha}}\})$ of $\hat{G}$. We also fix an F-splitting $(B_0^*, T_0^*, \{X_\alpha\})$ containing the pair (B_0^*, T_0^*). Then we may identify $\mathcal{T}$ with the dual group $\hat{T}_0^*$ of T_0^*. Let $(H, \mathcal{H}, s, \xi)$ be a set of endoscopic data for G. In this paper, we say that $(H, \mathcal{H}, s, \xi)$ and $(H', \mathcal{H}', s', \xi')$ are *equivalent* endoscopic data if there exists $g \in \hat{G}$ such that $g\xi(\mathcal{H})g^{-1} = \xi'(\mathcal{H}')$ and $gsg^{-1} \in s' \cdot Z_{\hat{G}}$. We fix an F-splitting $(B_{H,0}, T_{H,0}, \{Y_\alpha\})$ of H and a Γ-splitting $(\mathcal{B}_H, \mathcal{T}_H, \{\mathcal{Y}_{\check{\alpha}}\})$ of $\hat{H}$. We identify $\mathcal{T}_H$ with $\hat{T}_{H,0}$. By replacing the endoscopic data by equivalent data, we may assume that $s \in \mathcal{T}$, $\xi(\mathcal{T}_H) = \mathcal{T}$ and $\xi(\mathcal{B}_H) \subset \mathcal{B}$. Let

$$i_0^* : T_{H,0} \longrightarrow T_0^*$$

be the dual homomorphism of

$$\xi^{-1} : \hat{T}_0^* \cong \mathcal{T} \longrightarrow \mathcal{T}_H \cong \hat{T}_{H,0}.$$

We also write i_0^* for the corresponding morphism $R(H, T_{H,0}) \longrightarrow R(G, T_0^*)$. We denote by $A_{H,0}$ the split component of $T_{H,0}$.

Lemma 4.1. *There exists $\omega \in \Omega(G^*, T_0^*)$ such that $\omega \circ i_0^*(A_{H,0})$ is a standard subtorus of A_0^* and that*

$$\omega \circ i_0^*(R^+(H, T_{H,0})) \subset R^+(G^*, T_0^*).$$

Proof. Choose an admissible embedding $\mathrm{Int}\, g \circ i_0^* : T_{H,0} \longrightarrow T^* = \mathrm{Int}\, g(T_0^*)$ of $T_{H,0}$, where $g \in G^*(\overline{F})$. We may assume that A_{T^*} is a standard subtorus of A_0^*. Put $M^* = M_{A_{T^*}}$. Then there exists $m \in M^*(\overline{F})$ such that $mg \in \mathrm{Norm}(T_0^*, G^*)$. Let $\omega' \in \Omega(G^*, T_0^*)$ be the image of mg. Put

$$R'^+(H, T_{H,0}) = \{\alpha \in R(H, T_{H,0}) \mid \omega' \circ i_0^*(\alpha) \in R^+(G^*, T_0^*)\}.$$

Then $R'^+(H, T_{H,0})$ is preserved by Γ. Therefore, there exists an element $\omega_H \in \Omega(H, T_{H,0})_F$ such that $\omega_H(R^+(H, T_{H,0})) = R'^+(H, T_{H,0})$. Since $i_0^* \circ \Omega(H, T_{H,0}) \circ i_0^{*-1} \subset \Omega(G^*, T_0^*)$, we have $\omega' \cdot (i_0^* \circ \omega_H \circ i_0^{*-1}) \in \Omega(G^*, T_0^*)$. Put $\omega = \omega' \cdot (i_0^* \circ \omega_H \circ i_0^{*-1})$. Then ω satisfies the required properties. □

We identify $\Omega(G^*, T_0^*)$ with $\Omega(\hat{G}, \mathcal{T})$. We also identify $\Omega(H, T_H)$ with $\Omega(\hat{H}, \mathcal{T}_H)$. Let $\hat{n}_\omega \in \mathrm{Norm}(\mathcal{T}, \hat{G})$ be a representative for $\omega \in \Omega(\hat{G}, \mathcal{T})$. Since $(H, \mathcal{H}, s, \xi)$ and $(H, \mathcal{H}, \mathrm{Int}\, \hat{n}_\omega(s), \mathrm{Int}\, \hat{n}_\omega \circ \xi)$ are equivalent endoscopic data, we may assume that $i_0^*(A_{H,0})$ is a standard subtorus of A_0^* and that

$$s \in \mathcal{T},$$
$$\xi(\mathcal{T}_H) = \mathcal{T},$$
$$\xi(\mathcal{B}_H) \subset \mathcal{B}.$$

We put $A^H = i_0^*(A_{H,0})$ and $M^H = \mathrm{Cent}(A^H, G^*)$. Then $M^H \in \mathcal{L}^{G^*}$.

Lemma 4.2. *Let A_1 and A_2 be subtori of A_0^*. If $g \in G^*(\overline{F})$ satisfies $\mathrm{Int}\, g(A_1) = A_2$, then there exists $\omega \in \Omega(G^*, T_0^*)_F$ such that $\omega(A_1) = A_2$ and $\mathrm{res}_{A_1}(\omega^{-1} \circ \mathrm{Int}\, g) = \mathrm{id}_{A_1}$.*

Proof. Put $M_2^* = \mathrm{Cent}(A_2, G^*)$. Then M_2^* is quasi-split and T_0^* is a maximally split torus of M_2^*. Since A_1 and A_2 are split tori, we have $\mathrm{res}_{A_2}(\mathrm{Int}\, \sigma(g)g^{-1}) = \mathrm{id}_{A_2}$ for any $\sigma \in \Gamma$. This implies that $\sigma(g)g^{-1} \in M_2^*(\overline{F})$. Hence, by [Kot82, Corollary 2.2], there exists $m \in M_2^*(\overline{F})$ such that the torus $T^* = \operatorname{Int} mg(T_0^*)$ is defined over F and that the homomorphism $\operatorname{Int} mg : T_0^* \longrightarrow T^*$ is defined over F. This implies that T^* is also a maximally split torus of M_2^*. Therefore, we may assume that $T^* = T_0^*$. Then $mg \in \operatorname{Norm}(T_0^*, G^*)$. Let $\omega \in \Omega(G^*, T_0^*)_F$ be the image of mg, then it is immediate that $\omega(A_1) = A_2$ and $\operatorname{res}_{A_1}(\omega^{-1} \circ \operatorname{Int} g) = \operatorname{id}_{A_1}$. □

Lemma 4.3. *Let $\omega \in \Omega(H, A_{H,0})$, then there exists a unique $\omega_G \in \Omega(G^*, A_0^*)$ satisfying the following conditions:*

1) $\omega_G(A^H) = A^H$;
2) $\operatorname{res}_{A_{H,0}}(i_0^{*-1} \circ \omega_G \circ i_0^*) = \omega$;
3) $\omega_G(R^+(M^H, T_0^*)) \subset R^+(G^*, T_0^*)$.

Proof. Let $\omega' = i_0^* \circ \omega \circ i_0^{*-1}$, then we have $\omega' \in \Omega(G^*, T_0^*)$. Put $A_1 = A_2 = A^H$ and $g = n_{\omega'}$. Then apply Lemma 4.2. □

We define a homomorphism

$$i^* : \Omega(H, A_{H,0}) \longrightarrow \Omega(G^*, A_0^*)$$

by putting $i^*(\omega) = \omega_G$.

Let $R(G^*, A^H)_{\mathrm{sym}}$ be the subset of $\operatorname{res}_H(R(G^*, A_0^*)) - \{0\}$ consisting of the $R(G)$-symmetric roots on A^H. By applying Lemma 4.3 to the reflection s_α of $\alpha \in R(H, A_{H,0})$, we have the following corollary.

Corollary 4.4. *We have*

$$i_0^*(R(H, A_{H,0})) \subset R(G^*, A^H)_{\mathrm{sym}}.$$

For the sake of brevity, we assume that $\mathcal{H} = {}^L H$ until the end of this section. We denote by $H_{G\text{-reg}}$ the set of strongly G-regular elements in $H(F)$. For $\gamma^H \in H_{G\text{-reg}}$ and $\gamma^G \in G_{\mathrm{reg}}$, let $\Delta_{G,H}(\gamma^H, \gamma^G)$ be the Langlands–Shelstad transfer factor. Since we normalized the orbital integral, we define $\Delta_{G,H}$ by the product of Δ_{I}, Δ_{II} and Δ_{III} in this paper. Because we have to define the transfer of virtual characters, we assume the fundamental lemma for groups [Art96, Hypothesis 3.1] and for Lie algebras [Wal97, Conjecture 1.3] in this paper. Then [Wal97, Corollary 1.7] asserts that for each $f^G \in C_c^\infty(G)$, there exists $f^H \in C_c^\infty(H)$ such that

$$I^{\mathrm{st}}(\gamma^H, f^H) = \sum_{\gamma \in \Gamma(G)} \Delta_{G,H}(\gamma^H, \gamma) I(\gamma, f^G),$$

holds for all $\gamma^H \in H_{G\text{-reg}}$, where $\Gamma(G)$ is the set of conjugacy classes in G_{reg}. Hence, for $\theta_H \in \mathbb{C}[\Pi(H)]^{\mathrm{st}}$, we can define a linear form $\operatorname{Tran}_H^G(\theta_H)$ on $C_c^\infty(G)$ by the relation

$$\operatorname{Tran}_H^G(\theta_H)(f^G) = \theta_H(f^H).$$

Let X be a subset of G_{reg}. We say that $\theta \in \mathbb{C}[\Pi(G)]$ is *stable on X* if we have $I^G(\theta, \gamma) = I^G(\theta, \gamma')$ for all $\gamma, \gamma' \in X$ that are stably conjugate.

Lemma 4.5. *Let L be a standard Levi subgroup of H. If $\theta_L \in \mathbb{C}[\Pi(L)]$ is invariant under the adjoint action of $\operatorname{Norm}(L, H(F))$ and if $i_L^H(\theta_L)$ is stable on $L_{\mathrm{ell}} \cap H_{\mathrm{reg}}$, then θ_L is stable on L_{ell}.*

Proof. Let $\gamma \in L_{\mathrm{ell}} \cap H_{\mathrm{reg}}$ and $T = \operatorname{Cent}(\gamma, H)$. Let T' be a maximal torus of L such that $I_{\mathrm{conj}}^{H,L}(T', T) \neq \emptyset$. Let $i \in I_{\mathrm{conj}}^{H,L}(T', T)$, then there exists $h \in H(F)$ such that $i = \operatorname{Int} h$. Since $\gamma \in L_{\mathrm{ell}} \cap H_{\mathrm{reg}}$, we have $L = \operatorname{Cent}(A_T, H)$. Therefore, L is determined by γ. Since $i^{-1}(\gamma) \in L_{\mathrm{ell}}$, L is also determined by $i^{-1}(\gamma)$. Hence, we have $\operatorname{Int} h(L) = L$. Thus $h \in \operatorname{Norm}(L, H(F))$. Therefore, by (3.1), we have

$$I^H(i_L^H(\theta_L), \gamma) = \sum_{h \in \operatorname{Norm}(L,H(F))/L(F)} I^L(\theta_L, \operatorname{Int} h^{-1}(\gamma)).$$

Since θ_L is $\operatorname{Norm}(L, H(F))$-invariant, this shows that

$$I^H(i_L^H(\theta_L), \gamma) = n \cdot I^L(\theta_L, \gamma),$$

where $n = \sharp \operatorname{Norm}(L, H(F))/L(F)$. Since $i_L^H(\theta_L)$ is stable on $L_{\mathrm{ell}} \cap H_{\mathrm{reg}}$, this implies that θ_L is stable on L_{ell}. □

Proposition 4.6. *Assume the fundamental lemma for groups and for Lie algebras. Then for $\theta_H \in \mathbb{C}[\Pi(H)]^{\mathrm{st}}$, we have*

$$\operatorname{Tran}_H^G(\theta_H) \in \mathbb{C}[\Pi(G)].$$

Proof. For $L \in \mathcal{L}^H$, we write $\tilde{T}_{\mathrm{ell}}(L)$ for the set of orbits of (essential) elliptic triplets as in [Art96, p. 530]. We also define $\tilde{T}_{\mathrm{ell}}(L)_{\mathbb{C}}$ as in [Art96, p. 530]. We denote by $\mathbb{C}[\tilde{T}_{\mathrm{ell}}(L)]$ the subspace of $\mathbb{C}[\Pi(L)]$ generated by the virtual characters attached to $\tilde{T}_{\mathrm{ell}}(L)$ and by $\mathbb{C}[\tilde{T}_{\mathrm{ell}}(L)_{\mathbb{C}}]$ the subspace of $\mathbb{C}[\Pi(L)]$ generated by the virtual characters attached to $\tilde{T}_{\mathrm{ell}}(L)_{\mathbb{C}}$. Choose a subset $\tilde{\mathcal{L}^H}$ of $\mathcal{L}^H$ so that each standard Levi subgroup of H has one and only one associate standard Levi subgroup in $\tilde{\mathcal{L}^H}$. Since θ_H is a virtual character, we can write θ_H as $\theta_H = \sum_{L \in \tilde{\mathcal{L}^H}} i_L^H(\sigma_L)$ with $\sigma_L \in \mathbb{C}[\tilde{T}_{\mathrm{ell}}(L)_{\mathbb{C}}]$. Moreover, we may assume that σ_L is invariant under the action of $\operatorname{Norm}(L, H(F))$. We claim that $\sigma_L \in \mathbb{C}[\Pi(L)]^{\mathrm{st}}$. We prove this by induction on $r(H) - r(L)$. Let $L \in \tilde{\mathcal{L}^H}$. Then by the inductive assumption,

$$\theta_H - \sum_{\substack{L' \in \tilde{\mathcal{L}^H} \\ r(L') > r(L)}} i_{L'}^H(\sigma_{L'}) = \sum_{\substack{L' \in \tilde{\mathcal{L}^H} \\ r(L') \leqslant r(L)}} i_{L'}^H(\sigma_{L'})$$

is a stable virtual character. We see that $i_L^H(\sigma_L)$ is the only term which is not zero on $L_{\mathrm{ell}} \cap H_{\mathrm{reg}}$ in the right-hand side. Thus $i_L^H(\sigma_L)$ is stable on $L_{\mathrm{ell}} \cap H_{\mathrm{reg}}$. Therefore, Lemma 4.5 asserts that σ_L is stable on L_{ell}. Since σ_L is a finite linear combination of virtual characters attached to $\tilde{T}_{\mathrm{ell}}(L)_{\mathbb{C}}$, [Art96, Theorem 6.1] implies that σ_L is a stable virtual character. We have thus proved the claim. So, it is enough to show that $\operatorname{Tran}_H^G \circ i_L^H(\sigma_L) \in \mathbb{C}[\Pi(G)]$. Put $\mathbb{C}[\tilde{T}_{\mathrm{ell}}(L)]^{\mathrm{st}} = \mathbb{C}[\tilde{T}_{\mathrm{ell}}(L)] \cap \mathbb{C}[\Pi(L)]^{\mathrm{st}}$. It is not difficult to see that if $\operatorname{Tran}_H^G \circ i_L^H$ is not zero, then there exist a standard Levi subgroup M of G, a set of elliptic endoscopic data $(L, {}^L L, s_L, \xi_L)$ for M and a non-zero constant c such that

$$\operatorname{Tran}_H^G \circ i_L^H = c \cdot i_M^G \circ \operatorname{Tran}_L^M.$$

(Note that the transfer factor is defined up to a constant factor.) By applying [Art96, Theorem 6.2] to the set of endoscopic data $(L, {}^L L, s_L, \xi_L)$, we can show that

$$\operatorname{Tran}_H^G \circ i_L^H(\mathbb{C}[\tilde{T}_{\mathrm{ell}}(L)]^{\mathrm{st}}) \subset \mathbb{C}[\Pi(G)].$$

(By using [Art96, Lemma 5.2], we can show that the linear form $f \longrightarrow f'_{gr}(\phi')$ in [Art96, Theorem 6.2] is a virtual character.) It is easy to extend this to $\mathbb{C}[\tilde{T}_{\mathrm{ell}}(L)_{\mathbb{C}}]^{\mathrm{st}} = \mathbb{C}[\tilde{T}_{\mathrm{ell}}(L)_{\mathbb{C}}] \cap \mathbb{C}[\Pi(L)]^{\mathrm{st}}$. □

Let T^H be a maximal torus of H and T^G a maximal torus of G. We say that an isomorphism $i : T^H \longrightarrow T^G$ is *admissible* if i is defined over F and if there exist $h \in H(\overline{F})$ and $g \in G(\overline{F})$ such that $\operatorname{Int} g(T_0) = T^G$, $\operatorname{Int} h(T_{H,0}) = T^H$ and

$$i = \operatorname{Int} g \circ \varphi^{-1} \circ i_0^* \circ \operatorname{Int} h^{-1}.$$

We denote by $I^{G,H}(T^H, T^G)$ the set of admissible isomorphisms from T^H to T^G. Then $\Omega(H, T^H)_F$ and $\Omega(G, T^G)_F$ act on $I^{G,H}(T^H, T^G)$ from the right and left, respectively. Let $\gamma^H \in T^H(F)$ be a

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strongly G-regular element and $\gamma^G \in T^G(F)$ a strongly regular element. Then we say that γ^H is an *image* (or a (G,H)-*image*) of γ^G if there exists an admissible isomorphism $i \in I^{G,H}(T^H, T^G)$ such that $i(\gamma^H) = \gamma^G$. In this paper, we also say that T^H is an *image* of T^G if there exists an admissible isomorphism $T^H \longrightarrow T^G$. Let $\mathcal{T}^H$ be a system of representatives for the stable conjugacy classes of maximal tori of H. Since $\mathrm{Tran}_H^G(\theta_H)$ is a virtual character, we can consider a function $I^G(\mathrm{Tran}_H^G(\theta_H))$ on G_{reg}. By a routine calculation, we can prove the following formula from the definition of Tran_H^G.

Lemma 4.7. *For $\gamma \in G_{\mathrm{reg}}$ and $\theta_H \in \mathbb{C}[\Pi(H)]^{\mathrm{st}}$, we have*

$$I^G(\mathrm{Tran}_H^G(\theta_H), \gamma) = \sum_{T^H \in \mathcal{T}^H} \sum_{i \in I^{G,H}(T^H, G_\gamma)/\Omega(H,T^H)_F} \Delta_{G,H}(i^{-1}(\gamma), \gamma) I^H(\theta_H, i^{-1}(\gamma)).$$

5. Analogue of the formula of Bernstein–Zelevinski

In this section, we fix $M \in \mathcal{L}^G$. We put $M^* = \varphi(M)$. By Corollary 4.4, we have $i_0^*(R(H, A_{H,0})) \subset R(G^*, A^H)_{\mathrm{sym}}$. Therefore, we can define

$$D_{M^*,H} \subset \Omega(G^*, A_0^*)$$

as in § 2. We put $D_{M,H} = D_{M^*,H}$. Recall that we defined a homomorphism

$$i^* : \Omega(H, A_{H,0}) \longrightarrow \Omega(G^*, A_0^*)$$

by putting $i^*(\omega) = \omega_G$, where ω_G is the element in Lemma 4.3. Thus $i^*(\Omega(H, A_{H,0}))$ is the subgroup $\Omega(H)$ in § 2 corresponding to the root system $R(H) = i_0^*(R(H, A_{H,0}))$. Therefore, Proposition 2.2 asserts that $D_{M,H}$ is a system of representatives for

$$\Omega(M^*, A_0^*)\backslash\Omega(G^*, A_0^*)_{M,H}/i^*(\Omega(H, A_{H,0})),$$

where

$$\Omega(G^*, A_0^*)_{M,H} = \{\omega \in \Omega(G^*, A_0^*) \mid \omega \circ i_0^*(A_{H,0}) \supset A_{M^*}\}.$$

For $\omega \in D_{M,H}$, we put

$$M_\omega = \mathrm{Cent}((\omega \circ i_0^*)^{-1}(A_{M^*}), H).$$

Then it is easy to see that

$$i_0^*(R(M_\omega, A_{H,0})) = i_0^*(R(H, A_{H,0})) \cap \mathrm{res}_H\, \omega^{-1}(R(M^*, A_0^*)).$$

Thus Lemma 2.3 asserts that M_ω is a standard Levi subgroup of H. It is also easy to see that

$$\Omega(M_\omega, T_{H,0}) = \Omega(H, T_{H,0}) \cap (\omega \circ i_0^*)^{-1} \circ \Omega(M^*, T_0^*) \circ (\omega \circ i_0^*).$$

Let $\hat{M}_\omega$ be the dual group of M_ω. Since M_ω is a standard Levi subgroup, we can regard $\hat{M}_\omega$ as a standard Levi subgroup of $\hat{H}$. Let $\hat{n}_\omega \in \mathrm{Norm}(\mathcal{T}, \hat{G})$ be a representative for $\omega \in \Omega(G^*, T_0^*) = \Omega(\hat{G}, \mathcal{T})$. We put

$$\xi_\omega = \mathrm{Int}\, \hat{n}_\omega \circ \xi,$$
$$s'_\omega = \mathrm{Int}\, \hat{n}_\omega(s).$$

By using $\omega \circ i_0^*(R(M_\omega, T_{H,0})) = \omega \circ i_0^*(R(H, T_{H,0})) \cap R(M^*, T_0^*)$, we have $\xi_\omega(\hat{M}_\omega) = \mathrm{Cent}(s'_\omega, \hat{M})^0$, where ^{0} denotes the identity component. We can choose $a_\omega \in (Z_{\hat{M}}^\Gamma)^0$ such that $\mathrm{Cent}(a_\omega s'_\omega, \hat{G})^0 = \hat{M}_\omega$. We put $s_\omega = a_\omega s'_\omega$. Let $c : W_F \longrightarrow \mathcal{H}$ be a continuous splitting of

$$1 \longrightarrow \hat{H} \longrightarrow \mathcal{H} \longrightarrow W_F \longrightarrow 1.$$

By the proof of [KS99, Lemma 2.2.A], we may assume that $\mathrm{Int}\, c(W_F)$ preserves the pair $(\mathcal{B}_H, \mathcal{T}_H)$. We put $\mathcal{M}_\omega = \hat{M}_\omega \cdot c(W_F)$. Then $\mathcal{M}_\omega$ is a subgroup of $\mathcal{H}$.

LEMMA 5.1. *The sets* $(M_\omega, \mathcal{M}_\omega, s'_\omega, \xi_\omega)$ *and* $(M_\omega, \mathcal{M}_\omega, s_\omega, \xi_\omega)$ *are equivalent endoscopic data for* M, *and* $(M_\omega, \mathcal{M}_\omega, s_\omega, \xi_\omega)$ *is a set of endoscopic data for* G.

Proof. Let $w \in W_F$ and let $\sigma \in \Gamma$ be the image of w. We denote by σ_H the action of σ on $T_{H,0}$ and by σ_{G^*} the action of σ on T_0^*. Then there exists $\omega' \in \Omega(G^*, T_0^*)$ such that $i_0^* \circ \sigma_H \circ i_0^{*-1} = \omega' \circ \sigma_{G^*}$. Put $\omega'' = \omega\omega'\omega^{-1}$. Then we have $(\omega \circ i_0^*) \circ \sigma_H \circ (\omega \circ i_0^*)^{-1} = \omega'' \circ \sigma_{G^*}$. By $\omega \in D_{M,H}$, we have $\omega \circ i_0^*(A_{H,0}) \supset A_{M^*}$. Since $A_{H,0}$ and A_0^* are split tori, this shows that the action of ω'' on A_{M^*} is trivial. Thus $\omega'' \in \Omega(M^*, T_0^*)$. Since $\operatorname{Int} c(w)$ preserves the pair $(\mathcal{B}_H, \mathcal{T}_H)$, the action of $\operatorname{Int} c(w)$ on $\mathcal{T}_H$ is equal to the action of σ_H on $\mathcal{T}_H \cong \hat{T}_{H,0}$. Therefore, we have

$$\xi_\omega(c(w)) \in \operatorname{Norm}(\mathcal{T}, \hat{M}) \rtimes \sigma \subset {}^L M.$$

Hence, $\xi_\omega(\mathcal{M}_\omega) \subset {}^L M$. The other parts of the proof are easily verified. □

It is easy to see that the equivalence classes of the endoscopic data do not depend on the choice of $\hat{n}_\omega$ and c. We put $\mathcal{B}_{M_\omega} = \mathcal{B}_H \cap \hat{M}_\omega$. Then the restriction $(\mathcal{B}_{M_\omega}, \mathcal{T}_H, \{\mathcal{Y}_{\check{\alpha}}\})$ of the Γ-splitting of H is a Γ-splitting of $\hat{M}_\omega$. We have $s_\omega \in \mathcal{T}$ and $\xi_\omega(\mathcal{T}_H) = \mathcal{T}$. Moreover, since $\omega \in D_{M,H}$, we have $\omega \circ i_0^*(R^+(H, T_{H,0})) \subset R^+(G^*, T_0^*)$. This implies that $\xi_\omega(\mathcal{B}_{M_\omega}) \subset \mathcal{B}$. Let (H_1, ξ_{H_1}) be a z-pair (see [KS99, § 2.2]). Let $(\mathcal{B}_{H_1}, \mathcal{T}_{H_1}, \{\mathcal{Y}_{\check{\alpha}}\})$ be the Γ-splitting of $\hat{H}_1$ obtained from the Γ-splitting $(\mathcal{B}_H, \mathcal{T}_H, \{\mathcal{Y}_{\check{\alpha}}\})$. Then $\xi_{H_1} \circ c(W_F)$ preserves the pair $(\mathcal{B}_{H_1}, \mathcal{T}_{H_1})$. Let $M_{\omega,1}$ be the standard Levi subgroup of H_1 corresponding to M_ω. Then the restriction of the L-group data for H_1 defines an L-group data for $M_{\omega,1}$. Let ${}^L M_{\omega,1} \hookrightarrow {}^L H_1$ be the natural embedding. Then it is easy to see that $\xi_{H_1}(\mathcal{M}_\omega)$ is contained in ${}^L M_{\omega,1} \subset {}^L H_1$. Therefore, $(M_{\omega,1}, \xi_{H_1})$ is a z-pair for $\mathcal{M}_\omega$.

For the sake of brevity, we assume that $\mathcal{H} = {}^L H$ until the end of § 6. Then $\mathcal{M}_\omega$ is the image of the natural embedding ${}^L M_\omega \hookrightarrow {}^L H$. We identify $\mathcal{M}_\omega$ with ${}^L M_\omega$. Let $(M^* \cap B_0^*, T_0^*, \{X_\alpha\})$ be the restriction of the F-splitting $(B_0^*, T_0^*, \{X_\alpha\})$ to M^* and let $(M_\omega \cap B_{H,0}, T_{H,0}, \{Y_\alpha\})$ be the restriction of the F-splitting $(B_{H,0}, T_{H,0}, \{Y_\alpha\})$ to M_ω. We write $i_0^{M_\omega}$ for $\omega \circ i_0^*$. Then $i_0^{M_\omega}$ is the dual of ξ_ω^{-1}. We put

$$\Gamma(M_\omega, M) = \{(\gamma^{M_\omega}, \gamma^M) \in M_{\omega, G\text{-reg}} \times M_{G\text{-reg}} \mid \gamma^{M_\omega} \text{ is an } (M, M_\omega)\text{-image of } \gamma^M\}.$$

We denote by Δ_{M,M_ω} the Langlands–Shelstad transfer factor of M_ω. Note that Δ_{M,M_ω} is defined up to a constant factor.

LEMMA 5.2. *There exists* $c \in \mathbb{C}^\times$ *such that*

$$\Delta_{G,H}(\gamma^{M_\omega}, \gamma^M) = c \cdot \Delta_{M,M_\omega}(\gamma^{M_\omega}, \gamma^M)$$

for all $(\gamma^{M_\omega}, \gamma^M) \in \Gamma(M_\omega, M)$.

Proof. We write $\Delta'_{G,H}$ for the Langlands–Shelstad transfer factor of the set of endoscopic data $(H, {}^L H, s'_\omega, \xi_\omega)$. Since the relative transfer factor of $(H, {}^L H, s'_\omega, \xi_\omega)$ is equal to the relative transfer factor of $(H, {}^L H, s, \xi)$, it is enough to show that

$$\Delta'_{G,H}(\gamma^{M_\omega}, \gamma^M; \overline{\gamma}^{M_\omega}, \overline{\gamma}^M) = \Delta_{M,M_\omega}(\gamma^{M_\omega}, \gamma^M; \overline{\gamma}^{M_\omega}, \overline{\gamma}^M)$$

for all $(\gamma^{M_\omega}, \gamma^M), (\overline{\gamma}^{M_\omega}, \overline{\gamma}^M) \in \Gamma(M_\omega, M)$. We remark that $\xi_\omega(\mathcal{B}_H) \subset \mathcal{B}$. Put $T^{M_\omega} = \operatorname{Cent}(\gamma^{M_\omega}, M_\omega)$ and $T^M = \operatorname{Cent}(\gamma^M, M)$. Fix a maximal torus $T^* \subset M^*$ such that T^{M_ω} is an image of T^* and fix $i \in I^{M^*, M_\omega}(T^{M_\omega}, T^*)$. We regard i as an admissible embedding of $T^{M_\omega} \subset H$ into G^*. Choose $h \in M_\omega(\overline{F})$ and $m \in M^*(\overline{F})$ such that $\operatorname{Int} h(B_{H,0}, T_{H,0})$ and $\operatorname{Int} m(B_0^*, T_0^*)$ determine i. Then $\operatorname{Int} h(M_\omega \cap B_{H,0}, T_{H,0})$ and $\operatorname{Int} m(M^* \cap B_0^*, T_0^*)$ also determine i. Choose a-data $\{a_\alpha\}$ for $R(G^*, T^*)$. Define a-data for $R(M^*, T^*)$ by the restriction of $\{a_\alpha\}$ to $R(M^*, T^*)$. Let $\{\chi_\alpha\}$ be χ-data for $R(G^*, T^*)$. If $\alpha \in R(G^*, T^*) - R(M^*, T^*)$, then α is asymmetric. Therefore, we can choose $\{\chi_\alpha\}$ such that $\chi_\alpha \equiv 1$ for all $\alpha \in R(G^*, T^*) - R(M^*, T^*)$ (see [Art96, p. 521]). Define χ-data for $R(M^*, T^*)$ by the restriction of $\{\chi_\alpha\}$ to $R(M^*, T^*)$. We also fix data for $\overline{\gamma}^{M_\omega}$ and $\overline{\gamma}^M$ similarly.

Now, by using a similar argument to the proof of [Wal97, Lemma 6.5], we can show the required identities for Δ_{I}, Δ_{II} and Δ_{III_1}. Therefore, it remains to show the identity for Δ_{III_2}. Define $\mathbf{a}^G \in H^1(W_F, \hat{T}^*)$ for $(H, {}^LH, s'_\omega, \xi_\omega)$ and $\mathbf{a}^M \in H^1(W_F, \hat{T}^*)$ for $(M_\omega, {}^LM_\omega, s'_\omega, \xi_\omega)$ as in [LS87, § 3.5]. By the choice of splittings, χ-data, pairs and i, we can easily show that $\mathbf{a}^G = \mathbf{a}^M$. Let $\Delta^G_{\mathrm{III}_2}$ be the Δ_{III_2} factor of $(H, {}^LH, s'_\omega, \xi_\omega)$ and $\Delta^M_{\mathrm{III}_2}$ the Δ_{III_2} factor of $(M_\omega, {}^LM_\omega, s'_\omega, \xi_\omega)$, then the above identity implies that $\Delta^G_{\mathrm{III}_2}(\gamma^{M_\omega}, \gamma^M) = \Delta^M_{\mathrm{III}_2}(\gamma^{M_\omega}, \gamma^M)$. This completes the proof. □

We replace Δ_{M,M_ω} by $c \cdot \Delta_{M,M_\omega}$. Then we have

$$\Delta_{G,H}(\gamma^{M_\omega}, \gamma^M) = \Delta_{M,M_\omega}(\gamma^{M_\omega}, \gamma^M) \tag{5.1}$$

for all $(\gamma^{M_\omega}, \gamma^M) \in \Gamma(M_\omega, M)$.

Fix a maximal torus T^G of M. Assume that A_{T^G} is a standard subtorus of A_0. Let $\{T^H_1, \dots, T^H_r\}$ be a system of representatives for the stable conjugacy classes of maximal tori of H that are images of T^G. We may assume that $A_{T^H_1}, \dots, A_{T^H_r}$ are standard subtori of $A_{H,0}$. For $i = 1, \dots, r$ and $\omega \in D_{M,H}$, let $\{T^\omega_{i1}, \dots, T^\omega_{ir_{\omega,i}}\}$ be a system of representatives for the stable conjugacy classes of maximal tori of M_ω that are stably H-conjugate to T^H_i. For each T^ω_{ij}, fix $z^\omega_{ij} \in H(\overline{F})$ such that $\mathrm{Int}\, z^\omega_{ij}(T^H_i) = T^\omega_{ij}$ and that

$$\mathrm{Int}\, z^\omega_{ij} :\ T^H_i \longrightarrow T^\omega_{ij}$$

is defined over F. We put

$$Y = \bigcup_i I^{H,G}(T^H_i, T^G)/\Omega(H, T^H_i)_F.$$

Then we may regard Y as the set

$$\{(T^H_i, \tilde{\imath}^H) \mid 1 \leqslant i \leqslant r,\ \tilde{\imath}^H \in I^{H,G}(T^H_i, T^G)/\Omega(H, T^H_i)_F\}.$$

For $\omega \in D_{M,H}$, we put

$$\tilde{Y}_\omega = \bigcup_{i,j} I^{M_\omega,M}(T^\omega_{ij}, T^G)/\Omega(M_\omega, T^\omega_{ij})_F.$$

We also put

$$\tilde{Y} = \bigcup_{\omega \in D_{M,H}} \tilde{Y}_\omega.$$

Then $\tilde{Y}$ can be regarded as the set of $(\omega, T^\omega_{ij}, \tilde{\imath}^\omega)$, where $\omega \in D_{M,H}$, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant r_{\omega,i}$ and $\tilde{\imath}^\omega \in I^{M_\omega,M}(T^\omega_{ij}, T^G)/\Omega(M_\omega, T^\omega_{ij})_F$. It is easy to see that T^H_i is stably H-conjugate to $T^\omega_{i'j}$ if and only if $i = i'$. Moreover, if $(i,j) \neq (i',j')$, then T^ω_{ij} is not stably M_ω-conjugate to $T^\omega_{i'j'}$. We say that $(T^H_{i'}, \tilde{\imath}^H) \in Y$ *corresponds* to $(\omega, T^\omega_{ij}, \tilde{\imath}^\omega) \in \tilde{Y}$ if $i' = i$ and if there exist a representative i^H for $\tilde{\imath}^H$, a representative i^ω for $\tilde{\imath}^\omega$ and $\omega_H \in \Omega(H, T^H_{i'})_F$ such that

$$i^H = i^\omega \circ \mathrm{Int}\, z^\omega_{ij} \circ \omega_H.$$

Proposition 5.3. *The above correspondence is a one-to-one correspondence between Y and $\tilde{Y}$.*

It is enough to prove the one-to-one correspondence for each $i = 1, \dots, r$. Thus we fix $i \in \{1, \dots, r\}$. Put $T^H = T^H_i$, $T^\omega_j = T^\omega_{ij}$, $z^\omega_j = z^\omega_{ij}$ and $r_\omega = r_{\omega,i}$. Put $M_{T^H} = \mathrm{Cent}(A_{T^H}, H)$ and $M_{T^G} = \mathrm{Cent}(A_{T^G}, G)$. Then, by $T^G \subset M$, we have $M_{T^G} \subset M$. Since A_{T^H} is a standard subtorus of $A_{H,0}$, we have $T_{H,0} \subset M_{T^H}$. Let $i^H \in I^{H,G}(T^H, T^G)$, then there exist $g \in G(\overline{F})$ and $h \in M_{T^H}(\overline{F}) \subset H(\overline{F})$ such that $\mathrm{Int}\, h(T_{H,0}) = T^H$, $\mathrm{Int}\, g(T_0) = T^G$ and

$$i^H = \mathrm{Int}\, g \circ \varphi^{-1} \circ i^*_0 \circ \mathrm{Int}\, h^{-1},$$

since A_{T^H} is a standard subtorus of $A_{H,0}$. Fix $m \in M_{T^G}(\overline{F})$ such that $T^G = \operatorname{Int} m(T_0)$. Then we have $A_{T^G} \subset \operatorname{Int} m(A_0)$.

Lemma 5.4. *There exists $\omega' \in \Omega(G^*, A_0^*)$ such that*

$$i^H|_{A_{T^H}} = \operatorname{Int} m \circ \varphi^{-1} \circ \omega' \circ i_0^* \circ \operatorname{Int} h^{-1}|_{A_{T^H}}$$

and

$$\omega' \circ i_0^*(A_{H,0}) \supset A_{M^*}.$$

Proof. Put

$$\phi = \varphi \circ \operatorname{Int} m^{-1} \circ i^H \circ \operatorname{Int} h \circ i_0^{*-1} : T_0^* \longrightarrow T_0^*.$$

Since $\phi \circ i_0^*(A_{T^H}) = \varphi(A_{T^G})$, we have $\phi \circ i_0^*(A_{T^H}) \subset A_0^*$. Put $A_1 = i_0^*(A_{T^H}) \subset A_0^*$ and $A_2 = \phi \circ i_0^*(A_{T^H}) \subset A_0^*$. Since $\phi = \operatorname{Int}(\varphi(m^{-1}g))$, Lemma 4.2 asserts that there exists $\omega' \in \Omega(G^*, A_0^*)$ such that $\omega'(A_1) = A_2$ and $\omega'|_{A_1} = \phi|_{A_1}$. This implies the first relation. Since

$$A_{M^*} \subset \varphi(A_{T^G}) = A_2 = \phi \circ i_0^*(A_{T^H}) = \omega' \circ i_0^*(A_{T^H}),$$

ω' satisfies the second property. □

Proposition 2.2 asserts that the intersection of $D_{M,H}$ and $\Omega(M^*, A_0^*) \cdot \omega' \cdot i^*(\Omega(H, A_{H,0}))$ consists of a single element ω. We have $\omega\Omega(M^H, T_0^*)\omega^{-1} \subset \Omega(M^*, T_0^*)$. Recall that we have

$$i^*(\omega_H) \circ i_0^* \circ \omega_H^{-1} \circ i_0^{*-1} \in \Omega(M^H, T_0^*)$$

for any $\omega_H \in \Omega(H, A_{H,0}) = \Omega(H, T_{H,0})_F$.

Lemma 5.5. *There exist $\omega_H \in \Omega(H, T_{H,0})_F$ and $\omega_{M^*} \in \Omega(M^*, T_0^*)$ such that*

$$i^H = \operatorname{Int} m \circ \varphi^{-1} \circ \omega_{M^*} \cdot \omega \circ i_0^* \circ \omega_H \circ \operatorname{Int} h^{-1}.$$

Proof. It is easy to see that there exists $g' \in \operatorname{Norm}(T^G, G(\overline{F}))$ such that

$$i^H \circ (\operatorname{Int} m \circ \varphi^{-1} \circ \omega' \circ i_0^* \circ \operatorname{Int} h^{-1})^{-1} = \operatorname{Int} g'.$$

Since $i^H|_{A_{T^H}} = (\operatorname{Int} m \circ \varphi^{-1} \circ \omega' \circ i_0^* \circ \operatorname{Int} h^{-1})|_{A_{T^H}}$, we have $g' \in \operatorname{Cent}(A_{T^G}, G(\overline{F}))$. This implies that

$$\varphi(m^{-1}g'm) \in \operatorname{Norm}(T_0^*, M^*(\overline{F})).$$

Let $\omega'_{M^*} \in \Omega(M^*, T_0^*)$ be the image of $\varphi(m^{-1}g'm)$, then

$$i^H = \operatorname{Int} m \circ \varphi^{-1} \circ \omega'_{M^*} \cdot \omega' \circ i_0^* \circ \operatorname{Int} h^{-1}.$$

Now, Proposition 2.2 asserts that there exists $\omega_H \in \Omega(H, T_{H,0})_F$ such that

$$\omega'_{M^*} \cdot \omega' \in \Omega(M^*, A_0^*) \cdot \omega \cdot i^*(\omega_H).$$

Since $\Omega(M^*, A_0^*) \cdot \omega \cdot i^*(\omega_H) = \Omega(M^*, T_0^*)\omega \circ i_0^* \circ \omega_H \circ i_0^{*-1}$, there exists $\omega_{M^*} \in \Omega(M^*, T_0^*)$ such that

$$\omega'_{M^*} \cdot \omega' = \omega_{M^*} \cdot \omega \circ i_0^* \circ \omega_H \circ i_0^{*-1}.$$

□

Put

$$\phi = \omega_H \circ \operatorname{Int} h^{-1} \circ i^{H^{-1}} = i_0^{*-1} \circ \omega^{-1}\omega_{M^*}^{-1} \circ \varphi \circ \operatorname{Int} m^{-1}.$$

Then ϕ is a homomorphism from T^G to $T_{H,0}$. By $\phi = \omega_H \circ \operatorname{Int} h^{-1} \circ i^{H^{-1}}$, we have

$$\sigma(\phi) \circ \phi^{-1} = \omega_H \circ \operatorname{Int} \sigma(h)^{-1}h \circ \omega_H^{-1}$$

for any $\sigma \in \Gamma$. Thus
$$\sigma(\phi) \circ \phi^{-1} \in \Omega(H, T_{H,0}).$$
On the other hand, by using $\phi = i_0^{*-1} \circ \omega^{-1}\omega_{M^*}^{-1} \circ \varphi \circ \operatorname{Int} m^{-1}$, we have $\phi(A_M) = i_0^{*-1} \circ \omega^{-1}(A_{M^*}) \subset A_{H,0}$. Since A_M and $A_{H,0}$ are split tori, the restriction of $\sigma(\phi) \circ \phi^{-1}$ to $i_0^{*-1} \circ \omega^{-1}(A_{M^*})$ is identity. Thus
$$\sigma(\phi) \circ \phi^{-1} \in (\omega \circ i_0^*)^{-1} \circ \Omega(M^*, T_0^*) \circ (\omega \circ i_0^*).$$
Consequently, we have
$$\sigma(\phi) \circ \phi^{-1} \in \Omega(M_\omega, T_{H,0}).$$
Therefore [Kot82, Corollary 2.2] asserts that there exists $z \in M_\omega(\overline{F})$ such that the homomorphism $\operatorname{Int} z \circ \phi$ from T^G to $\operatorname{Int} z(T_{H,0})$ is defined over F. Since this implies that T_H is stably H-conjugate to $\operatorname{Int} z(T_{H,0})$, we may assume that $\operatorname{Int} z(T_{H,0})$ is equal to $T_j^\omega \in \{T_1^\omega, \dots, T_{r_\omega}^\omega\}$. Now, put
$$i^\omega = \phi^{-1} \circ \operatorname{Int} z^{-1}.$$
Then we have
$$i^\omega = \operatorname{Int} m \circ \varphi^{-1} \circ \omega_{M^*}\omega \circ i_0^* \circ \operatorname{Int} z^{-1} = \operatorname{Int} m \circ \varphi^{-1}(\omega_{M^*}) \circ \varphi^{-1} \circ i_0^{M_\omega} \circ \operatorname{Int} z^{-1}.$$
This implies that $i^\omega \in I^{M_\omega, M}(T_j^\omega, T^G)$. Since
$$i^{\omega -1} \circ i^H = \operatorname{Int} z \circ \omega_H \circ \operatorname{Int} h^{-1} \in \operatorname{Int} z_j^\omega \circ \Omega(H, T^H)_F,$$
we conclude that $(T^H, \tilde{\imath}^H) \in Y$ corresponds to $(\omega, T_j^\omega, \tilde{\imath}^\omega) \in \tilde{Y}_\omega$. Conversely, let $(\omega, T_j^\omega, \tilde{\imath}^\omega)$ be an element of $\tilde{Y}$ and i^ω a representative for $\tilde{\imath}^\omega$. If we put $i^H = i^\omega \circ \operatorname{Int} z_j^\omega$, then $(T^H, \tilde{\imath}^H) \in Y$ corresponds to $(\omega, T_j^\omega, \tilde{\imath}^\omega)$.

So, it remains to show that for each element in Y, there exists only one element in $\tilde{Y}$ that corresponds to it and vice-versa. Suppose that $(T^H, \tilde{\imath}^H) \in Y$ and $(T^H, \tilde{\imath}'^H) \in Y$ correspond to the same $(\omega, T_j^\omega, \tilde{\imath}^\omega) \in \tilde{Y}$. Then it is easy to see that $\tilde{\imath}^H = \tilde{\imath}'^H$. Conversely, suppose that $(\omega, T_j^\omega, \tilde{\imath}^\omega) \in \tilde{Y}$ and $(\omega', T_{j'}^{\omega'}, \tilde{\imath}'^{\omega'}) \in \tilde{Y}$ correspond to the same $(T^H, \tilde{\imath}^H) \in Y$. Let i^ω be a representative for $\tilde{\imath}^\omega$ and i^H a representative for $\tilde{\imath}^H$, then there exist $m \in M(\overline{F})$ and $h \in M_\omega(\overline{F})$ such that $\operatorname{Int} h(T_{H,0}) = T_j^\omega$, $\operatorname{Int} m(T_0) = T^G$ and
$$i^\omega = \operatorname{Int} m \circ \varphi^{-1} \circ \omega \circ i_0^* \circ \operatorname{Int} h^{-1},$$
and there exists $\omega_H \in \Omega(H, T^H)_F$ such that
$$i^H = i^\omega \circ \operatorname{Int} z_j^\omega \circ \omega_H.$$
Choose $i'^{\omega'}$, m', h' and ω'_H similarly. Then we have
$$i_0^{*-1} \circ \omega'^{-1} \circ \varphi \circ \operatorname{Int} m'^{-1} m \circ \varphi^{-1} \circ \omega \circ i_0^* = \operatorname{Int} h'^{-1} z_{j'}^{\omega'} \circ \omega'_H \omega_H^{-1} \circ \operatorname{Int} z_j^{\omega -1} h. \tag{5.2}$$
Therefore,
$$\operatorname{Int} h'^{-1} z_{j'}^{\omega'} \circ \omega'_H \omega_H^{-1} \circ \operatorname{Int} z_j^{\omega -1} h \circ (\omega \circ i_0^*)^{-1}(A_{M^*}) = (\omega' \circ i_0^*)^{-1}(A_{M^*}).$$
Put $A_1 = (\omega \circ i_0^*)^{-1}(A_{M^*})$ and $A_2 = (\omega' \circ i_0^*)^{-1}(A_{M^*})$. Then $\omega, \omega' \in D_{M,H}$ implies that $A_1, A_2 \subset A_{H,0}$. Since H is quasi-split and since
$$\operatorname{Int} h'^{-1} z_{j'}^{\omega'} \circ \omega'_H \omega_H{}^{-1} \circ \operatorname{Int} z_j^{\omega -1} h \in \operatorname{Int} H(\overline{F}),$$
Lemma 4.2 asserts that there exists $\omega''_H \in \Omega(H, A_{H,0})$ such that
$$\operatorname{Int} h'^{-1} z_{j'}^{\omega'} \circ \omega'_H \omega_H^{-1} \circ \operatorname{Int} z_j^{\omega -1} h|_{A_1} = \omega''_H|_{A_1}.$$
Then
$$i_0^{*-1} \circ \omega'^{-1} \circ \varphi \circ \operatorname{Int} m'^{-1} m \circ \varphi^{-1} \circ \omega \circ i_0^*|_{A_1} = \omega''_H|_{A_1}.$$

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Since $m, m' \in M(\overline{F})$, this shows that $i_0^{*-1} \circ \omega'^{-1}\omega \circ i_0^*|_{A_1} = \omega''_H|_{A_1}$. Therefore, we have $i_0^* \circ \omega''_H \circ i_0^{*-1}|_{\omega^{-1}(A_{M^*})} = \omega'^{-1}\omega|_{\omega^{-1}(A_{M^*})}$. Now, by using $\omega^{-1}(A_{M^*}) \subset i_0^*(A_{H,0})$ and $i^*(\omega''_H)|_{i_0^*(A_{H,0})} = i_0^* \circ \omega''_H \circ i_0^{*-1}|_{i_0^*(A_{H,0})}$, we have

$$i^*(\omega''_H)|_{\omega^{-1}(A_{M^*})} = i_0^* \circ \omega''_H \circ i_0^{*-1}|_{\omega^{-1}(A_{M^*})}.$$

Hence, $\omega' \cdot i^*(\omega''_H) \cdot \omega^{-1}|_{A_{M^*}} = \mathrm{id}_{A_{M^*}}$. This implies that

$$\omega' \cdot i^*(\omega''_H) \cdot \omega^{-1} \in \Omega(M^*, A_0^*).$$

Therefore, Proposition 2.2 asserts that $\omega = \omega'$. Now, by (5.2), we have

$$\mathrm{Int}\, h'^{-1} z_{j'}^{\omega'} \circ \omega'_H \omega_H{}^{-1} \circ \mathrm{Int}\, z_j^{\omega -1} h \in (\omega \circ i_0^*)^{-1} \circ \Omega(M^*, T_0^*) \circ (\omega \circ i_0^*).$$

On the other hand, we have $\mathrm{Int}\, h'^{-1} z_{j'}^{\omega'} \circ \omega'_H \omega_H{}^{-1} \circ \mathrm{Int}\, z_j^{\omega -1} h \in \Omega(H, T_{H,0})$. Therefore, we have $\mathrm{Int}\, h'^{-1} z_{j'}^{\omega'} \circ \omega'_H \omega_H{}^{-1} \circ \mathrm{Int}\, z_j^{\omega -1} h \in \Omega(M_\omega, T_{H,0})$. This implies that

$$\mathrm{Int}\, z_{j'}^{\omega'} \circ \omega'_H \omega_H{}^{-1} \circ \mathrm{Int}\, z_j^{\omega -1} \in \mathrm{Int}\, M_\omega(\overline{F}). \tag{5.3}$$

Since $\mathrm{Int}\, z_{j'}^{\omega'} \circ \omega'_H \omega_H{}^{-1} \circ \mathrm{Int}\, z_j^{\omega -1}$ is an isomorphism from T_j^ω to $T_{j'}^\omega$ defined over F, this shows that T_j^ω is stably M_ω-conjugate to $T_{j'}^\omega$. Hence we have $T_j^\omega = T_{j'}^\omega$ and $z_j^\omega = z_{j'}^\omega$. By using (5.3) and $i^H = i^\omega \circ \mathrm{Int}\, z_j^\omega \circ \omega_H = i'^\omega \circ \mathrm{Int}\, z_j^\omega \circ \omega'_H$, we have

$$i'^{\omega -1} \circ i^\omega \in \Omega(H, T_j^\omega)_F \cap \mathrm{Int}\, M_\omega(\overline{F}) = \Omega(M_\omega, T_j^\omega)_F.$$

This implies that $\tilde{\imath}'^\omega = \tilde{\imath}^\omega$. Thus we have shown that

$$(\omega, T_j^\omega, \tilde{\imath}^\omega) = (\omega', T_{j'}^{\omega'}, \tilde{\imath}'^{\omega'}).$$

This completes the proof of Proposition 5.3.

The following theorem is an analogue of [BZ77, Lemma 2.12].

THEOREM 5.6. *Assume the fundamental lemma for groups and for Lie algebras. Then we have*

$$r_M^G \circ \mathrm{Tran}_H^G = \sum_{\omega \in D_{M,H}} \mathrm{Tran}_{M_\omega}^M \circ r_{M_\omega}^H.$$

Proof. Let $\gamma^G \in M_{G\text{-reg}}$. Put $T^G = \mathrm{Cent}(\gamma^G, G)$. We may assume that A_{T^G} is a standard subtorus of A_0. Let Y and $\tilde{Y}$ be as above. Then Y and $\tilde{Y}$ are finite sets. Let $(\omega, T_{ij}^\omega, \tilde{\imath}^\omega) \in \tilde{Y}$, then (5.1) asserts that $\Delta_{G,H}(i^{\omega -1}(\gamma^G), \gamma^G) = \Delta_{M,M_\omega}(i^{\omega -1}(\gamma^G), \gamma^G)$, where i^ω is a representative for $\tilde{\imath}^\omega$. On the other hand, if $(T_i^H, \tilde{\imath}^H) \in Y$ corresponds to $(\omega, T_{ij}^\omega, \tilde{\imath}^\omega)$, then $i^{\omega -1}(\gamma^G)$ is stably H-conjugate to $i^{H^{-1}}(\gamma^G)$, where i^H is a representative for $\tilde{\imath}^H$. Therefore,

$$\Delta_{G,H}(i^{H^{-1}}(\gamma^G), \gamma^G) = \Delta_{M,M_\omega}(i^{\omega -1}(\gamma^G), \gamma^G).$$

Fix $a \in A_M^-$. Put $a^* = \varphi(a)$. Then $a^* \in A_{M^*}^-$. For $\omega \in D_{M,H}$, we have $i^{\omega -1}(a) = (\omega \circ i_0^*)^{-1}(a^*) \in A_{M_\omega}^-$. Let $\theta_H \in \mathbb{C}[\Pi(H)]^{\mathrm{st}}$, then by Lemmas 3.1 and 4.7, we have

$$\begin{aligned} I^M(\mathrm{Tran}_{M_\omega}^M \circ r_{M_\omega}^H(\theta_H), a^n\gamma^G) &= \sum_{(T_{ij}^\omega, \tilde{\imath}^\omega) \in \tilde{Y}_\omega} \Delta_{M,M_\omega}(i^{\omega -1}(a^n\gamma^G), a^n\gamma^G) \cdot I^{M_\omega}(r_{M_\omega}^H(\theta_H), i^{\omega -1}(a^n\gamma^G)) \\ &= \sum_{(T_{ij}^\omega, \tilde{\imath}^\omega) \in \tilde{Y}_\omega} \Delta_{M,M_\omega}(i^{\omega -1}(a^n\gamma^G), a^n\gamma^G) \cdot I^H(\theta_H, i^{\omega -1}(a^n\gamma^G)) \end{aligned}$$

for sufficiently large n. Since $i^{\omega -1}(a^n\gamma^G)$ is stably H-conjugate to $i^{H^{-1}}(a^n\gamma^G)$, we have

$$I^H(\theta_H, i^{\omega -1}(a^n\gamma^G)) = I^H(\theta_H, i^{H^{-1}}(a^n\gamma^G)).$$

Therefore, by Lemmas 3.1 and 4.7 and Proposition 5.3, we have

$$\begin{aligned} I^M(r_M^G \circ \mathrm{Tran}_H^G(\theta_H), a^n\gamma^G) &= I^G(\mathrm{Tran}_H^G(\theta_H), a^n\gamma^G), \\ &= \sum_{(T_i^H, \tilde{\imath}^H)\in Y} \Delta_{G,H}(i^{H^{-1}}(a^n\gamma^G), a^n\gamma^G)\cdot I^H(\theta_H, i^{H^{-1}}(a^n\gamma^G)), \\ &= \sum_{\omega\in D_{M,H}} \sum_{(T_{ij}^\omega, \tilde{\imath}^\omega)\in \tilde{Y}_\omega} \Delta_{M,M_\omega}(i^{\omega-1}(a^n\gamma^G), a^n\gamma^G)\cdot I^H(\theta_H, i^{\omega-1}(a^n\gamma^G)), \\ &= \sum_{\omega\in D_{M,H}} I^M(\mathrm{Tran}_{M_\omega}^M \circ r_{M_\omega}^H(\theta_H), a^n\gamma^G), \end{aligned}$$

for sufficiently large n. It is then immediate from Lemma 3.2 that

$$I^M(r_M^G \circ \mathrm{Tran}_H^G(\theta_H), \gamma^G) = \sum_{\omega\in D_{M,H}} I^M(\mathrm{Tran}_{M_\omega}^M \circ r_{M_\omega}^H(\theta_H), \gamma^G).$$

□

6. Commutativity

By Corollary 3.4 and Proposition 4.6, we can define $\mathbf{D}_G \circ \mathrm{Tran}_H^G$ and $\mathrm{Tran}_H^G \circ \mathbf{D}_H$. By Lemma 5.1, we may regard $(M_\omega, \mathcal{M}_\omega, s_\omega, \xi_\omega)$ as a set of endoscopic data for G. Let Δ_{G,M_ω} be the Langlands–Shelstad transfer factor. If $\Gamma(M_\omega, M) = \emptyset$, then we can show that $\Delta_{G,M_\omega} \equiv 0$. If $\Gamma(M_\omega, M) \neq \emptyset$, then we normalize Δ_{G,M_ω} so that

$$\Delta_{G,M_\omega}(\gamma^{M_\omega}, \gamma^M) = \Delta_{M,M_\omega}(\gamma^{M_\omega}, \gamma^M),$$

for all $(\gamma^{M_\omega}, \gamma^M) \in \Gamma(M_\omega, M)$. (As in Lemma 5.2, we can show that the relative transfer factors Δ_{G,M_ω} and Δ_{M,M_ω} are equal on $\Gamma(M_\omega, M)$.) Then it is not difficult to show that

$$i_M^G \circ \mathrm{Tran}_{M_\omega}^M = \mathrm{Tran}_{M_\omega}^G$$

and that

$$\mathrm{Tran}_H^G \circ i_{M_\omega}^H = \mathrm{Tran}_{M_\omega}^G .$$

Thus we have the following lemma.

Lemma 6.1. *We have*

$$i_M^G \circ \mathrm{Tran}_{M_\omega}^M = \mathrm{Tran}_H^G \circ i_{M_\omega}^H .$$

Recall that M_0 is a minimal Levi subgroup of G. We put $M_0^* = \varphi(M_0) \in \mathcal{L}^{G^*}$.

Lemma 6.2. *The standard Levi subgroup M_0^* itself is the only associate standard Levi subgroup of M_0^*.*

Proof. Put $X_*(A_{M_0^*}) = \mathrm{Hom}(\mathbb{G}_m, A_{M_0^*})$ and $\mathfrak{a}_{M_0^*} = X_*(A_{M_0^*}) \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\omega \in \Omega(G, A_0)$, then Lemma 4.2 asserts that there exists $\omega^* \in \Omega(G^*, A_0^*)$ such that $\omega = \mathrm{res}_{A_0}(\varphi^{-1} \circ \omega^* \circ \varphi)$. This implies that for each chamber C of $\mathfrak{a}_{M_0^*}$, there exists

$$\omega^* \in \Omega(G^*, A_0^*)_{\mathfrak{a}_{M_0^*}} = \{\omega' \in \Omega(G^*, A_0^*) \mid \omega'(\mathfrak{a}_{M_0^*}) = \mathfrak{a}_{M_0^*}\}$$

such that $\omega^*(C) = C^+_{M_0^*}$ and $\omega^*(S(M_0^*)) = S(M_0^*)$. Hence Lemma 2 shows that M_0^* itself is the only associate standard Levi subgroup of M_0^*. □

Therefore, Lemma 2.5 implies the following lemma.

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Lemma 6.3. *If $M^* \in \mathcal{L}^{G^*}$ satisfies $\omega(A_{M^*}) \subset A_{M_0^*}$ for some $\omega \in \Omega(G^*, A_0^*)$, then we have*

$$M^* \supset M_0^*.$$

Lemma 6.4. *Let $L \in \mathcal{L}^H$ and $\gamma^H \in L_{G\text{-reg}}$. If γ^H is an image of $\gamma^G \in G_{\mathrm{reg}}$, then there exists $\omega \in \Omega(G^*, A_0^*)$ such that*

$$\omega \circ i_0^*(A_L) \subset \varphi(A_0) = A_{M_0^*}.$$

Proof. Let $T^H = \mathrm{Cent}(\gamma^H, H)$ and $T^G = \mathrm{Cent}(\gamma^G, G)$. We may assume that A_{T^H} is a standard subtorus of $A_{H,0}$ and A_{T^G} is a standard subtorus of A_0. Put $M_{T^H} = \mathrm{Cent}(A_{T^H}, H) \in \mathcal{L}^H$ and $M = M_{T^G} = \mathrm{Cent}(A_{T^G}, G) \in \mathcal{L}^G$. Since γ^H is an image of γ^G, there exists $i^H \in I^{G,H}(T^H, T^G)$ such that $i^H(\gamma^H) = \gamma^G$. Then Lemma 5.5 asserts that there exist $\omega_H \in \Omega(H, T_{H,0})_F$, $\omega \in D_{M,H}$, $\omega_M \in \Omega(M, T_0)$, $m \in M(\overline{F})$ and $h \in M_{T^H}(\overline{F})$ such that $\mathrm{Int}\, m(T_0) = T^G$, $\mathrm{Int}\, h(T_{H,0}) = T^H$ and

$$i^H = \mathrm{Int}\, m \circ \omega_M \circ \varphi^{-1} \circ \omega \circ i_0^* \circ \omega_H \circ \mathrm{Int}\, h^{-1}.$$

Hence, we have $A_{T^G} = i^H(A_{T^H}) = \varphi^{-1} \circ \omega \circ i_0^* \circ \omega_H(A_{T^H})$. By using $i_0^* \circ \omega_H \circ i_0^{*-1} \in i^*(\omega_H) \cdot \Omega(M^H, T_0^*)$ and $i_0^*(A_{T^H}) \subset i_0^*(A_{H,0})$, we have $\varphi^{-1} \circ \omega \circ i_0^* \circ \omega_H(A_{T^H}) = \varphi^{-1} \circ \omega \cdot i^*(\omega_H) \circ i_0^*(A_{T^H})$. Thus

$$\omega \cdot i^*(\omega_H) \circ i_0^*(A_L) \subset \omega \cdot i^*(\omega_H) \circ i_0^*(A_{T^H}) = \varphi(A_{T^G}) \subset \varphi(A_0). \qquad \square$$

For $M^* \in \mathcal{L}^{G^*}$ and $L \in \mathcal{L}^H$, we define $D_{M^*,H,L}$ as in § 2.

Corollary 6.5. *Let $M^* \in \mathcal{L}^{G^*}$. If $D_{M^*,H,L} \neq \emptyset$ and if there exists $\gamma^H \in L_{G\text{-reg}}$ that is an image of an element $\gamma^G \in G_{\mathrm{reg}}$, then we have*

$$M^* \supset M_0^*.$$

Proof. Let $\omega \in D_{M^*,H,L}$, then we have $\omega^{-1}(A_{M^*}) \subset i_0^*(A_L)$. On the other hand, Lemma 6.4 asserts that there exists $\omega' \in \Omega(G^*, A_0^*)$ such that $\omega' \circ i_0^*(A_L) \subset A_{M_0^*}$. Thus we have $\omega' \omega^{-1}(A_{M^*}) \subset A_{M_0^*}$. It is then immediate from Lemma 6.3 that $M^* \supset M_0^*$. $\square$

We put $a(G) = \dim A_0$, $a(G^*) = \dim A_0^*$ and $a(H) = \dim A_{H,0}$. Then for $M \in \mathcal{L}^G$, we have $r(\varphi(M)) - r(M) = a(G^*) - a(G)$. We put

$$a_{M^*,H,L} = \sharp D_{M^*,H,L}$$

as in § 2.

Theorem 6.6. *Assume the fundamental lemma for groups and for Lie algebras. Then we have*

$$\mathbf{D}_G \circ \mathrm{Tran}_H^G = (-1)^{a(G)-a(H)}\, \mathrm{Tran}_H^G \circ \mathbf{D}_H.$$

Proof. By Theorem 5.6, we have

$$\begin{aligned}\mathbf{D}_G \circ \mathrm{Tran}_H^G &= \sum_{M \in \mathcal{L}^G} (-1)^{r(M)} i_M^G \circ r_M^G \circ \mathrm{Tran}_H^G \\ &= \sum_{M \in \mathcal{L}^G} \sum_{\omega \in D_{M,H}} (-1)^{r(M)} i_M^G \circ \mathrm{Tran}_{M_\omega}^M \circ r_{M_\omega}^H.\end{aligned}$$

Since $i_M^G \circ \mathrm{Tran}_{M_\omega}^M = \mathrm{Tran}_H^G \circ i_{M_\omega}^H$, this is equal to

$$\sum_{M \in \mathcal{L}^G} \sum_{\omega \in D_{M,H}} (-1)^{r(M)} \mathrm{Tran}_H^G \circ i_{M_\omega}^H \circ r_{M_\omega}^H,$$
$$= (-1)^{a(G)-a(G^*)} \sum_{\substack{M^* \in \mathcal{L}^{G^*} \\ M^* \supset M_0^*}} (-1)^{r(M^*)} \sum_{L \in \mathcal{L}^H} \sum_{\omega \in D_{M^*,H,L}} \mathrm{Tran}_H^G \circ i_L^H \circ r_L^H,$$
$$= (-1)^{a(G)-a(G^*)} \sum_{\substack{M^* \in \mathcal{L}^{G^*} \\ M^* \supset M_0^*}} (-1)^{r(M^*)} \sum_{L \in \mathcal{L}^H} a_{M^*,H,L} \mathrm{Tran}_H^G \circ i_L^H \circ r_L^H.$$

By using Corollary 6.5 and Theorem 2.4, we can show that this is equal to

$$(-1)^{a(G)-a(G^*)} \sum_{L \in \mathcal{L}^H} \sum_{M^* \in \mathcal{L}^{G^*}} (-1)^{r(M^*)} a_{M^*,H,L} \mathrm{Tran}_H^G \circ i_L^H \circ r_L^H$$
$$= (-1)^{a(G)-a(G^*)} (-1)^{a(G^*)-a(H)} \sum_{L \in \mathcal{L}^H} (-1)^{r(L)} \mathrm{Tran}_H^G \circ i_L^H \circ r_L^H$$
$$= (-1)^{a(G)-a(H)} \mathrm{Tran}_H^G \circ \mathbf{D}_H. \qquad \square$$

Now, we treat the general case. Let (H_1, ξ_{H_1}) be a z-pair for the set of endoscopic data $(H, \mathcal{H}, s, \xi)$ as in § 5. Then for $\omega \in D_{M,H}$, we can define a z-pair $(M_{\omega,1}, \xi_{H_1})$ for $(M_\omega, \mathcal{M}_\omega, s_\omega, \xi_\omega)$ as in § 5. Let Z_1 be the kernel of the morphism $H_1 \longrightarrow H$ and λ_{H_1} the quasi-character of $Z_1(F)$ defined in [KS99, § 2.2]. Let $\Pi(H_1, \lambda_{H_1})$ be the set of equivalence classes of irreducible admissible representations of $H_1(F)$ whose central characters on $Z_1(F)$ are equal to λ_{H_1}. We denote by $\mathbb{C}[\Pi(H_1, \lambda_{H_1})]$ the subspace of $\mathbb{C}[\Pi(H_1)]$ generated by $\Pi(H_1, \lambda_{H_1})$. We put $\mathbb{C}[\Pi(H_1, \lambda_{H_1})]^{\mathrm{st}} = \mathbb{C}[\Pi(H_1, \lambda_{H_1})] \cap \mathbb{C}[\Pi(H_1)]^{\mathrm{st}}$. Then we have

$$\mathrm{Tran}_{H_1}^G (\mathbb{C}[\Pi(H_1, \lambda_{H_1})]^{\mathrm{st}}) \subset \mathbb{C}[\Pi(G)],$$

as in Proposition 4.6. On the other hand, by [KS99, Lemma 5.1.C] we have

$$\Delta_{G,H_1}(z\gamma^{H_1}, \gamma^G) \cdot I^{H_1}(\theta_{H_1}, z\gamma^{H_1}) = \Delta_{G,H_1}(\gamma^{H_1}, \gamma^G) \cdot I^{H_1}(\theta_{H_1}, \gamma^{H_1}),$$

for all $z \in Z_1(F)$ and $\theta_{H_1} \in \mathbb{C}[\Pi(H_1, \lambda_{H_1})]^{\mathrm{st}}$. Therefore, by arguments similar to the proofs of Theorems 5.6 and 6.6, we have the following theorem.

Theorem 6.7. *Assume the fundamental lemma for groups and for Lie algebras. Then we have*

$$r_M^G \circ \mathrm{Tran}_{H_1}^G = \sum_{\omega \in D_{M,H}} \mathrm{Tran}_{M_{\omega,1}}^M \circ r_{M_{\omega,1}}^{H_1},$$
$$\mathbf{D}_G \circ \mathrm{Tran}_{H_1}^G = (-1)^{a(G)-a(H)} \mathrm{Tran}_{H_1}^G \circ \mathbf{D}_{H_1}.$$

7. Functoriality

In this section, we discuss the relation between the Zelevinski involutions and the Arthur conjecture [Art89, Conjecture 6.1]. So we assume [Art89, Conjecture 6.1], Hypothesis 1.1 and the fundamental lemma for groups and for Lie algebras in this section. Let

$$\psi : W_F \times SU_2(\mathbb{C}) \times SL_2(\mathbb{C}) \longrightarrow {}^L G$$

be an Arthur parameter and let $\Pi_\psi(G)$ be the A-packet of ψ. We denote the second factor by $SU_2(\mathbb{C})$ in order to distinguish this from the third factor $SL_2(\mathbb{C})$, which is used to define the Arthur parameters in [Art89]. We put

$$S_\psi = \mathrm{Cent}(\psi, \hat{G}),$$
$$\mathbb{S}_\psi = S_\psi / S_\psi^0.$$

Let $s_\psi = \psi(1 \times 1 \times (-1)) \in S_\psi$. We denote by $\mathbb{C}[\Pi_\psi(G)]$ the subspace generated by $\Pi_\psi(G)$. We put $\mathbb{C}[\Pi_\psi(G)]^{\mathrm{st}} = \mathbb{C}[\Pi_\psi(G)] \cap \mathbb{C}[\Pi(G)]^{\mathrm{st}}$. For the set of endoscopic data $(H, \mathcal{H}, s, \xi)$ corresponding to a semisimple element $s \in S_\psi$, we choose a z-pair (H_1, ξ_{H_1}). Then $\xi_{H_1} \circ \psi$ is an Arthur parameter on H_1, which we also denote by ψ. Let $\theta_\psi^{H_1} \in \mathbb{C}[\Pi_\psi(H_1)]^{\mathrm{st}}$ be the distribution ψ_H in [Art89, Conjecture 6.1]. As in [Art89, Conjecture 6.1], we define the function δ. Then

$$\mathrm{Tran}_{H_1}^G(\theta_\psi^{H_1}) = \sum_{\pi \in \Pi_\psi(G)} \delta(s_\psi s, \pi) \cdot \pi,$$

where H is the endoscopic group corresponding to $s \in S_\psi$. Let $\rho = \rho_\psi$ be the normalizing function in [Art89, Conjecture 6.1]. Then

$$\langle \overline{s}, \pi | \rho \rangle = \delta(s, \pi) \rho(s)^{-1},$$

where $\overline{s} \in \mathbb{S}_\psi$ is the image of $s \in S_\psi$. We identify $SU_2(\mathbb{C})$ with $SL_2(\mathbb{C})$. We define $d(\psi)$ by

$$d(\psi)(w \times t \times u) = \psi(w \times u \times t), \quad w \times t \times u \in W_F \times SU_2(\mathbb{C}) \times SL_2(\mathbb{C}).$$

Then $d(\psi)$ is the Arthur parameter constructed from ψ by interchanging the role of $SU_2(\mathbb{C})$ and $SL_2(\mathbb{C})$. We have $S_\psi = S_{d(\psi)}$ and $\mathbb{S}_\psi = \mathbb{S}_{d(\psi)}$. If G is quasi-split, then we put $G^* = G$ and

$$\mathbb{S}_\psi^* = S_\psi / S_\psi^0 \cdot Z_{\hat{G}}^\Gamma.$$

We fix Whittaker data χ for G^* (see [KS99, § 5.3]). Let ϕ_ψ be the corresponding Langlands parameter on G^* and $\Pi_{\phi_\psi}(G^*)$ the L-packet of ϕ_ψ. We determine the base point $\pi_\chi \in \Pi_{\phi_\psi}(G^*)$ as in [Art89, § 6]. Let $(H, \mathcal{H}, s, \xi)$ be the set of endoscopic data corresponding to $s \in S_\psi$. If $z \in Z_{\hat{G}}^\Gamma$, then $(H, \mathcal{H}, s, \xi)$ and $(H, \mathcal{H}, sz, \xi)$ are equivalent endoscopic data. Therefore, Hypothesis 1.1 implies that for $\pi \in \Pi_\psi(G^*)$,

$$\langle \overline{s}, \pi | \pi_\chi \rangle = \delta(s, \pi) \delta(s, \pi_\chi)^{-1}$$

depends only on the image of s in $\mathbb{S}_\psi^*$. (We also write $\overline{s}$ for the image of s in $\mathbb{S}_\psi^*$.) Thus we may regard $\langle \cdot, \pi | \pi_\chi \rangle$ as an irreducible character of $\mathbb{S}_\psi^*$.

Let $\{M\}$ be the set of associate standard Levi subgroups of M. We say that $\pi \in \Pi(G)$ is *of type* $\{M\}$ if $r_M^G(\pi)$ is a non-zero linear combination of supercuspidal representations of $M(F)$. If π is of type $\{M\}$, then we put $r_\pi = r(M)$. For $\pi \in \Pi(G)$, we define $\mathbf{d}_G(\pi)$ by

$$\mathbf{d}_G(\pi) = (-1)^{r_\pi} \mathbf{D}_G(\pi).$$

Then we have $r_\pi = r_{\mathbf{d}_G(\pi)}$. The following proposition, which is conjectured by Kato [Kat93], is proved by Aubert [Aub95, Aub96].

Proposition 7.1. *For $\pi \in \Pi(G)$, we have $\mathbf{d}_G(\pi) \in \Pi(G)$.*

Lemma 7.2. *Conjecture 1.4 implies Conjecture 1.2.*

Proof. First, we treat the case that G is quasi-split. Put $G = G^*$. We prove

$$\mathbf{d}_{G^*}(\Pi_\psi(G^*)) = \Pi_{d(\psi)}(G^*)$$

by induction on $r(G^*)$. If $\mathbb{S}_\psi^* = \{1\}$, then there is nothing to prove. Suppose that $\mathbb{S}_\psi^* \neq \{1\}$. Let $\pi \in \Pi_\psi(G^*)$, then since $\langle \cdot, \pi | \pi_\chi \rangle$ is an irreducible character, there exists $\overline{s} \in \mathbb{S}_\psi^*$ such that $\overline{s} \neq \overline{1}$ and $\langle \overline{s}, \pi | \pi_\chi \rangle \neq 0$. Let $(H, \mathcal{H}, s, \xi)$ be the set of endoscopic data corresponding to $s \in S_\psi$. Choose a z-pair (H_1, ξ_{H_1}). Then the coefficient of π in $\mathrm{Tran}_{H_1}^{G^*}(\theta_\psi^{H_1})$ is not zero. On the other hand, by the inductive assumption, we have $\mathbf{d}_{H_1}(\Pi_\psi(H_1)) = \Pi_{d(\psi)}(H_1)$. Hence, Corollary 3.4 asserts that $\mathbf{D}_{H_1}(\theta_\psi^{H_1}) \in \mathbb{C}[\Pi_{d(\psi)}(H_1)]^{\mathrm{st}}$. Therefore, Hypothesis 1.1 implies that there exists $c_\psi^{H_1} \in \mathbb{C}^\times$ such that

$$\mathbf{D}_{H_1}(\theta_\psi^{H_1}) = c_\psi^{H_1}\theta_{d(\psi)}^{H_1}.$$

By Theorem 6.7, we have

$$\mathbf{D}_{G^*} \circ \operatorname{Tran}_{H_1}^{G^*}(\theta_\psi^{H_1}) = (-1)^{a(G^*)-a(H)} c_\psi^{H_1} \operatorname{Tran}_{H_1}^{G^*}(\theta_{d(\psi)}^{H_1}).$$

Since the coefficient of π in $\operatorname{Tran}_{H_1}^{G^*}(\theta_\psi^{H_1})$ is not zero, this shows that the coefficient of $\mathbf{d}_{G^*}(\pi)$ in $\operatorname{Tran}_{H_1}^{G^*}(\theta_{d(\psi)}^{H_1})$ is not zero. Therefore, $\mathbf{d}_{G^*}(\pi) \in \Pi_{d(\psi)}(G^*)$. Thus we have $\mathbf{d}_{G^*}(\Pi_\psi(G^*)) = \Pi_{d(\psi)}(G^*)$.

Now, we turn to the general case. Let $\pi \in \Pi(G)$, then [Art89, Conjecture 6.1(iii)] shows that we have $\pi \in \Pi_\psi(G)$ if and only if the coefficient of π in $\operatorname{Tran}_{G^*}^{G}(\theta_\psi^{G^*})$ is not zero. By Theorem 6.6, we have

$$\mathbf{D}_G \circ \operatorname{Tran}_{G^*}^{G}(\theta_\psi^{G^*}) = (-1)^{a(G^*)-a(G)} c_\psi^{G^*} \operatorname{Tran}_{G^*}^{G}(\theta_{d(\psi)}^{G^*}).$$

Therefore, $\mathbf{d}_G(\Pi_\psi(G)) = \Pi_{d(\psi)}(G)$. □

Now, we assume Conjecture 1.4. By comparing the coefficient of $\mathbf{d}_G(\pi)$ in

$$\mathbf{D}_G \circ \operatorname{Tran}_{H_1}^{G}(\theta_\psi^{H_1}) = (-1)^{a(G)-a(H)} c_\psi^{H_1} \operatorname{Tran}_{H_1}^{G}(\theta_{d(\psi)}^{H_1}),$$

we have

$$\delta(s_\psi s, \pi)(-1)^{r_\pi} = (-1)^{a(G)-a(H)} c_\psi^{H_1} \delta_d(s_{d(\psi)} s, \mathbf{d}_G(\pi)), \tag{7.1}$$

where $\delta_d = \delta_{d(\psi)}$. We assume that $G = G^*$. By dividing the formula (7.1) by that of π_χ, we have

$$\langle \overline{s}_\psi \overline{s}, \pi | \pi_\chi \rangle (-1)^{r_\pi - r_{\pi_\chi}} = \langle \overline{s}_{d(\psi)} \overline{s}, \mathbf{d}_{G^*}(\pi) | \pi_{d,\chi} \rangle \langle \overline{s}_{d(\psi)} \overline{s}, \mathbf{d}_{G^*}(\pi_\chi) | \pi_{d,\chi} \rangle^{-1}, \tag{7.2}$$

where $\pi_{d,\chi}$ is the base point in $\Pi_{d(\psi)}(G^*)$. Since $\delta(s_\psi s, \pi_\chi) \neq 0$, we have $\delta_d(s_{d(\psi)} s, \mathbf{d}_{G^*}(\pi_\chi)) \neq 0$ and

$$\langle \overline{s}_{d(\psi)} \overline{s}, \mathbf{d}_{G^*}(\pi_\chi) | \pi_{d,\chi} \rangle \neq 0$$

for all $s \in S_{d(\psi)}$. If we put $\overline{s} = \overline{s}_\psi$ and $\pi = \mathbf{d}_{G^*}(\pi_{d,\chi})$, then we have

$$\langle \overline{1}, \mathbf{d}_{G^*}(\pi_{d,\chi}) | \pi_\chi \rangle (-1)^{r_{\pi_{d,\chi}} - r_{\pi_\chi}} = \langle \overline{s}_{d(\psi)} \overline{s}_\psi, \mathbf{d}_{G^*}(\pi_\chi) | \pi_{d,\chi} \rangle^{-1}.$$

Since $\overline{s}_{d(\psi)} \overline{s}_\psi$ is an element of order 1 or 2 contained in the center of $\mathbb{S}_{d(\psi)}$, this implies that

$$\langle \overline{1}, \mathbf{d}_{G^*}(\pi_{d,\chi}) | \pi_\chi \rangle = \langle \overline{1}, \mathbf{d}_{G^*}(\pi_\chi) | \pi_{d,\chi} \rangle = 1.$$

Thus we have the following lemma.

Lemma 7.3. *The character $\langle \cdot, \mathbf{d}_{G^*}(\pi_\chi) | \pi_{d,\chi} \rangle$ is a one-dimensional character of $\mathbb{S}_{d(\psi)}$.*

Moreover, since

$$\langle \overline{s}, \pi | \pi_\chi \rangle = \pm \langle \overline{s}, \mathbf{d}_{G^*}(\pi) | \pi_{d,\chi} \rangle \langle \overline{s}, \mathbf{d}_{G^*}(\pi_\chi) | \pi_{d,\chi} \rangle^{-1},$$

we have the following proposition.

Proposition 7.4. *Let $G = G^*$. Assume the fundamental lemma for groups and for Lie algebras, the Arthur conjecture [Art89, Conjecture 6.1], Hypothesis 1.1 and Conjecture 1.4. Then*

$$\langle \overline{s}, \pi | \pi_\chi \rangle = \langle \overline{s}, \mathbf{d}_{G^*}(\pi) | \pi_{d,\chi} \rangle \langle \overline{s}, \mathbf{d}_{G^*}(\pi_\chi) | \pi_{d,\chi} \rangle^{-1}.$$

This is the formula in Conjecture 1.3. By an easy calculation, we have

$$\mathbf{D}_{G^*}\Bigg(\sum_{\pi \in \Pi_\psi(G^*)} \langle \overline{s}_\psi, \pi | \pi_\chi \rangle \cdot \pi \Bigg) = (-1)^{r_{\pi_\chi}} \langle \overline{s}_{d(\psi)}, \mathbf{d}_{G^*}(\pi_\chi) | \pi_{d,\chi} \rangle^{-1} \Bigg(\sum_{\pi \in \Pi_{d(\psi)}(G^*)} \langle \overline{s}_{d(\psi)}, \pi | \pi_{d,\chi} \rangle \cdot \pi \Bigg).$$

Since $\theta_\psi^{G^*} = \sum_{\pi\in\Pi_\psi(G^*)} \delta(s_\psi,\pi)\cdot\pi$, the relation $\mathbf{D}_{G^*}(\theta_\psi^{G^*}) = c_\psi^{G^*}\theta_{d(\psi)}^{G^*}$ implies that

$$c_\psi^{G^*} = (-1)^{r_{\pi_\chi}}\delta(s_\psi,\pi_\chi)\delta_d(s_{d(\psi)},\pi_{d,\chi})^{-1}\langle\overline{s}_{d(\psi)},\mathbf{d}_{G^*}(\pi_\chi)|\pi_{d,\chi}\rangle^{-1}.$$

Therefore, by using $|\delta(s_\psi,\pi_\chi)| = 1$ and $|\delta_d(s_{d(\psi)},\pi_{d,\chi})| = 1$ (see [Art89, Conjecture 6.1 (iv)]), we have $|c_\psi^{G^*}| = 1$.

By dividing (7.1) by $\rho(s_\psi s)$, we have

$$\langle\overline{s}_\psi\overline{s},\pi|\rho\rangle(-1)^{r_\pi} = (-1)^{a(G)-a(H)}c_\psi^{H_1}\rho(s_\psi s)^{-1}\rho_d(s_{d(\psi)}s)\langle\overline{s}_{d(\psi)}\overline{s},\mathbf{d}_G(\pi)|\rho_d\rangle.$$

Hence, by $|c_\psi^{H_1}| = 1$ and [Art89, Conjecture 6.1 (iii)], we have

$$\langle\overline{1},\pi|\rho\rangle = \langle\overline{1},\mathbf{d}_G(\pi)|\rho_d\rangle.$$

Appendix. Proofs of the results in § 2

In this section, we prove the results in § 2. We keep the notation in § 2. We begin with the following lemma.

Lemma A.1. *If $\alpha \in R^+(G;\mathfrak{a}_M)$ is $R(G)$-symmetric, then there exist*

$$\tilde\alpha_1,\dots,\tilde\alpha_r \in R^+(G)$$

satisfying the following conditions:

1) $(\tilde\alpha_i,\tilde\alpha_j) = 0$, *if* $i \neq j$;
2) $\mathrm{res}_{\mathfrak{a}_M}(\tilde\alpha_i) \in \mathbb{R}_+^\times\alpha,\quad i = 1,\dots,r$;
3) $s_{\tilde\alpha_1}\cdots s_{\tilde\alpha_r} \in \Omega(G)_{\mathfrak{a}_M}$;
4) $s_\alpha = \mathrm{res}_{\mathfrak{a}_M}(s_{\tilde\alpha_1}\cdots s_{\tilde\alpha_r})$.

Before proving Lemma 1, we prepare some lemmas. For $S(M') \subset S(G)$, we put

$$C_{M'}^+ = \{a \in \mathfrak{a}_{M'} \mid \tilde\alpha(a) > 0 \text{ for all } \tilde\alpha \in S(G) - S(M')\}.$$

We denote by $\overline{C_{M'}^+}$ the closure of $C_{M'}^+$. For $S(M'), S(M'') \subset S(G)$, we put

$$\Omega(M',M'') = \{\omega \in \Omega(G) \mid \omega(S(M'')) = S(M')\}.$$

Each connected component of

$$\mathfrak{a}_{M'} - \left(\bigcup_{\alpha\in R(G)-R(M')} \ker\alpha\right)$$

is called a chamber of $\mathfrak{a}_{M'}$. The following lemma is [Cas74, Proposition 1.2.2].

Lemma A.2. *If C is a chamber of $\mathfrak{a}_{M'}$, then there exist unique $S(M'')$ and $\omega \in \Omega(M',M'')$ such that*

$$\omega(C_{M''}^+) = C.$$

Lemma A.3. *Let $\alpha \in R^+(G;\mathfrak{a}_M)$. If $\ker\alpha \cap \overline{C_M^+}$ contains a non-empty open subset of $\ker\alpha$, then there exists a simple root $\tilde\alpha \in S(G)$ such that*

$$\mathrm{res}_{\mathfrak{a}_M}(\tilde\alpha) \in \mathbb{R}_+^\times\alpha.$$

Proof. Put $S(M') = \{\tilde\beta \in S(G) \mid \tilde\beta(a) = 0 \text{ for all } a \in C_\alpha\}$, where $C_\alpha = \ker\alpha\cap\overline{C_M^+}$. Since $C_\alpha \subset \overline{C_M^+}$, we have $\tilde\beta(a) \geqslant 0$ for all $a \in C_\alpha$ and $\tilde\beta \in S(G)$. Thus

$$R(M') = \{\tilde\beta \in R(G) \mid \tilde\beta(a) = 0 \text{ for all } a \in C_\alpha\}.$$

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Since this shows that a root $\tilde{\alpha}'$ satisfying $\mathrm{res}_{\mathfrak{a}_M}(\tilde{\alpha}') = \alpha$ is contained in $R(M')$, we have $R(M') \supsetneq R(M)$. Thus $S(M') \supsetneq S(M)$. Take a simple root $\tilde{\alpha} \in S(M') - S(M)$. Then $\tilde{\alpha}$ satisfies $\mathrm{res}_{\mathfrak{a}_M}(\tilde{\alpha}) \in \mathbb{R}_+^\times \alpha$. □

Let C be a chamber of $\mathfrak{a}_M$ whose closure $\overline{C}$ contains a non-empty open subset of $\ker \alpha$. Then Lemma 2 asserts that there exist unique $S(M')$ and $\omega \in \Omega(M, M')$ such that $\omega(C_{M'}^+) = C$. By Lemma 3, we can choose a simple root $\tilde{\alpha}$ such that

$$\mathrm{res}_{\mathfrak{a}_{M'}}(\tilde{\alpha}) = \mathbb{R}_+^\times \omega^{-1}\alpha.$$

Since we may replace $S(M)$ by $S(M')$ and α by $\omega^{-1}\alpha$, it is enough to prove Lemma 1 under the condition $\alpha \in \mathrm{res}_{\mathfrak{a}_M}(S(G))$. Therefore we assume that there exists a simple root $\tilde{\alpha}$ such that $\alpha = \mathrm{res}_{\mathfrak{a}_M}(\tilde{\alpha})$. Put $S(M_\alpha) = S(M) \cup \{\tilde{\alpha}\}$. We denote by ω_-^α the longest element of $\Omega(M_\alpha)$.

Lemma A.4. *Let α and $\tilde{\alpha}$ be as above, then α is $R(G)$-symmetric if and only if*

$$\omega_-^\alpha \tilde{\alpha} = -\tilde{\alpha}.$$

Moreover, if α is $R(G)$-symmetric, then we have

$$\mathrm{res}_{\mathfrak{a}_M}(\omega_-^\alpha) = s_\alpha.$$

Proof. The condition for $\omega \in \Omega(G)$ to satisfy $\mathrm{res}_{\mathfrak{a}_M}(\omega) = s_\alpha$ is that $\omega(R(M)) = R(M)$, $\omega\tilde{\alpha} < 0$ and $\omega(R(M_\alpha)) = R(M_\alpha)$. It is easy to see that such an element ω exists if and only if we have $\omega_-^\alpha \tilde{\alpha} = -\tilde{\alpha}$. Moreover, if $\omega_-^\alpha \tilde{\alpha} = -\tilde{\alpha}$, then the condition for ω to satisfy $\mathrm{res}_{\mathfrak{a}_M}(\omega) = s_\alpha$ is that $\omega \in \omega_-^\alpha \cdot \Omega(M)$. So the lemma is proved. □

Let R be a root system and ω_- the longest element of the Weyl group of R with respect to a positive root system R^+. Then it is not hard to check that there exist mutually orthogonal roots $\tilde{\alpha}_1, \dots, \tilde{\alpha}_p \in R^+$ such that

$$s_{\tilde{\alpha}_1} \cdots s_{\tilde{\alpha}_p} = \omega_-.$$

Now, we prove Lemma 1. Suppose that α is $R(G)$-symmetric. By Lemma 4, we have

$$\mathrm{res}_{\mathfrak{a}_M}(\omega_-^\alpha) = s_\alpha.$$

For $R = R(M_\alpha)$ and $\omega_- = \omega_-^\alpha$, take $\tilde{\alpha}_1, \dots, \tilde{\alpha}_p$ as above. We may arrange the index so that $\tilde{\alpha}_1, \dots, \tilde{\alpha}_r$ are not contained in $R^+(M)$ and that $\tilde{\alpha}_{r+1}, \dots, \tilde{\alpha}_p$ are contained in $R^+(M)$. Then we have

$$\mathrm{res}_{\mathfrak{a}_M}(s_{\tilde{\alpha}_1} \cdots s_{\tilde{\alpha}_r}) = \mathrm{res}_{\mathfrak{a}_M}(\omega_-^\alpha) = s_\alpha,$$
$$\mathrm{res}_{\mathfrak{a}_M} \tilde{\alpha}_1, \dots, \mathrm{res}_{\mathfrak{a}_M} \tilde{\alpha}_r \in \mathbb{R}_+^\times \alpha.$$

This proves Lemma 1.

Until the end of this section, we fix $S(M^H)$, $R(H)$ as in § 2.

Lemma A.5 (Lemma 2.1). *Let $\omega \in \tilde{D}_M^{-1}$ and $\omega' \in \Omega(G)_H$. Let $\tilde{\alpha} \in R^+(G)$ be a positive root such that $\mathrm{res}_H(\tilde{\alpha}) \neq 0$. If $\omega\omega'\tilde{\alpha} > 0$, then for any $\tilde{\alpha}' \in R^+(G)$ satisfying $\mathrm{res}_H(\tilde{\alpha}') \in \mathbb{R}_+^\times \mathrm{res}_H(\tilde{\alpha})$, we have $\omega\omega'\tilde{\alpha}' > 0$ and if $\omega\omega'\tilde{\alpha} < 0$,then for any $\tilde{\alpha}' \in R^+(G)$ satisfying $\mathrm{res}_H(\tilde{\alpha}') \in \mathbb{R}_+^\times \mathrm{res}_H(\tilde{\alpha})$, we have $\omega\omega'\tilde{\alpha}' < 0$.*

Proof. Take $a_+ \in C_M^+$. Suppose that $\tilde{\alpha}' \in R^+(G)$ satisfies $\mathrm{res}_H(\tilde{\alpha}') = c \cdot \mathrm{res}_H(\tilde{\alpha})$, where $c \in \mathbb{R}_+^\times$. Then we have $(c \cdot \tilde{\alpha} - \tilde{\alpha}')(\mathfrak{a}^H) = 0$. By the definition of $\tilde{D}_M$, we have $\omega'^{-1}\omega^{-1}(\mathfrak{a}_M) \subset \mathfrak{a}^H$. Thus, $\omega\omega'(c \cdot \tilde{\alpha} - \tilde{\alpha}')(\mathfrak{a}_M) = 0$. Therefore,

$$c \cdot \omega\omega'\tilde{\alpha}(a_+) = \omega\omega'\tilde{\alpha}'(a_+).$$

Now, suppose that $\omega\omega'\tilde{\alpha} > 0$. If $\omega\omega'\tilde{\alpha}(a_+) > 0$, then $\omega\omega'\tilde{\alpha}'(a_+) > 0$. This implies that $\omega\omega'\tilde{\alpha}' > 0$. If $\omega\omega'\tilde{\alpha}(a_+) = 0$, then $\omega\omega'\tilde{\alpha}'(a_+) = 0$. Thus $\omega\omega'\tilde{\alpha}, \omega\omega'\tilde{\alpha}' \in R(M)$. Since $\omega \in \tilde{D}_M^{-1}$ and

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since $\omega\omega'\tilde{\alpha} \in R^+(M)$, we have $\omega'\tilde{\alpha} > 0$. This implies that $c^{-1}\cdot \mathrm{res}_H(\omega'\tilde{\alpha}') = \mathrm{res}_H(\omega'\tilde{\alpha}) \in R^+(G; \mathfrak{a}^H)$. Therefore, $\omega'\tilde{\alpha}' > 0$. Since $\omega \in \tilde{D}_M^{-1}$ and since $\omega\omega'\tilde{\alpha}' \in R(M)$, this shows that $\omega\omega'\tilde{\alpha}' > 0$. □

In the following, we prove Proposition 2.2. Obviously, we have $\omega \in \Omega(G)_{M,H}$ if and only if $\omega\Omega(M^H)\omega^{-1} \subset \Omega(M)$.

Lemma A.6. *Let $\omega \in \Omega(G)_{M,H}$, then we have*

$$\Omega(M)\omega\Omega(H) \cap D_{M,H} \neq \emptyset.$$

Proof. It is easy to see that $\tilde{D}_M^{-1}$ is a system of representatives for $\Omega(M)\backslash\Omega(G)_{M,H}$. Therefore, we may assume that $\omega \in \tilde{D}_M^{-1}$. Put $\tilde{R}^+(H) = \{\tilde{\alpha} \in R(G) \mid \mathrm{res}_H(\tilde{\alpha}) \in R^+(H)\}$. By Lemma 2.1, we can choose $\omega_0 \in \omega\Omega(H)$ such that $\omega_0(\tilde{R}^+(H)) > 0$. Now, we prove that $\Omega(M)\omega_0 \cap D_{M,H} \neq \emptyset$ by induction on $l_M(\omega_0^{-1})$ under the condition $\omega_0(\tilde{R}^+(H)) > 0$. If $l_M(\omega_0^{-1}) = 0$, then we have $\omega_0 \in D_{M,H}$ and the lemma is proved. Suppose that $l_M(\omega_0^{-1}) > 0$. Then there exists a simple root $\tilde{\alpha} \in S(M)$ such that $\omega_0^{-1}\tilde{\alpha} < 0$. Since $\omega_0(\tilde{R}^+(H)) > 0$, we have $\mathrm{res}_H(\omega_0^{-1}\tilde{\alpha}) \notin R(H)$. Therefore, $\omega_0(\tilde{R}^+(H)) > 0$ implies that $s_{\tilde{\alpha}}\omega_0(\tilde{R}^+(H)) > 0$. Since $l_M(\omega_0^{-1}s_{\tilde{\alpha}}) < l_M(\omega_0^{-1})$, we have $\Omega(M)s_{\tilde{\alpha}}\omega_0 \cap D_{M,H} \neq \emptyset$, by the inductive assumption. Because $s_{\tilde{\alpha}} \in \Omega(M)$, this shows that $\Omega(M)\omega_0 \cap D_{M,H} \neq \emptyset$. □

Proposition A.7 (Proposition 2.2). *The subset $D_{M,H} \subset \Omega(G)_{M,H}$ is a system of representatives for $\Omega(M)\backslash\Omega(G)_{M,H}/\Omega(H)$.*

Proof. It is enough to show that

$$\sharp(\Omega(M)\omega_0\Omega(H) \cap D_{M,H}) = 1$$

for $\omega_0 \in D_{M,H}$. Let $\omega \in \Omega(M)\omega_0\Omega(H) \cap D_{M,H}$, then we can write ω as $\omega = \omega_M\omega_0\omega_H$ with $\omega_M \in \Omega(M)$ and $\omega_H \in \Omega(H)$. We claim that $\omega \in \omega_0\Omega(H)\Omega(M^H)$. We prove this by induction on $l_M(\omega_M^{-1})$. If $l_M(\omega_M^{-1}) = 0$, then $\omega_M = 1$. Thus $\omega \in \omega_0\Omega(H)$. Suppose that $l_M(\omega_M^{-1}) > 0$ and $l_H(\omega_H) = 0$, then we have $\mathrm{res}_H(\omega_H) = 1$. Thus, $\omega_0(\mathfrak{a}^H) \supset \mathfrak{a}_M$ implies that $\omega_0\omega_H\omega_0^{-1} \in \Omega(M)$. Combining this with $l_M(\omega_0^{-1}) = l_M(\omega^{-1}) = 0$ and $\omega = \omega_M \cdot (\omega_0\omega_H\omega_0^{-1}) \cdot \omega_0$, we have $\omega_M \cdot (\omega_0\omega_H\omega_0^{-1}) = 1$. Thus, $\omega = \omega_0$. Suppose that $l_M(\omega_M^{-1}) > 0$ and $l_H(\omega_H) > 0$. Then there exists a positive root $\alpha \in R^+(H)$ such that $\omega_H\alpha < 0$. Then, by Lemma 2.1, we have $\omega_0\omega_H\tilde{\alpha} < 0$ for any $\tilde{\alpha} \in R^+(G)$ satisfying $\mathrm{res}_H\,\tilde{\alpha} \in \mathbb{R}_+^\times\alpha$. On the other hand, since $\omega \in D_{M,H}$, we have $\omega_M\omega_0\omega_H\tilde{\alpha} > 0$. Because ω_M changes the positivity of $\omega_0\omega_H\tilde{\alpha}$, we have $\omega_0\omega_H\tilde{\alpha} \in R(M)$. Put $\tilde{\alpha}' = \omega_H\tilde{\alpha}$ and $\alpha' = \omega_H\alpha$. Since $\mathrm{res}_H(\tilde{\alpha}')$ is $R(G)$-symmetric, we can choose $\tilde{\alpha}_1', \dots, \tilde{\alpha}_r' \in R^+(G)$ as in Lemma 1. For $i = 1, \dots, r$, put $\tilde{\beta}_i = \omega_0\tilde{\alpha}_i'$. Since $\mathrm{res}_H(\omega_H^{-1}\tilde{\alpha}_i') \in \mathbb{R}_+^\times\alpha$, the above argument shows that $\tilde{\beta}_i \in R^-(M)$ and that $\omega_M\tilde{\beta}_i \in R^+(M)$. Put $\tilde{s}_{\alpha'} = s_{\tilde{\alpha}_1'} \cdots s_{\tilde{\alpha}_r'} = \omega_0^{-1}s_{\tilde{\beta}_1} \cdots s_{\tilde{\beta}_r}\omega_0$. Then we have $\tilde{s}_{\alpha'} \in \Omega(H)\Omega(M^H)$ and

$$\omega = \omega_M\omega_0\omega_H = \omega_M s_{\tilde{\beta}_1} \cdots s_{\tilde{\beta}_r} \cdot \omega_0 \cdot \tilde{s}_{\alpha'}\omega_H.$$

Since $l_M(s_{\tilde{\beta}_r}^{-1} \cdots s_{\tilde{\beta}_1}^{-1}\omega_M^{-1}) < l_M(\omega_M^{-1})$, we have

$$\omega \in \omega_0\Omega(H)\Omega(M^H)$$

by the inductive assumption. Now, we can write ω as $\omega = \omega_0\omega_H'$ with $\omega_H' \in \Omega(H)\Omega(M^H)$. Since $l_H(\omega) = l_H(\omega_0) = 0$, we have $\omega_H' \in \Omega(M^H)$. Thus, $\omega_0\omega_H'\omega_0^{-1} \in \Omega(M)$. Therefore, $l_M(\omega^{-1}) = l_M(\omega_0^{-1}) = 0$ shows that $\omega_0\omega_H'\omega_0^{-1} = 1$, since $\omega = \omega_0\omega_H'\omega_0^{-1} \cdot \omega_0$. Hence, $\omega = \omega_0$. □

We denote by $\mathbb{C}[\Omega(G)]$ the group ring of $\Omega(G)$. We define $\tilde{\xi}_M \in \mathbb{C}[\Omega(G)]$ by

$$\tilde{\xi}_M = \sum_{\omega \in \tilde{D}_M} \omega.$$

In the following, we prove that

$$\sum_{R(M) \in \mathcal{L}^G} (-1)^{r(M)}\tilde{\xi}_M = (-1)^{r(M^H)}\omega_-^{M^H}\omega_-^G,$$

where ω_-^G is the longest element of $\Omega(G)$ and $\omega_-^{M^H}$ is the longest element of $\Omega(M^H)$. We have $\omega \in \tilde{D}_M$ if and only if $R^+(M^H) \subset \omega(R^+(M)) \subset R^+(G)$. Therefore, we have

$$\sum_{R(M)\in\mathcal{L}^G} (-1)^{r(M)}\tilde{\xi}_M = \sum_{R(M)\in\mathcal{L}^G} (-1)^{r(M)} \Bigg(\sum_{\substack{\omega\in\Omega(G)\\ R^+(M^H)\subset\omega(R^+(M))\subset R^+(G)}} \omega \Bigg)$$
$$= \sum_{\omega\in\Omega(G)} \Bigg(\sum_{\substack{R(M)\in\mathcal{L}^G\\ \omega^{-1}(R^+(M^H))\subset R^+(M)\subset\omega^{-1}(R^+(G))}} (-1)^{r(M)} \Bigg)\omega.$$

For $\omega \in \Omega(G)$, put

$$J_\omega = \{\tilde{\alpha} \in S(G) \mid \omega\tilde{\alpha} > 0, \text{ and } n_{\tilde{\beta}\tilde{\alpha}} = 0 \text{ for all } \tilde{\beta} \in S(M^H)\},$$
$$I_\omega = \{\tilde{\alpha} \in S(G) \mid n_{\tilde{\beta}\tilde{\alpha}} \neq 0 \text{ for some } \tilde{\beta} \in S(M^H)\},$$

where $n_{\tilde{\beta}\tilde{\alpha}}$ is defined by $\omega^{-1}\tilde{\beta} = \sum_{\tilde{\alpha}\in S(G)} n_{\tilde{\beta}\tilde{\alpha}} \cdot \tilde{\alpha}$.

Lemma A.8. *Let $\omega \in \Omega(G)$, then we have*

$$I_\omega = \omega^{-1}(S(M^H))$$

if and only if we have

$$\omega^{-1}(R^+(M^H)) \subset R^+(M) \subset \omega^{-1}(R^+(G))$$

for some $R(M) \in \mathcal{L}^G$.

Proof. Suppose that there exists $R(M) \in \mathcal{L}^G$ such that

$$\omega^{-1}(R^+(M^H)) \subset R^+(M) \subset \omega^{-1}(R^+(G)).$$

Then by $\omega^{-1}(R^+(M^H)) \subset R^+(M)$, we have $I_\omega \subset S(M)$. Therefore, $R^+(M) \subset \omega^{-1}(R^+(G))$ shows that $I_\omega \subset \omega^{-1}(R^+(G))$. Thus, we have $\omega^{-1}(R^+(M^H)) > 0$ and $\omega(I_\omega) > 0$. Now, by considering the definition of I_ω, we can easily show that $\omega^{-1}(S(M^H)) \subset S(G)$. Conversely, assume that $I_\omega = \omega^{-1}(S(M^H))$. Let $R(M_{I_\omega})$ be the standard subroot system of $R(G)$ corresponding to I_ω. Then we have

$$\omega^{-1}(R^+(M^H)) \subset R^+(M_{I_\omega}) \subset \omega^{-1}(R^+(G)).$$ □

If $I_\omega = \omega^{-1}(S(M^H))$, then we have $I_\omega \cup J_\omega = \{\tilde{\alpha} \in S(G) \mid \omega\tilde{\alpha} > 0\}$.

Lemma A.9. *If $\omega \in \Omega(G)$ satisfies $I_\omega = \omega^{-1}(S(M^H))$, then we have*

$$\sum_{\substack{R(M)\in\mathcal{L}^G\\ \omega^{-1}(R^+(M^H))\subset R^+(M)\subset\omega^{-1}(R^+(G))}} (-1)^{r(M)} = \begin{cases} 0, & \text{if } J_\omega \neq \emptyset,\\ (-1)^{r(M^H)}, & \text{if } J_\omega = \emptyset. \end{cases}$$

Proof. Since I_ω is the set of simple roots of the smallest standard subroot system of $R(G)$ containing $\omega^{-1}(R^+(M^H))$, the left-hand side of the formula is equal to

$$\sum_{\substack{R(M)\in\mathcal{L}^G\\ I_\omega\subset S(M)\subset(I_\omega\cup J_\omega)}} (-1)^{r(M)} = \sum_{J\subset J_\omega} (-1)^{\sharp I_\omega}(-1)^{\sharp J}.$$ □

The above argument shows that

$$\sum_{R(M)\in\mathcal{L}^G} (-1)^{r(M)}\tilde{\xi}_M = (-1)^{r(M^H)} \sum \omega,$$

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where the sum in the right-hand side runs over $\omega \in \Omega(G)$ satisfying $I_\omega = \omega^{-1}(S(M^H))$ and $J_\omega = \emptyset$. We have $I_\omega = \omega^{-1}(S(M^H))$ and $J_\omega = \emptyset$ if and only if $\omega(I_\omega) = S(M^H)$ and $\omega(S(G) - I_\omega) < 0$. Now, it is easy to see that $\omega = \omega_-^{M^H}\omega_-^G$ is the only element that satisfies this condition. Thus, we have proved the following lemma.

Lemma A.10. *We have*

$$\sum_{R(M)\in\mathcal{L}^G} (-1)^{r(M)}\tilde{\xi}_M = (-1)^{r(M^H)}\omega_-^{M^H}\omega_-^G.$$

For

$$\xi = \sum_{\omega\in\Omega(G)} a_\omega \cdot \omega \in \mathbb{C}[\Omega(G)],$$

we define $[\xi]_H \in \mathbb{C}[\Omega(G)_H]$ by

$$[\xi]_H = \sum_{\omega\in\Omega(G)_H} a_\omega \cdot \omega.$$

For $R(L) \in \mathcal{L}^H$, set $\tilde{R}(L) = \{\tilde{\alpha} \in R(G) \mid \mathrm{res}_H(\tilde{\alpha}) \in R(L)\}$ and $\tilde{R}^+(L) = \tilde{R}(L) \cap R^+(G;\mathfrak{a}^H)$. We put

$$D_L = \{\omega \in \Omega(G) \mid \omega(\tilde{R}^+(L)) > 0\},$$
$$\xi_L = \sum_{\omega\in D_L} \omega.$$

For $\omega \in \Omega(G)_H$, put $J_\omega^H = \{\alpha \in S(H) \mid \omega\alpha > 0\}$. Then,

$$\sum_{R(L)\in\mathcal{L}^H} (-1)^{r(L)}[\xi_L]_H = \sum_{\omega\in\Omega(G)_H} \sum_{\substack{R(L)\in\mathcal{L}^H \\ \omega(S(L))>0}} (-1)^{r(L)} \cdot \omega = \sum_{\omega\in\Omega(G)_H} \sum_{S(L)\subset J_\omega^H} (-1)^{r(L)} \cdot \omega = \sum_{\substack{\omega\in\Omega(G)_H \\ J_\omega^H=\emptyset}} \omega.$$

Thus we have

$$\sum_{R(L)\in\mathcal{L}^H} (-1)^{r(L)}[\xi_L]_H = \sum_{\substack{\omega\in\Omega(G)_H \\ \omega(R^+(H))<0}} \omega.$$

Lemma A.11. *We have*

$$[\omega_-^{M^H}\omega_-^G\xi_H]_H = \sum_{R(L)\in\mathcal{L}^H} (-1)^{r(L)}[\xi_L]_H.$$

Proof. We have

$$[\omega_-^G\xi_H]_H = \sum_{\substack{\omega\in\Omega(G)_H \\ \omega(R^+(H))<0}} \omega,$$

since $\omega \in \Omega(G)_H$ satisfies $(\omega_-^G)^{-1}\omega \in D_H$ if and only if $\omega(R^+(H)) < 0$. On the other hand, the action of $\omega_-^{M^H} \in \Omega(G)_H$ on $\omega(R(H))$ is trivial. Hence we have

$$[\omega_-^{M^H}\omega_-^G\xi_H]_H = \omega_-^{M^H}[\omega_-^G\xi_H]_H = \omega_-^{M^H} \cdot \sum_{\substack{\omega\in\Omega(G)_H \\ \omega(R^+(H))<0}} \omega = \sum_{\substack{\omega\in\Omega(G)_H \\ \omega(R^+(H))<0}} \omega. \qquad \square$$

Lemma A.12 (Lemma 2.3). *Let $\omega \in D_{M,H}$, then*

$$R(H) \cap \mathrm{res}_H(\omega^{-1}(R(M))) \in \mathcal{L}^H.$$

Proof. For $\alpha_i \in S(H)$, fix $\tilde{\alpha}_i \in R^+(G)$ such that $\mathrm{res}_H(\tilde{\alpha}_i) = \alpha_i$. Let $\alpha \in R^+(H)$. Choose $\tilde{\alpha} \in R^+(G)$ such that $\mathrm{res}_H(\tilde{\alpha}) = \alpha$. Write $\alpha = \sum_{\alpha_i\in S(H)} n_i\cdot\alpha_i$. Then $\mathrm{res}_{\omega(\mathfrak{a}^H)}(\omega\tilde{\alpha}) = \sum_{\alpha_i\in S(H)} n_i\cdot\mathrm{res}_{\omega(\mathfrak{a}^H)}(\omega\tilde{\alpha}_i)$.

Since $\omega \in D_{M,H}$, we have $\omega\alpha_i > 0$. Thus, $\omega\tilde{\alpha}_i = \sum_{\tilde{\beta}_j \in S(G)} m_{ij} \cdot \tilde{\beta}_j$, where $m_{ij} \geqslant 0$. Hence, by $\omega(\mathfrak{a}^H) \supset \mathfrak{a}_M$, we have

$$\mathrm{res}_{\mathfrak{a}_M}(\omega\tilde{\alpha}) = \sum_i n_i \cdot \mathrm{res}_{\mathfrak{a}_M}(\omega\tilde{\alpha}_i) = \sum_{i,j} n_i m_{ij} \cdot \mathrm{res}_{\mathfrak{a}_M}(\tilde{\beta}_j).$$

Assume $\tilde{\alpha} \in \omega^{-1}(R(M))$, then we have $n_i m_{ij} = 0$ for any $\tilde{\beta}_j \in S(G) - S(M)$ and $\alpha_i \in S(M)$. Therefore, for each $\tilde{\alpha}_i$ satisfying $n_i \neq 0$, we have $\tilde{\alpha}_i \in \tilde{R}(H) \cap \omega^{-1}(R(M))$. Hence, $R(H) \cap \mathrm{res}_H(\omega^{-1}(R(M)))$ is standard. $\square$

In the following, we prove that

$$\sum_{R(M) \in \mathcal{L}^G} (-1)^{r(M)} a_{M,H,L} = (-1)^{r(M^H)}(-1)^{r(L)}.$$

We prove this by imitating the proof in [Car93, § 2.7]. Let $\Omega(L) \subset \Omega(H)$ be the Weyl group of $R(L)$, which we regard as a subgroup of $\Omega(G)$. We put $\mathfrak{a}_L = \{a \in \mathfrak{a}^H \mid \alpha(a) = 0 \text{ for all } \alpha \in S(L)\}$. If $\omega \in D_{M,H,L}$, then we have

$$\Omega(H) \cap \omega^{-1}\Omega(M)\omega = \Omega(L).$$

For $\alpha \in R^+(H)$, put $\mathrm{ext}(\alpha) = \{\tilde{\alpha} \in R^+(G) \mid \mathrm{res}_H(\tilde{\alpha}) = \alpha\}$. Let $\omega \in \Omega(G)$. In the following lemma, we say that $l_L(\omega)$ *can be defined* if either $\omega(\mathrm{ext}(\alpha)) \subset R^+(G)$ or $\omega(\mathrm{ext}(\alpha)) \subset R^-(G)$ holds for each $\alpha \in R^+(L)$. We write $\omega\alpha > 0$ if $\omega(\mathrm{ext}(\alpha)) \subset R^+(G)$ and $\omega\alpha < 0$ if $\omega(\mathrm{ext}(\alpha)) \subset R^-(G)$. If $l_L(\omega)$ can be defined, then we put

$$l_L(\omega) = \sharp\{\alpha \in R^+(L) \mid \omega\alpha < 0\}.$$

Since $l_L(\omega^{-1})$ can be defined for any $\omega \in \Omega(H)$, we have

$$\Omega(H) \cap D_L^{-1} = \{\omega \in \Omega(H) \mid l_L(\omega^{-1}) = 0\}.$$

Lemma A.13. *For $\omega \in D_{M,H,L}$, we have*

$$\Omega(M)\omega\Omega(H) = \{\omega_M\omega\omega_H \mid \omega_M \in \Omega(M),\ \omega_H \in \Omega(H) \cap D_L^{-1}\}.$$

Moreover, if $l_H(\omega_M\omega\omega_H)$ can be defined, then we have

$$l_H(\omega_M\omega\omega_H) \geqslant l_H(\omega_H).$$

Proof. Let $\omega_M \in \Omega(M)$ and $\omega_H \in \Omega(H)$. By induction on $l_L(\omega_H^{-1})$, we prove that

$$\omega_M\omega\omega_H \in \Omega(M)\omega(\Omega(H) \cap D_L^{-1}).$$

If $l_L(\omega_H^{-1}) = 0$, then nothing remains to be proved. Suppose that $l_L(\omega_H^{-1}) > 0$. Then there exists $\alpha \in S(L)$ such that $\omega_H^{-1}\alpha < 0$. Choose $\tilde{\alpha} \in \omega^{-1}(R(M))$ such that $\mathrm{res}_H(\tilde{\alpha}) = \alpha$. Since α is $R(G)$-symmetric, we can take $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_r$ as in Lemma 1. Then for $i = 1, \ldots, r$, we have $\mathrm{res}_{\omega^{-1}(\mathfrak{a}_M)}(\tilde{\alpha}_i) \in \mathbb{R}_+^\times \mathrm{res}_{\omega^{-1}(\mathfrak{a}_M)}(\tilde{\alpha})$, since $\mathfrak{a}^H \supset \omega^{-1}(\mathfrak{a}_M)$. This implies that $\tilde{\alpha}_i(\omega^{-1}(\mathfrak{a}_M)) = \tilde{\alpha}(\omega^{-1}(\mathfrak{a}_M)) = 0$. Thus $\tilde{\alpha}_i \in \omega^{-1}(R(M))$. Put $\tilde{s}_\alpha = s_{\tilde{\alpha}_1} \cdots s_{\tilde{\alpha}_r}$. Then we have $\omega \cdot \tilde{s}_\alpha = s_{\omega\tilde{\alpha}_1} \cdots s_{\omega\tilde{\alpha}_r} \cdot \omega$. Let $\omega' = \tilde{s}_\alpha s_\alpha^{-1} \in \Omega(M^H)$, then

$$\omega_M\omega\omega_H = \omega_M\omega\tilde{s}_\alpha\tilde{s}_\alpha\omega_H = \omega_M s_{\omega\tilde{\alpha}_1} \cdots s_{\omega\tilde{\alpha}_r} \cdot \omega\omega'\omega^{-1} \cdot \omega \cdot s_\alpha\omega_H.$$

Since $l_L(\omega_H^{-1}s_\alpha^{-1}) = l_L(\omega_H^{-1}) - 1$ and since $s_{\omega\tilde{\alpha}_1} \cdots s_{\omega\tilde{\alpha}_r}$ and $\omega\omega'\omega^{-1}$ are contained in $\Omega(M)$, this shows that $\omega_M\omega\omega_H \in \Omega(M)\omega(\Omega(H) \cap D_L^{-1})$ by the inductive assumption.

Next, suppose that $l_H(\omega_M\omega\omega_H)$ can be defined. Since $\omega_H \in \Omega(H) \cap D_L^{-1}$, we have

$$l_H(\omega_H) = \{\alpha \in R^+(H) - \omega_H^{-1}(R^+(L)) \mid \omega_H\alpha < 0\}$$

and

$$\omega_H(R^+(H) \cap \omega_H^{-1}\,\mathrm{res}_H(\omega^{-1}(R(M)))) = R^+(L).$$

On the other hand, let $\alpha \in R^+(H) - \mathrm{res}_H(\omega^{-1}(R(M)))$. Then we have $\omega\alpha > 0$. Since $\omega\alpha \notin \mathrm{res}_{\omega(\mathfrak{a}^H)}(R(M))$, we have $\omega\tilde{\alpha} \notin R(M)$ for any $\tilde{\alpha} \in R(G)$ satisfying $\mathrm{res}_H(\tilde{\alpha}) = \alpha$. This implies that $\omega_M\omega\alpha > 0$, since $\omega_M \in \Omega(M)$. Therefore, we have shown that if $\alpha \in \omega_H(R^+(H) - \omega_H^{-1}(R^+(L))) = \omega_H(R^+(H) - \omega_H^{-1}\,\mathrm{res}_H(\omega^{-1}(R(M))))$ is positive, then $\omega_M\omega\alpha$ is positive and if it is negative, then $\omega_M\omega\alpha$ is negative. Thus we have proved that

$$l_H(\omega_M\omega\omega_H) = l_H(\omega_H) + \sharp\{\alpha \in R^+(L) \mid \omega_M\omega\alpha < 0\}. \qquad \square$$

Corollary A.14. *If $\omega \in D_{M,H}$, then*

$$D_H \cap \Omega(M)\omega\Omega(H) \subset \Omega(M)\omega.$$

Proof. Let $\omega \in D_{M,H,L}$ and $\omega_M\omega\omega_H \in D_H \cap \Omega(M)\omega\Omega(H)$, where $\omega_M \in \Omega(M)$ and $\omega_H \in \Omega(H) \cap D_L^{-1}$. Since $\omega_M\omega\omega_H \in D_H$, $l_H(\omega_M\omega\omega_H)$ can be defined. Moreover, we have

$$0 = l_H(\omega_M\omega\omega_H) \geqslant l_H(\omega_H).$$

Thus, $l_H(\omega_H) = 0$. Since $\omega_H \in \Omega(H)$, this shows that $\omega_H = 1$. $\square$

Let $x \in \Omega(G)_H$ and $\omega \in D_{M,H}$. Since $x\omega^{-1} \in \Omega(G)_{M,H}^{-1}$, we can write $x\omega^{-1}$ as $x\omega^{-1} = d_M\omega_M$ with $d_M \in \tilde{D}_M$ and $\omega_M \in \Omega(M)$. Then d_M and ω_M are uniquely determined by x and ω.

Lemma A.15. *We have*

$$\tilde{D}_M^{-1}x \cap D_H \cap \Omega(M)\omega\Omega(H) = \begin{cases} \{d_M^{-1}x\}, & \text{if } d_M^{-1}x \in D_H, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Proof. Suppose that $d \in \tilde{D}_M$ satisfies $d^{-1}x \in D_H \cap \Omega(M)\omega\Omega(H)$. Then Corollary 14 asserts that there exists $\omega'_M \in \Omega(M)$ such that $d^{-1}x = \omega'_M\omega$. This implies that $x\omega^{-1} = d\omega'_M$ and shows that $d = d_M$. Therefore, $\tilde{D}_M^{-1}x \cap D_H \cap \Omega(M)\omega\Omega(H)$ is equal to $\{d_M^{-1}x\}$ or $\emptyset$. Since $d_M^{-1}x = \omega_M\omega$, the element $d_M^{-1}x$ is contained in $\tilde{D}_M^{-1}x \cap D_H \cap \Omega(M)\omega\Omega(H)$ if and only if $d_M^{-1}x \in D_H$. $\square$

Put

$$R(L) = R(H) \cap \mathrm{res}_H(\omega^{-1}(R(M))).$$

Then $\omega \in D_{M,H,L}$.

Lemma A.16. *We have $d_M^{-1}x \in D_H$ if and only if $x \in D_L$.*

Proof. Since $\mathfrak{a}^H \supset \omega^{-1}(\mathfrak{a}_M)$, we have $\tilde{R}^+(L) = \tilde{R}^+(H) \cap \omega^{-1}(R(M))$. Because $d_M^{-1}x\omega^{-1} = \omega_M$ preserves $R(M)$, this implies that

$$d_M^{-1}x(\tilde{R}^+(L)) = d_M^{-1}x(\tilde{R}^+(H)) \cap R(M).$$

Now, suppose $d_M^{-1}x \in D_H$, then we have $d_M^{-1}x(\tilde{R}^+(L)) \subset R^+(M)$. Therefore, $x(\tilde{R}^+(L)) \subset R^+(G)$. Conversely, suppose that $x(\tilde{R}^+(L)) \subset R^+(G)$. Then we have $d_M^{-1}x(\tilde{R}^+(L)) \subset R^+(G)$. Now, we consider $d_M^{-1}x(\tilde{R}^+(H) - \tilde{R}^+(L))$. Since $d_M^{-1}x = \omega_M\omega$, this is equal to $\omega_M\omega(\tilde{R}^+(H) - \tilde{R}^+(L))$. Because $\omega \in D_{M,H,L}$, we have $\omega(\tilde{R}^+(H) - \tilde{R}^+(L)) \subset R^+(G) - R^+(M)$. Since ω_M preserves the positivities of the roots in $R^+(G) - R^+(M)$, this shows that $d_M^{-1}x(\tilde{R}^+(H) - \tilde{R}^+(L)) \subset R^+(G)$. Thus, $d_M^{-1}x \in D_H$. $\square$

Lemma A.17. *Let $x \in \Omega(G)_H$, then we have*

$$\sharp(\tilde{D}_M^{-1}x \cap D_H) = \sum a_{M,H,L},$$

where the sum in the right-hand side runs over $R(L) \in \mathcal{L}^H$ satisfying $x \in D_L$.

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Proof. If $d \in \tilde{D}_M$, then $d^{-1}x \in \Omega(G)_{M,H}$. Therefore, Proposition 2.2 asserts that there exists a unique $\omega \in D_{M,H}$ such that $d^{-1}x \in \Omega(M)\omega\Omega(H)$. Hence, by Lemma 15, we have

$$\sharp(\tilde{D}_M^{-1}x \cap D_H) = \sharp\{\omega \in D_{M,H} \mid \tilde{D}_M^{-1}x \cap D_H \cap \Omega(M)\omega\Omega(H) \neq \emptyset\}.$$

Since $D_{M,H} = \bigcup_{R(L)\in\mathcal{L}^H} D_{M,H,L}$, the required equation follows immediately from Lemma 16. □

Proposition A.18. *We have*

$$[\tilde{\xi}_M \xi_H]_H = \sum_{R(L)\in\mathcal{L}^H} a_{M,H,L}[\xi_L]_H.$$

Proof. Let $x \in \Omega(G)_H$, then the coefficient of x in $[\tilde{\xi}_M\xi_H]_H$ is equal to the number of the pair $(d_M, \omega_H) \in \tilde{D}_M \times D_H$ satisfying $x = d_M\omega_H$. Therefore, the coefficient of x in $[\tilde{\xi}_M\xi_H]_H$ is $\sharp(\tilde{D}_M^{-1}x \cap D_H)$, and Lemma 17 asserts that this is equal to the coefficient of x in the right-hand side of the formula. □

Theorem A.19 (Theorem 2.4). *We have*

$$\sum_{R(M)\in\mathcal{L}^G} (-1)^{r(M)} a_{M,H,L} = (-1)^{r(M^H)} \cdot (-1)^{r(L)}.$$

Proof. By Lemma 10, we have

$$\omega_-^{M^H}\omega_-^G = (-1)^{r(M^H)} \sum_{R(M)\in\mathcal{L}^G} (-1)^{r(M)}\tilde{\xi}_M.$$

Therefore, by using Lemma 11 and Proposition 18, we have

$$\begin{aligned}\sum_{R(L)\in\mathcal{L}^H} (-1)^{r(L)}[\xi_L]_H &= [\omega_-^{M^H}\omega_-^G\xi_H]_H \\ &= (-1)^{r(M^H)} \sum_{R(M)\in\mathcal{L}^G} (-1)^{r(M)}[\tilde{\xi}_M\xi_H]_H \\ &= (-1)^{r(M^H)} \sum_{R(M)\in\mathcal{L}^G} (-1)^{r(M)} \sum_{R(L)\in\mathcal{L}^H} a_{M,H,L}[\xi_L]_H.\end{aligned}$$

It is easy to see that $[\xi_L]_H$ are linearly independent. Hence, by comparing the coefficient of $[\xi_L]_H$, we get the required formula. □

For $R(M), R(M') \in \mathcal{L}^G$, we put

$$D_{M,M'} = \{\omega \in \Omega(G) \mid l_M(\omega^{-1}) = l_{M'}(\omega) = 0\}.$$

Then $D_{M,M'}$ is a set of representatives for $\Omega(M)\backslash\Omega(G)/\Omega(M')$.

Lemma A.20 (Lemma 2.5). *Let $R(M), R(M_0) \in \mathcal{L}^G$. Assume that $R(M_0)$ has no other associate standard subroot system than $R(M_0)$ itself. If $R(M)$ satisfies $\omega(\mathfrak{a}_M) \subset \mathfrak{a}_{M_0}$ for some $\omega \in \Omega(G)$, then we have*

$$R(M) \supset R(M_0).$$

Proof. Choose a chamber C of $\mathfrak{a}_{M_0}$ such that the closure $\overline{C}$ of C contains a non-empty open subset of $\omega(\mathfrak{a}_M)$. By Lemma 2 we can take $\omega' \in \Omega(G)_{\mathfrak{a}_{M_0}}$ such that $\omega'(C) = C^+_{M_0}$. Then $\omega'\omega(\mathfrak{a}_M)$ is a standard subspace and there exists $S(M') \subset S(G)$ such that $\omega'\omega(R(M)) = R(M')$. Take $\omega_0 \in D_{M,M'} \cap \Omega(M)\omega^{-1}\omega'^{-1}\Omega(M')$. Since $\omega_0(R(M')) = R(M)$, we have $\omega_0(S(M')) = S(M)$. Therefore,

$$\omega_0(S(M_0)) \subset \omega_0(S(M')) = S(M) \subset S(G).$$

This implies that $\omega_0(R(M_0))$ is an associate standard subroot system of $R(M_0)$. Hence $R(M_0) = \omega_0(R(M_0)) \subset R(M)$. □

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