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PERFECT MCLAIN GROUPS ARE SUPERPERFECT

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It is shown that if a McLain group is perfect, then it is superperfect. The proof involves demonstrating that any dense linearly ordered set has the apparently stronger property of being superdense.

1. Introduction

The study of McLain groups $M(S, F)$ offers an attractive interplay between group theory and combinatorial set theory. This arises from the choice of a linearly ordered set S in the definition. Recall (from, for example, [4], (6.2)) that this involves considering the vector space V over the field F whose basis elements v_x are indexed by elements of S. $M(S, F)$ is then the group of linear transformations of V into itself generated by those transformations of the form $1 + ae_{xy}$ ($a \in F$, $x < y$ in S), where e_{xy} sends v_x to v_y and annihilates the rest of the basis. A simple example of the interaction referred to is the well-known result (proved, for convenience, in §3 below).

PROPOSITION 1.1. *$M(S, F)$ is perfect if and only if S is dense.*

Here we follow standard terminology, calling a group *perfect* if generated by commutators of its elements, and a set S *dense* if whenever $x < z$ in S then there exists y in (x, z) (that is, $x < y < z$).

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Our aim is to provide a significant strengthening of this result, as follows. (A perfect group M is said to be *superperfect* if every homomorphism onto M having central kernel is split.)

THEOREM 1.2. *If $M(S, F)$ is perfect, then it is superperfect.*

The homological interpretation of this result is that (for trivial integer coefficients) $H_1(M(S, F)) = 0$ implies $H_2(M(S, F)) = 0$. This naturally leads one to ask whether it also implies that $H_i(M(S, F)) = 0$ for all $i \geq 1$. In other words:

Are perfect McLain groups acyclic?

Although we do not answer the question, this note does suggest that its answer may well lie in a deeper understanding of the nature of dense linearly ordered sets. For the proof of Theorem 1.2 given below consists first in showing that dense linearly ordered sets have an apparently stronger property. We say that S is *superdense* if it admits a *superdense filtration* $S_0 \subset S_1 \subset S_2 \ldots$ (so that $S = \cup S_i$), that is, one for which, whenever $x < z$ in any S_i, there exists y in $S_{i+1} \backslash S_i$ with $x < y < z$. Section 2 below is devoted to proving:

PROPOSITION 1.3. *If S is dense, then it is superdense.*

This extra structure is employed in Section 3 to give a group-theoretic proof of the following:

PROPOSITION 1.4. *If S is superdense, then $M(S, F)$ is superperfect.*

Our theorem is of course immediate from the three propositions. It should be remarked that the theorem can also be established without invoking the notion of superdensity [2], albeit at the expense of rather more group theory. The proof of Proposition 1.4 can be seen as group-theoretically simpler, in much the same way as the proof of superperfectness of Steinberg groups in [1] simplifies that in [3] by exploiting the canonical filtration of the index set (the natural numbers), and so avoiding the need to establish the equivalence of all possible candidates for splittings. Also, of course, Section 2 is likely to be of interest in its own right.

2. Set theory

In this section we prove Proposition 1.3. Suppose that S is a dense linearly ordered set. We shall say a sequence $\{C_i\}$ of non-empty subsets $C_0, C_1, \ldots$ of S is a *d-sequence* for S if

(i) $C_i \cap C_j = \emptyset$ whenever $i \neq j$,

(ii) C_0 is *topologically dense* in $\cup C_i$ (that is, intersects each interval (x, y) of $\cup C_i$), and

(iii) given $x < y$ in any C_i, then each C_j $(j > i)$ intersects (x, y).

Consider the collection C of d-sequences. It is non-void, because the density of S ensures that, up to order-isomorphism (and all countable linearly ordered sets without endpoints are, after Cantor, order-isomorphic), a copy of the rationals $\mathbb{Q}$ may be embedded in S. Then a d-sequence $\{Q_i\}$ is defined inductively by setting $Q_0 = \mathbb{Z}[2^{-1}]$, and, for $j \geq 1$, $Q_j = \mathbb{Z}\left[2^{-1}, 3^{-1}, 5^{-1}, \ldots, p_{j+1}^{-1}\right] \backslash Q_{j-1}$, where p_k denotes the kth prime number. (Thus $\mathbb{Z}\left[2^{-1}, 3^{-1}, \ldots, p_k^{-1}\right]$ comprises those rationals that may be expressed as finite sums of the form $q = \sum_{i \leq k} \left(\sum_n a_{n,i} p_i^{-1}\right)$ with each $a_{n,i}$ an integer. So $\cup Q_i = \mathbb{Q}$.)

Now partially order C by defining $\{C_i\} \leq \{C_i'\}$ if $C_i \leq C_i'$ for all $i = 0, 1, 2, \ldots$. Given any chain $\{C_i^\delta\}_{\delta \in \Delta}$, define $\left\{C_i^\Delta\right\}$ by $C_i^\Delta = \bigcup_{\delta \in \Delta} C_i^\delta$. Then the C_i^Δ's are pairwise disjoint, for otherwise $x \in C_i^\Delta \cap C_j^\Delta$ implies that $x \in C_i^\delta \cap C_j^\delta$ for some $\delta \in \Delta$. Similarly we may verify that properties (ii) and (iii) hold for $\left\{C_i^\Delta\right\}$. Hence each chain has an upper bound.

An application of Zorn's lemma now yields a maximal d-sequence $\{M_i\}$. We claim that $\cup M_i = S$. To prove this, we distinguish two cases.

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CASE 1. Some interval (a, b) of S is disjoint from $M = \cup M_i$.

In this event, as (a, b) is dense, we again embed $\mathbb{Q}$ in (a, b) . As before, let $\{Q_i\}$ be a d-sequence for $\mathbb{Q}$, and consider $\{M_i \cup Q_i\}$. This sequence clearly satisfies condition (i), leaving (ii) and (iii) to be checked in order to demonstrate that it is a d-sequence. First, $M_0 \cup Q_0$ is dense in $\cup(M_i \cup Q_i)$, for suppose $w \in M$ and $z \in \cup Q_i$ with $w < z$. Then there exists $x \in \cup Q_i$ such that $x < z$. However, since $(a, b) \cap M = \emptyset$, this implies that $w < a < x < z$. From the fact that $\{Q_i\}$ is a d-sequence, there exists $y \in (x, z) \cap Q_0$, giving $w < a < x < y < z$. This clinches (ii). Similar reasoning accounts for (iii). Thus $\{M_i \cup Q_i\}$ is indeed a d-sequence, contradicting the maximality of $\{M_i\}$. Hence this case cannot occur.

CASE 2. Each interval of S intersects M .

Here M is topologically dense in S . As M_0 is topologically dense in M , it must therefore also be topologically dense in S . Take any element $a \in S \setminus \bigcup_{i\geq 1} M_i$. Define a new sequence $\{M_i'\}$ via $M_0' = M_0 \cup \{a\}$ and $M_i' = M_i$ for $i \geq 1$. Again we assert that this is a d-sequence. Only (iii) needs to be checked. Therefore suppose $x \in M_0$ with $a < x$, say. Because M_0 is topologically dense in S , we may find $y \in (x, a) \cap M_0$. From the fact that $\{M_i\}$ is a d-sequence, for any $j \geq 1$, $M_j' = M_j$ intersects (x, y) and thereby (x, a) . So $\{M_i'\}$ is a d-sequence too. Maximality of $\{M_i\}$ now ensures that each $M_i' = M_i$; in particular $a \in M_0$. So $M = S$, as earlier claimed.

Now define (for $i = 0, 1, \ldots$) $S_i = \bigcup_{j=0}^{i} M_j$. Then $S_0 \subset S_1 \subset \ldots$ is evidently a superdense filtration of S .

The proof above may be adapted to show that if S is dense, then $S = \cup T_i$ where the T_i's are topologically dense, pairwise disjoint sub-orderings of S . (In particular, for $S = Q$, our given d-sequence has this property.) On the other hand, dense S may have a superdense filtration $S_0 \subset S_1 \subset \ldots$ with *no* S_i topologically dense in S. (For example, given any non-trivial linearly ordered set L, consider $S = \bigoplus_{\mathbb{N}} L$ (summed finitely and ordered lexicographically), and define S_i to comprise those elements of S that are everywhere zero after the $(i+1)$st coordinate. In particular, this applies to $S = \mathbb{Q}$ by choice of $L = \mathbb{Q}$ again.) It is an open question whether such a filtration is possible for arbitrary dense S.

3. Group theory

We first establish Proposition 1.1. Both for this and subsequent arguments it is helpful to have a presentation for $M(S, F)$. We already have $\{1+ae_{xy} \mid a \in F, x < y \text{ in } S\}$ as a generating set. Moreover the composition laws $e_{xy}e_{yz} = e_{xz}$ and $e_{xy}e_{zt} = 0$ if $y \neq z$ readily lead to relations (for arbitrary $a, b \in F$ and $x, y, z, t \in S$)

$$(3.1) \qquad (1+ae_{xy})(1+be_{xy}) = 1 + (a+b)e_{xy} ,$$

$$(3.2) \qquad [1+ae_{xy}, 1+be_{yz}] = 1 + abe_{xz} ,$$

and

$$(3.3) \qquad [1+ae_{xy}, 1+be_{zt}] = 1 \text{ if } x \neq t \text{ and } y \neq z .$$

(Here $[g, h]$ is the commutator $ghg^{-1}h^{-1}$.)

To decide whether or not any other relation is a consequence of these, consider any finite product of the generators which is not made trivial by (3.1)-(3.3). The given relations enable such a product to be rewritten in the form

$$(1+ae_{x_0y_0}) \prod_{x>x_0} (1+b_xe_{xy_0}) \prod_{\substack{x \\ y<y_0}} (1+c_{xy}e_{xy})$$

for suitable $x_0 < y_0$ in S and non-zero a in F. However, as a product of transformations this expression sends the basis element v_{x_0} to

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some linear combination which includes v_{y_0} with coefficient a, and is therefore non-trivial. This means

LEMMA 3.4. *$M(S, F)$ is generated by all $1 + ae_{xy}$ with $a \in F$, $x < y$ in S, subject only to the relations* (3.1)-(3.3).

It follows immediately from (3.2) that $M = M(S, F)$ is perfect whenever S is dense. On the other hand, if S is not dense, then let (x_0, y_0) be empty in S. Map M to the additive group of F by sending $1 + ae_{x_0y_0}$ to a, and all other generators $1 + ae_{xy}$ to $0 \in F$. In view of (3.1)-(3.3) this defines a non-trivial homomorphism from M to an abelian group; thus no element $1 + ae_{x_0y_0}$ can be a product of commutators and M cannot be perfect. This clinches Proposition 1.1.

For consideration of Proposition 1.4 we first perform a simplification, based on the fact that the homology of a group is the direct limit of the homology of its finitely generated subgroups. Note that, for a fixed superdense filtration $S_0 \subset S_1 \subset S_2 \subset \dots$ on S, any finite subset of S is contained in some subset S' of S such that each $S'_i = S_i \cap S$ is finite and $S'_0 \subset S'_1 \subset S'_2 \subset \dots$ is a superdense filtration on S'. It therefore suffices to show that each such $M(S', F)$ is superperfect; accordingly, since each such S' is dense in itself, we may as well assume that our original S and its superdense filtration have this "locally-finite" form.

In order to prove Proposition 1.4 we start with an arbitrary epimorphism $\phi : G \to M$ whose kernel K lies in the centre $Z(G)$ of G. We produce a splitting $\sigma : M \to G$ (that is, $(1+ae_{xy})\sigma\phi = 1 + ae_{xy}$ for all a, x, y). To this end we use the superdense filtration $S_0 \subset S_1 \subset S_2 \quad \dots$ on S, to filter M as $\bigcup_{i\geq 0} M_i$ with $M_0 \leq M_1 \leq \dots$ where M_i is the subgroup generated by all elements of the form $1 + ae_{xy}$, $a \in F$, $x, y \in S_i$. For each $i \geq 1$ we shall define $\sigma_i : M_{i-1} \to G$ splitting ϕ over M_{i-1}. These splittings are compatible (that is, combine as a splitting over the inductive limit M of the subgroups M_i) because of the following lemma applied to the fact (obvious from (3.2)) that $M_{i-1} \leq [M_i, M_i]$.

LEMMA 3.5. *Let* $\iota : [H, H] \to H$ *and* $\kappa : J \to J/Z(J)$ *denote the obvious inclusion and projection homomorphisms. If* $\alpha, \beta : H \to J$ *have* $\alpha\kappa = \beta\kappa$, *then* $\iota\alpha = \iota\beta$.

This lemma is a consequence of the observation that, because α and β agree modulo $Z(J)$, then for any $h, h' \in H$,

$$[(h)\alpha, (h')\alpha] = [(h)\beta Z(J), (h')\beta Z(J)] = [(h)\beta, (h')\beta] .$$

Our task, once σ_i has been defined, is to show that it is a homomorphism, in other words, respects the relations (3.1)-(3.3). For the definition first, for each x in S_{i-1} choose an element y of $S_i \backslash S_{i-1}$ such that $x < y < z$ whenever $x < z$ in S_{i-1} (possible, since the superdense filtration is also locally finite). Then, with g_{wt}^a denoting the K-coset $\left(1+ae_{wt}\right)\phi^{-1}$ in G , let

$$\left(1+ae_{xz}\right)\sigma_i = \left[g_{xy}^1, g_{yz}^a\right] .$$

By virtue of the centrality of K , the right-hand expression determines a single element of G . In particular, because $g_{yz}^0 = K$, so

$$(1)\sigma = \left[g_{xy}^1, g_{yz}^0\right] = 1 .$$

Next, recall the following facts from [1, p. 68].

In any group G , *for* $u, v, w \in G$,

$[[u, v], w]$

(a) $\quad = [u, v][w, v][v, wu]$

(b) $\quad = \left[[u, v], [w, v]\right]\left[[v, w], u\right]$ *if* $[u, w] \in Z(G)$

(c) $\quad = 1$ *if also* $[v, w] \in Z(G)$.

Here (a) is obtained by straightforward multiplication, and (b) by substitution for $[v, wu] = [v, uw]$ in (a). Let us use (c) to prove that σ respects (3.3), by noting that, for $s, t, u, v \in S_i$,

(d) $$\left[g^a_{st},\ g^b_{uv}\right] = 1 \quad \text{if} \quad s \neq v \quad \text{and} \quad t \neq u\ .$$

To see this, take $w \in S_{i+1} \backslash S_i$ with $s < w < t$ and observe that $g^a_{st} = \left[g^1_{sw},\ g^a_{wt}\right]K$. Next, by using (a) (after rearrangement) and (d) in turn, we have that

$$\begin{aligned}
\left(1+ae_{xz}\right)\sigma_i\left(1+be_{xz}\right)\sigma_i &= \left[g^1_{xy},\ g^a_{yz}\ g^b_{yz}\right]\left[g^a_{yz},\ \left[g^1_{xy},\ g^b_{yz}\right]^{-1}\right] \\
&= \left[g^1_{xy},\ g^{a+b}_{yz}\right]\left[g^a_{yz},\ g^{-b}_{xz}\right] \\
\text{(e)} \qquad &= \left(1+(a+b)e_{xz}\right)\sigma_i\ .
\end{aligned}$$

Thus σ_i also respect relations of type (3.1). Lastly, for (3.2) we apply (b), then (d), then (e), as follows:

$$\begin{aligned}
\left[\left(1+ae_{xz}\right)\sigma_i,\ \left(1+be_{zt}\right)\sigma_i\right] &= \left[\left[g^1_{xy},\ g^a_{yz}\right],\ g^b_{zt}\right] \\
&= \left[\left[g^1_{xy},\ g^a_{yz}\right],\ \left[g^b_{zt},\ g^a_{yz}\right]\right]\left[\left[g^b_{zt},\ g^a_{yz}\right],\ g^1_{xy}\right] \\
&= \left[g^a_{xz},\ g^{-ab}_{yt}\right]\left[g^1_{xy},\ g^{-ab}_{yt}\right]^{-1} \\
&= \left(\left(1-abe_{xt}\right)\sigma_i\right)^{-1} \\
&= \left(1+abe_{xt}\right)\sigma_i\ .
\end{aligned}$$

Thus σ is a homomorphism after all, completing the proof of Proposition 1.4 and hence Theorem 1.2.

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