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LARGE SETS NOT CONTAINING IMAGES OF A GIVEN SEQUENCE

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ABSTRACT. In the first part we construct a subset H of positive measure in the unit interval and a zero-sequence $\{a_n\}$ so that H contains no homothetic copy of $\{a_n\}$. In Theorem 2 we prove that if $\varepsilon > 0$ and a zero-sequence $\{a_n\}$ are given then there exists a set A of measure less than ε so that $\bigcup_{n=1}^{\infty} (A + a_n)$ covers the interval. An application of this result is Theorem 3: for any sequence $\{a_n\}$ and $\varepsilon > 0$ there is a set H of measure $1-\varepsilon$ such that for no N and c is $\{a_n + c\}_{n \geq N}$ contained by H.

1. **Introduction.** The aim of this paper is to give two generalizations of the following theorem of D. Borwein and S. Z. Ditor [1]: there exists a set $H \subseteq [0, 1]$ of positive measure and a sequence $\{a_n\}$ converging to 0 such that if $x \in [0, 1]$ then $x + a_n \notin H$ for infinitely many n. We prove that there even exists a set H working simultaneously for all of the $\{\lambda a_n\}$'s, where $\{a_n\}$ is a certain specified sequence. We then prove that for any given $\{a_n\}$ we can actually construct a set H with the original property. A common generalization of these two theorems would give the solution of an old and quite challenging problem of P. Erdős [2]: for any given $\{a_n\}$ there is a set of positive measure not containing a subset similar to our sequence.

2. **Similarities**

PROPOSITION. *Assume that n is a natural number $n \geq 2$, $\varepsilon > 1/n$. Put $A = [0, 1-\varepsilon]$, $B = [1+\varepsilon, 2]$. If S is an $n+1$-term arithmetical progression in $A \cup B$ then either $S \subseteq A$ or $S \subseteq B$.*

Proof. Assume otherwise. As there is a gap of length 2ε between A and B, the difference of S is at least 2ε. The total length of S i.e. the difference between its last and first member is at least $2\varepsilon n > 2$, a contradiction.

THEOREM 1. *For any given $\varepsilon > 0$ there exist a set $H \subseteq [0, 1]$ of measure $1-\varepsilon$ and a sequence $\{a_n\}$ converging to 0 such that for any given $x \in [0, 1]$ and $\lambda \neq 0$, $x + \lambda a_n \notin H$ for infinitely many n.*

Proof. Let us define $K_0 = [0, 1]$ and for every n omit a central subinterval of relative length ε_n of every interval from K_n. Clearly K_{n+1} will be the union of

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2^{n+1} intervals of equal length and $\lambda(K_{n+1}) = (1-\varepsilon_0)\cdots(1-\varepsilon_n)$. Our set will be of form $H = \bigcap_{n=0}^{\infty} K_n$ for a suitably chosen sequence $\varepsilon_n \searrow 0$.

For $k = 1, 2, \ldots$ construct an arithmetic progression of n_k terms S_k of length 2^{-k} in $(0, 2^{-k+1})$, where $n_k > 1/\varepsilon_{2k}$. Clearly $\bigcup_{k=1}^{\infty} S_k$ is a zero-sequence. If λS_k is a subset of H then by repeated application of the proposition, λS_k is in one of the intervals of K_{2k} which has length $<2^{-2k}$; so $\lambda < 2^{-k}$. Thus, if $x \in [0, 1]$, $\lambda \neq 0$ are given, and k is big enough, $x + \lambda S_k$ has a point missing from H.

Theorem 2. *If $\varepsilon > 0$ and $a_n \searrow 0$ are given, there is a finite subsequence $a_{j_1} > a_{j_2} > \cdots > a_{j_s}$ and a set $H \subset [0, 1]$ with $\lambda(H) \leq \varepsilon$ and*

$$[0, 1] \subseteq H \cup \bigcup_{i=1}^{s} (H + a_{j_i}).$$

(Here $H + a$ denotes $\{x + a : x \in H\}$.)

Proof. Note first that by the well-known properties of Lebesgue-measure the statement of the theorem is equivalent to: there is, for any given $\varepsilon > 0$, a set H of measure less then ε for which

$$[0, 1] \subseteq H \cup \bigcup_{j=1}^{\infty} (H + a_j).$$

We shall prove, however, the finite version.

Our strategy will be the following: for $\varepsilon = \varepsilon_0 = 1$ the statement is clear. Then we shall inductively define $1 > \varepsilon_1 > \varepsilon_2 > \cdots$ and so on, and prove that if the statement holds for ε_k (and for every sequence) it is true for ε_{k+1}. As $\varepsilon_k \searrow 0$ this "infinite descent" gives the desired result.

Assume that the theorem is proved for $\varepsilon = \varepsilon_k$ and that H is a set witnessing this fact:

$$[0, 1] = H \cup \bigcup_{i=1}^{p} (H + a_i).$$

As the construction proceeds we shall see that the set H will be always a finite union of disjoint intervals—not too serious a restriction, in fact. Let δ be a sufficiently small positive number (say $\delta < \varepsilon_k^2/12$). If H is the union of N disjoint intervals and $a_j < \delta/2Np$ then we can divide H into two parts: $H = H^* \cup H^{**}$ where $\lambda(H^{**}) < \delta/p$ and H^* is the union of disjoint intervals each of length $2a_j$.

Let us define L as the union of the left-halves of these intervals, M as the union of the right-halves. Clearly $H^* = L \cup M$ and $M = L + a_j$.

Now we are going to estimate the measure of $X = L \cup \bigcup_{i=1}^{j} (L + a_i)$ ($j \geq p$ is assumed throughout). Since $H \cup \bigcup_{i=1}^{p} (H + a_i)$ covers $[0, 1]$ and $H^* \cup \bigcup_{i=1}^{p} (H^* + a_i)$ covers all $[0, 1]$ but a set of measure at most δ and $[H^* \cup$

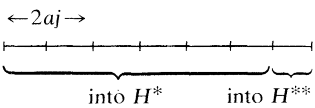

Figure 1

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$\bigcup_{i=1}^{p}(H^*+a_i)]-X$ is covered by certain intervals of length a_j, for every such interval J, $J-a_j$ is in X and neither J nor $J-a_j$ is in H^*. This gives

$$\lambda(X)>\tfrac{1}{2}[1-2pa_j+\lambda(H^*)]>\tfrac{1}{2}[1+\varepsilon_k-\delta].$$

(Loosely speaking, $X\cup(X+a_j)$ covers almost all of [0, 1], almost all of H^* twice.)

The set $Y=[0,1]-X$ is a union of finitely many disjoint intervals. Deleting from it a set of measure less than δ we get the union T of disjoint intervals having equal length.

In any one of these intervals, say in K, one can construct a set Q of measure $\varepsilon_k\lambda(K)$ and choose an index s such that $K\subseteq Q\cup\bigcup_{i=1}^{s}(Q+a_i)$ because our theorem is valid for $\varepsilon=\varepsilon_k$.

The appropriately chosen translated images of Q will give a set R with $\lambda(R)=\varepsilon_k\lambda(T)$ and $T=R\cup\bigcup_{i=1}^{s}(R+a_i)$.

Set $\tilde{H}=H^{**}\cup L\cup R\cup(Y-T)$.

We can easily deduce $\tilde{H}\cup\bigcup_{i=1}^{s+j}(\tilde{H}+a_i)\supseteq X\cup Y=[0,1]$.

The measure of $\tilde{H}$ is $\lambda(\tilde{H})<2\delta+\frac{1}{2}\varepsilon_k+\varepsilon_k\frac{1}{2}(1-\varepsilon_k+\delta)=\varepsilon_k-\frac{1}{2}\varepsilon_k^2+(2+\varepsilon_k/2)\delta$. We have already chosen δ less than $\frac{1}{12}\varepsilon_k^2$. Also $\varepsilon_{k+1}<\varepsilon_k-\varepsilon_k^2/3$. Thus $\varepsilon_0,\varepsilon_1,\ldots$ is clearly a zero-sequence supporting our claim.

THEOREM 3. *For any given zero-sequence* $a_n\searrow 0$ *and* $\varepsilon>0$ *there is a set* $A\subset[0,1]$ *with* $\lambda(A)>1-\varepsilon$ *possessing the property: if* $x\in[0,1]$ *then* $\{n:x+a_n\notin A\}$ *is infinite.*

Proof. We are going to construct the complement of the desired A. Using theorem 2 we inductively construct sets $H_1,H_2,\ldots$ of measure $\lambda(H_k)=\varepsilon/2^k$ and subsequent strings of elements from the sequence $\{a_{j_{k-1}+1},\ldots,a_{j_k}\}$ with [0, 1] covered by $H_k\cup\bigcup_{j_{k-1}+1}^{j_k}(H_k+a_i)$. So if $x\in A=[0,1]-(\bigcup_{k=1}^{\infty}H_k)$, for every k there is an i with $j_{k-1}<i\le j_k$ and $x-a_i\in H_k\subseteq[0,1]-A$. This gives a set A of measure $1-\varepsilon$ with the property: if $x\in A$ then $\{j:x-a_j\notin A\}$ is infinite. Taking a closed A with this property, one can remove the condition $x\in A$, and $\{1-x:x\in A\}$ will be a good set.

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