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Abstract

We construct the quantized enveloping algebra of any simple Lie algebra of type $\mathbb{ADE}$ as the quotient of a Grothendieck ring arising from certain cyclic quiver varieties. In particular, the dual canonical basis of a one-half quantum group with respect to Lusztig's bilinear form is contained in the natural basis of the Grothendieck ring up to rescaling. This paper expands the categorification established by Hernandez and Leclerc to the whole quantum groups. It can be viewed as a geometric counterpart of Bridgeland's recent work for type $\mathbb{ADE}$.

Contents

1. Introduction

1.1 History

For any given symmetric Cartan datum, let $\mathfrak{g}$ be the associated Kac–Moody Lie algebra and $\mathbf{U}_t(\mathfrak{g})$ the corresponding quantized enveloping algebra. There have been several different approaches to the categorical realizations of $\mathbf{U}_t(\mathfrak{g})$.

One-half quantum group. The earliest and best-developed theories are categorifications of a one-half quantum group. Notice that $\mathbf{U}_t(\mathfrak{g})$ has the triangular decomposition $\mathbf{U}_t(\mathfrak{g}) = \mathbf{U}_t(\mathfrak{n}^+) \otimes \mathbf{U}_t(\mathfrak{h}) \otimes \mathbf{U}_t(\mathfrak{n}^-)$. Let Q denote a quiver associated with $\mathfrak{g}$ which has no oriented cycles. For any field k, let kQ denote the path algebra associated with Q.

(1) In 1990, Ringel showed in [Rin90] that the positive (respectively negative) one-half quantum group $\mathbf{U}_t(\mathfrak{n}^+)$ (respectively $\mathbf{U}_t(\mathfrak{n}^-)$) can be realized as a subalgebra of the *Hall algebra* of the abelian category $\mathbb{F}_qQ - \mathsf{mod}$, where $\mathbb{F}_q$ is any finite field and $\mathbb{F}_qQ - \mathsf{mod}$ the category of the left modules of the path algebra $\mathbb{F}_qQ$. Let us call this Hall algebra approach an *additive categorification* of $\mathbf{U}_t(\mathfrak{n}^+)$, because the product of any Chevalley generator with itself is translated into the direct sum of a simple $\mathbb{F}_qQ$-module with itself.

(2) Lusztig has given a geometric construction of $\mathbf{U}_t(\mathfrak{n}^+)$ (cf. [Lus90, Lus91]), by considering the Grothendieck ring arising from certain perverse sheaves over the varieties of $\mathbb{C}Q$-modules.

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This geometric approach is very powerful. In particular, the perverse sheaves provide us with a positive basis[1] of $\mathbf{U}_t(\mathfrak{n}^+)$, which is called *the canonical basis*; cf. also [Kas91] for the *crystal basis*. We can view Lusztig's construction as a *monoidal categorification* of $\mathbf{U}_t(\mathfrak{n}^+)$, because the addition and the multiplication in $\mathbf{U}_t(\mathfrak{n}^+)$ are translated into the direct sum and the derived tensor of perverse sheaves.

We recommend to the reader the survey papers of Schiffmann [Sch06, Rou09] for the results (1) and (2).

(3) Recently, the quiver Hecke algebras (or KLR-algebras; cf. [KL09, Rou08]) provide us with a monoidal categorification of the one-half quantum group $\mathbf{U}_t(\mathfrak{n}^+)$. A link between Lusztig's approach and the quiver Hecke algebras has been established in [VV11].

(4) Finally, assume that $\mathfrak{g}$ is of type $\mathbb{ADE}$. Hernandez and Leclerc showed that a certain subcategory of finite-dimensional representations of the quantum affine algebra $\mathbf{U}_q(\widehat{\mathfrak{g}})$ provides a monoidal categorification of $\mathbf{U}_t(\mathfrak{n}^+)$; cf. [HL15]. By [Nak01] and [HL15, § 9], their construction can be understood in terms of graded quiver varieties and then be compared with the work of Lusztig.

We remark that the categorifications in (3) and (4) are compatible with the (dual) canonical basis obtained in (2). Moreover, by (4), the categorification in the present paper is compatible with the dual canonical basis.

Whole quantum group. We can define the algebra $\widetilde{\mathbf{U}}_t(\mathfrak{g})$ as a variant of the whole quantum group $\mathbf{U}_t(\mathfrak{g})$, cf. § 2.1, which has the triangle decomposition $\mathbf{U}_t(\mathfrak{n}^+) \otimes \widetilde{\mathbf{U}}_t(\mathfrak{h}) \otimes \mathbf{U}_t(\mathfrak{n}^-)$. This variant plays a crucial role in Bridgeland's work [Bri13], which we shall briefly recall. The whole quantum group $\mathbf{U}_t(\mathfrak{g})$ is obtained from $\widetilde{\mathbf{U}}_t(\mathfrak{g})$ by a reduction at the Cartan part $\widetilde{\mathbf{U}}_t(\mathfrak{h})$.

There have been various attempts to make a Hall algebra construction of the whole quantum group; cf. for example [Kap98, PX97, PX00, XXZ06]. The complete result was obtained in the recent work of Bridgeland.

THEOREM [Bri13]. *Fix a finite field $\mathbb{F}_q$. Let $\widetilde{U}_{\sqrt{q}}(\mathfrak{g})[(K_i)^{-1}, (K_i')^{-1}]_{i\in I}$ denote the localization of $\widetilde{U}_{\sqrt{q}}(\mathfrak{g})$ at the Cartan part. Then it is isomorphic to the localization of the Ringel Hall algebra of the 2-periodic complexes of projective $\mathbb{F}_qQ$-modules at the contractible complexes.*

The usual quantum group $\mathbf{U}_t(\mathfrak{g})$ can be obtained from the above construction as the natural quotient of $\widetilde{\mathbf{U}}_t(\mathfrak{g})[(K_i)^{-1}, (K_i')^{-1}]_{i\in I}$.

In the work of Bridgeland, the realizations of the half-quantum groups $\mathbf{U}_t(\mathfrak{n}^+)$ and $\mathbf{U}_t(\mathfrak{n}^-)$ can be identified with those in Ringel's approach. The Cartan part $\widetilde{\mathbf{U}}_t(\mathfrak{h})$ is generated by certain complexes homotopic to zero, which are redundant information in the study of the corresponding triangulated category. In the sense of § 1.1, this Hall algebra approach can be viewed as an additive categorification of $\mathbf{U}_t(\mathfrak{g})$.

Also, by the works of Khovanov, Lauda, Rouquier, and Webster, cf. [KL10, Rou08, Web10, Web13], the quiver Hecke algebras provide a monoidal categorification of the modified quantum group $\dot{\mathbf{U}}_t(\mathfrak{g})$, which is a different variant of the whole quantum group $\mathbf{U}_t(\mathfrak{g})$ [Lus93].

Finally, we notice that Fang and Rosso have constructed the whole quantum group in the spirit of quantum shuffle algebras; cf. [FR12].

1.2 Main construction and result

In this paper, we give a geometric construction of the whole quantum group for the Lie algebra $\mathfrak{g}$ of Dynkin type $\mathbb{A}$, $\mathbb{D}$, $\mathbb{E}$, inspired by the following papers.

[1] By a positive basis, we mean a basis whose structure constants are non-negative.

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Inspired by [KS13, Theorem 2.7], we use some cyclic quiver varieties associated with roots of unity to replace the abelian category of 2-periodic complexes in [Bri13].

Then, the work of Hernandez and Leclerc [HL15, § 9] establishes a construction of the half-quantum groups $\mathbf{U}_t(\mathfrak{n}^+)$ and $\mathbf{U}_t(\mathfrak{n}^-)$, which can be compared with Lusztig's work by [HL15, § 9]. The techniques developed in [HL15] and [LP13] will be crucial in our proofs.

We construct the Cartan part $\widetilde{\mathbf{U}}_t(\mathfrak{h})$ from certain strata of cyclic quiver varieties, which are identified with the stratum $\{0\}$ in Nakajima's *transverse slice theorem* [Nak01, 3.3.2]. The analog of these strata for graded quiver varieties provides redundant information in the study of quantum affine algebras [Nak01]. So, our construction of the Cartan part shares the same spirit as that of Bridgeland's work:

The Cartan part is categorified by redundant information.

In the sense of our previous discussion in categorification (2), this geometric construction can be viewed as a monoidal categorification of $\widetilde{\mathbf{U}}_t(\mathfrak{g})$, which contains the Hernandez–Leclerc categorification of $\mathbf{U}_t(\mathfrak{n})$. In particular, we obtain a positive basis of $\widetilde{\mathbf{U}}_t(\mathfrak{g})$, which, up to rescaling, contains the dual canonical basis of a one-half quantum group with respect to Lusztig's bilinear form.

We refer the reader to § 3.1 for the detailed construction and Theorems 3.1.6 and 3.1.7 for the rigorous statements of the results.

1.3 Remarks

This paper could be viewed as a geometric counterpart of Bridgeland's work for the type $\mathbb{ADE}$. It is natural to compare this geometric construction with the Hall algebra construction of Bridgeland. The details might appear elsewhere.

On the other hand, by choosing the shifted simple modules in derived categories as in [HL15, § 8.2], the analogous construction in the present paper remains effective over graded quiver varieties associated with a generic q. Details might appear elsewhere. The corresponding Grothendieck ring should then be compared with the semi-derived Hall algebra associated with the quiver Q in the sense of Gorsky [Gor13]. However, this straightforward generalization is not a unique approach. A completely different construction might appear in Gorsky's future work.

In this paper, the twisted product defined for the Grothendieck ring is different from those used by [Nak04] or by [Her04a, HL15]. It is worth mentioning that our twisted product agrees with the non-commutative multiplication of [Her04a, HL15] on the one-half quantum group $\mathbf{U}_t(\mathfrak{n}^+)$, as we shall prove in the last section. We refer the reader to Example 3.2.3 for a comparison of various products.

Our Grothendieck ring $\widetilde{\mathbf{U}}_t(\mathfrak{g})$ is defined over some cyclic quiver varieties, which are closely related to the Grothendieck ring of finite-dimensional representations of the quantum affine algebra $\mathbf{U}_q(\widehat{\mathfrak{g}})$ at the root of unity q. In particular, in [Nak04], Nakajima has used these cyclic quiver varieties to study the t-analog of the q-characters on the latter Grothendieck ring. However, to the best knowledge of the author, there exists no twisted product in the literature such that the Cartan part of $\widetilde{\mathbf{U}}_t(\mathfrak{g})$ consists of center elements, which prevents a direct reduction of $\widetilde{\mathbf{U}}_t(\mathfrak{g})$ to the t-deformed Grothendieck ring of representations of $\mathbf{U}_q(\widehat{\mathfrak{g}})$ considered in [Nak04, Her04b].

Finally, the present paper is just a first step of this geometric approach. In particular, the reduction of the Cartan part discussed here follows a straightforward algebraic approach, which was used by Bridgeland. We shall use the corresponding geometric realization to study quantum groups in a future work.

2. Preliminaries

2.1 Quantum groups

We recall the basic facts concerning the quantum groups and refer the reader to Schiffmann's paper [Sch06] or Lusztig's book [Lus93] for more details. We shall follow the notation used in [KQ14, Kim12].

Let n be any given positive integer and define the index set $I = \{1, \dots, n\}$. Fix a symmetric root datum. Denote the Cartan matrix by $C = (a_{ij})_{i,j\in I}$ and the positive simple roots by $\{\alpha_i, i \in I\}$. Let $\mathfrak{g}$ be the corresponding Kac–Moody Lie algebra.

Let t be an indeterminate. We define $[n]_t = (t^n - t^{-n})/(t - t^{-1})$, $[n]_t! = [1]_t \cdot [2]_t \cdot \dots \cdot [n]_t$. Let $\widetilde{\mathbf{U}}_t(\mathfrak{g})$ be the $\mathbb{Q}(t)$-algebra generated by the *Chevalley generators* E_i, K_i, K'_i, F_i, $i \in I$, which are subject to the following relations:

$$\sum_{k=0}^{1-a_{ij}} (-1)^k E_i^{(k)} E_j E_i^{(1-a_{ij}-k)} = 0,$$
$$\sum_{k=0}^{1-a_{ij}} (-1)^k F_i^{(k)} F_j F_i^{(1-a_{ij}-k)} = 0,$$
$$[E_i, F_j] = \delta_{ij} \frac{K_i - K'_i}{t - t^{-1}},$$
$$[K_i, K_j] = [K_i, K'_j] = [K'_i, K'_j] = 0,$$
$$K_i E_j = t^{a_{ij}} E_j K_i,$$
$$K_i F_j = t^{-a_{ij}} F_j K_i,$$
$$K'_i E_j = t^{-a_{ij}} E_j K'_i,$$
$$K'_i F_j = t^{a_{ij}} F_j K'_i,$$

where $E_i^{(k)} = E_i^k/[k]_t!$ and $F_i^{(k)} = F_i^k/[k]_t!$.

The *quantum group* $\mathbf{U}_t(\mathfrak{g})$ is defined as the quotient algebra of $\widetilde{\mathbf{U}}_t(\mathfrak{g})$ with respect to the ideal generated by the elements $K_i * K'_i - 1$, $i \in I$.

Let $\widetilde{\mathbf{U}}_t(\mathfrak{n}^+)$ be the subalgebra of $\widetilde{\mathbf{U}}_t(\mathfrak{g})$ generated by E_i, $i \in I$, $\widetilde{\mathbf{U}}_t(\mathfrak{h})$ the subalgebra of $\widetilde{\mathbf{U}}_t(\mathfrak{g})$ generated by K_i, K'_i, and $\widetilde{\mathbf{U}}_t(\mathfrak{n}^-)$ the subalgebra of $\widetilde{\mathbf{U}}_t(\mathfrak{g})$ generated by F_i. The subalgebras $\mathbf{U}_t(\mathfrak{n}^+)$, $\mathbf{U}_t(\mathfrak{h})$, and $\mathbf{U}_t(\mathfrak{n}^-)$ of $\mathbf{U}_t(\mathfrak{g})$ are defined similarly. Then both $\widetilde{\mathbf{U}}_t(\mathfrak{g})$ and $\mathbf{U}_t(\mathfrak{g})$ have triangular decompositions:

$$\widetilde{\mathbf{U}}_t(\mathfrak{g}) = \widetilde{\mathbf{U}}_t(\mathfrak{n}^+) \otimes \widetilde{\mathbf{U}}_t(\mathfrak{h}) \otimes \widetilde{\mathbf{U}}_t(\mathfrak{n}^-),$$
$$\mathbf{U}_t(\mathfrak{g}) = \mathbf{U}_t(\mathfrak{n}^+) \otimes \mathbf{U}_t(\mathfrak{h}) \otimes \mathbf{U}_t(\mathfrak{n}^-).$$

From the definitions, we have $\mathbf{U}_t(\mathfrak{n}^+) = \widetilde{\mathbf{U}}_t(\mathfrak{n}^+)$, $\mathbf{U}_t(\mathfrak{n}^-) = \widetilde{\mathbf{U}}_t(\mathfrak{n}^-)$, and $\mathbf{U}_t(\mathfrak{h}) = \widetilde{\mathbf{U}}_t(\mathfrak{h})/(K_i * K'_i - 1)_i$.

The *Kashiwara bilinear form* $(\ ,\ )_K$ on $\mathbf{U}_t(\mathfrak{n}^+)$ has the property $(E_i, E_j)_K = \delta_{ij}$; cf. [Kas91, § 3.4]. The *Lusztig bilinear form* $(\ ,\ )_L$ on $\mathbf{U}_t(\mathfrak{n}^+)$ has the property $(E_i, E_j)_L = \delta_{ij}(1 - t^2)^{-1}$; cf. [Lus93, 1.2.5]. In general, by [Lec04, 2.2], for any homogeneous elements $x, y \in \mathbf{U}_t(\mathfrak{n}^+)_\beta$, where $\beta = \sum_{i\in I} \beta_i \alpha_i$, $\beta_i \in \mathbb{N}$, we have

$$(x, y)_K = (1 - t^2)^{\sum_i \beta_i} \cdot (x, y)_L. \tag{1}$$

We let $A_t(\mathfrak{n}^+)$ denote the quantum coordinate ring which is the graded dual $\mathbb{Q}(t)$-vector space of $\mathbf{U}_t(\mathfrak{n}^+)$ endowed with a restricted multiplication; cf. [GLS13, § 4] and also [Kim12, § 3].

Proposition 2.1.1 [GLS13, Proposition 4.1]. *There exists an algebra isomorphism Ψ from $\mathbf{U}_t(\mathfrak{n}^+)$ to $A_t(\mathfrak{n}^+)$ such that any element x is sent to the linear map $(x,\ )_K$.*

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2.2 Graded and cyclic quiver varieties

In this paper, we consider the quivers[2] Q of type $\mathbb{A}$, $\mathbb{D}$, $\mathbb{E}$.

Choose any $q \in \mathbb{C}^*$ such that $q \neq 1$. It generates a cyclic subgroup $\langle q \rangle$ in the multiplicative group $(C^*, *)$. We assume that either $\langle q \rangle$ is an infinite group or its cardinality is divisible by 2.

Because the underlying graph of the Dynkin quiver Q is a tree, we can choose a height function $\xi : I \to \langle q \rangle$ such that $\xi(k) = q * \xi(i)$ whenever there is an arrow from k to i in Q.

Define $\widehat{I} = \{(i, a) \in I \times \langle q \rangle | \xi(i) * a^{-1} \in \langle q^2 \rangle\}$. The reader is referred to Example 2.2.1 for an example.

Let σ denote the automorphism of $I \times \langle q \rangle$ such that $\sigma(i, a) = (i, q^{-1}a)$. Then $I \times \langle q \rangle$ is the disjoint union of $\widehat{I}$ and $\sigma\widehat{I}$. We use τ to denote the automorphism σ^2 on $I \times \langle q \rangle$.

We always use x to denote the elements in $\sigma\widehat{I}$. We use v, w to denote the finitely supported elements in $\mathbb{N}^{\sigma\widehat{I}}$, $\mathbb{N}^{\widehat{I}}$, respectively. Let $e_{i,a}$ denote the characteristic function of (i, a), which is also viewed as the unit vector supported at (i, a). We have $\sigma^* e_{\sigma(i,a)} = e_{i,a}$. For any given v, w, we denote the associated $I \times \langle q \rangle$-graded vector spaces by $V = \bigoplus_{i,a} V(i, a) = \bigoplus \mathbb{C}^{v(i,a)}$ and $W = \bigoplus_{i,a} W(i, a) = \bigoplus \mathbb{C}^{w(i,a)}$.

The *q-Cartan matrix* C_q is a linear map from $\mathbb{Z}^{\sigma\widehat{I}}$, $\mathbb{Z}^{\widehat{I}}$, such that for any $(i, a) \in \sigma\widehat{I}$, we have

$$C_q e_{i,a} = e_{i,qa} + e_{i,q^{-1}a} + \sum_{j \in I, j \neq i} a_{ij} e_{j,a}. \tag{2}$$

A pair (v, w) is called *l-dominant* if $w - C_q v \geqslant 0$.

We shall define graded/cyclic quiver varieties. Details could be found in [Nak01] (cf. also [Nak11, Qin14, KQ14]).

Let Ω denote the set of the arrows of Q. Similarly, let $\overline{\Omega}$ denote the set of the arrows of the opposite quiver Q^{op}. For each arrow h, we let $s(h)$ and $t(h)$ denote its source and target, respectively. Define

$$E^q(\Omega; v, w) = \bigoplus_{(i,a) \in \sigma\widehat{I}} \; \bigoplus_{h \in \Omega : s(h)=i, t(h)=j} \mathsf{Hom}(V(i, a), V(j, aq^{-1})), \tag{3}$$

$$L^q(w, v) = \bigoplus_{x \in \sigma\widehat{I}} \mathsf{Hom}(W(\sigma^{-1}x), V(x)), \tag{4}$$

$$L^q(v, w) = \bigoplus_{x \in \sigma\widehat{I}} \mathsf{Hom}(V(x), W(\sigma x)). \tag{5}$$

Define the vector space $\mathsf{Rep}^q(Q; v, w)$ to be

$$\mathsf{Rep}^q(Q; v, w) = E^q(\Omega; v, w) \oplus E^q(\overline{\Omega}; v, w) \oplus L^q(w, v) \oplus L^q(v, w), \tag{6}$$

whose elements are denoted by

$$\begin{aligned}&\left(\bigoplus_h B_h, \bigoplus_{\overline{h}} B_{\overline{h}}, (\alpha_i)_{i \in I}, (\beta_i)_{i \in I}\right)\\ &\quad = \left(\bigoplus_{h \in \Omega}\left(\bigoplus_a B_{h,a}\right), \bigoplus_{\overline{h} \in \overline{\Omega}}\left(\bigoplus_b B_{\overline{h},b}\right), \left(\bigoplus_a \alpha_{i,a}\right)_{i \in I}, \left(\bigoplus_b \beta_{i,b}\right)_{i \in I}\right).\end{aligned}$$

[2] The quiver Q used in this paper should be compared with the opposite quiver Q^{op} used in [KQ14].

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$$\begin{tikzcd}
& & 3 \arrow[dl, "h_2"'] \\
& 2 \arrow[dl, "h_1"'] & \\
1 & &
\end{tikzcd}$$

Figure 1. A quiver of type A_3.

$$\begin{tikzcd}
& \cdots & \boxed{(3,q^2)} & (3,q^3) & \boxed{(3,q^4)} & (3,q^5) \\
\cdots & \boxed{(2,q)} & (2,q^2) & \boxed{(2,q^3)} & (2,q^4) & \cdots \\
\boxed{(1,1)} & (1,q) & \boxed{(1,q^2)} \arrow[l, dashed, "\sigma"'] & (1,q^3) \arrow[l, dashed, "\sigma"'] & \cdots &
\end{tikzcd}$$

Figure 2. (Colour online) Vertices $I \times \langle q \rangle$.

The group $\mathrm{GL}_v = \prod_{x \in \sigma \widehat{I}} \mathrm{GL}_{v(x)}$ naturally acts on $\mathsf{Rep}^q(Q; v, w)$. We have the map μ as the natural analog of the moment map such that

$$\mu\left(\left(\bigoplus_h B_h, \bigoplus_{\overline{h}} B_{\overline{h}}, (\alpha_i), (\beta_i)\right)\right) = \sum_{h \in \Omega, \overline{h}' \in \overline{\Omega} : s(h) = t(\overline{h}')} B_h B_{\overline{h}'} - \sum_{h \in \Omega, \overline{h}' \in \overline{\Omega} : s(\overline{h}') = t(h)} B_{\overline{h}'} B_h + \sum_{i \in I} \alpha_i \beta_i;$$

cf. [Nak01] for details.

For the GL_v-variety $\mu^{-1}(0)$, we construct Mumford's GIT[3] quotient $\mathcal{M}^q(v, w)$ and the categorical quotient $\mathcal{M}_0^q(v, w)$. There is a natural proper morphism π from the GIT quotient $\mathcal{M}^q(v, w)$ to the categorical quotient $\mathcal{M}_0^q(v, w)$.

Example 2.2.1. Let the quiver Q be given by Figure 1. We can choose the height function ξ such that $\xi(i) = q^{i-1}$. Then $I \times \langle q \rangle$ is given by Figure 2, where the vertices in square boxes belong to $\widehat{I}$ and the other vertices belong to $\sigma \widehat{I}$.

Then the vector space $\mathsf{Rep}^q(Q; v, w)$ is described in Figure 3, whose rows and columns are indexed by I-degrees (vertices) and $\langle q \rangle$-degrees (heights), respectively.

In this example, the analog of the moment map μ takes the form

$$(\alpha_1 \beta_1 + B_{h_1} B_{\overline{h}_1}) \oplus (\alpha_2 \beta_2 + B_{h_2} B_{\overline{h}_2} - B_{\overline{h_1}} B_{h_1}) \oplus (\alpha_3 \beta_3 - B_{\overline{h}_2} B_{h_2}).$$

[3] GIT stands for 'geometric invariant theory'.

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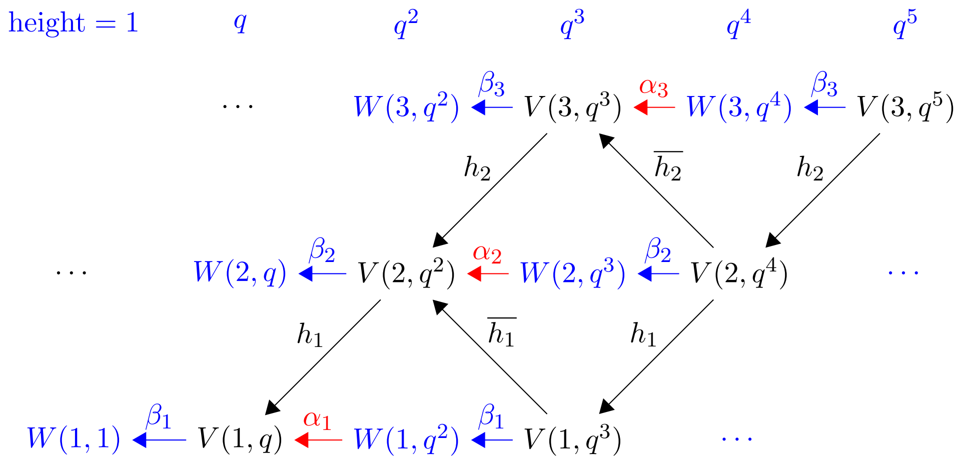

FIGURE 3. (Colour online) Vector space $\mathsf{Rep}^q(Q; v, w)$.

First, assume that q is not a root of unity. Then the quotients $\mathcal{M}^q(v,w)$ and $\mathcal{M}_0^q(v,w)$ do not depend on q. They will be called the *graded quiver varieties* and denoted by $\mathcal{M}^\bullet(v,w)$ and ${\mathcal{M}_0}^\bullet(v,w)$, respectively. Let ${\mathcal{M}_0}^\bullet(w)$ denote the natural union $\bigcup_v {\mathcal{M}_0}^\bullet(v,w)$. This is a finite-dimensional affine variety with a stratification into the *regular strata*

$$ {\mathcal{M}_0}^\bullet(w) = \bigsqcup_{v: w - C_q v \geqslant 0} {\mathcal{M}_0}^{\bullet\mathrm{reg}}(v,w). $$

Similarly, assume that q equals ϵ, which is a root of unity. The varieties $\mathcal{M}^\epsilon(v,w)$ and ${\mathcal{M}_0}^\epsilon(v,w)$ will be called the *cyclic quiver varieties*. Let ${\mathcal{M}_0}^\epsilon(w)$ denote the natural union $\bigcup_v {\mathcal{M}_0}^\epsilon(v,w)$.

PROPOSITION 2.2.2 [Nak01, § 2.5]. *Assume that the quiver Q is of Dynkin type $\mathbb{A}$, $\mathbb{D}$, $\mathbb{E}$. Then the union ${\mathcal{M}_0}^\epsilon(w)$ is finite dimensional with a stratification into the regular strata*

$$ {\mathcal{M}_0}^\epsilon(w) = \bigsqcup_{v: w - C_q v \geqslant 0} {\mathcal{M}_0}^{\epsilon\mathrm{reg}}(v,w). \tag{7} $$

The properties of the cyclic quiver varieties are similar to those of the graded quiver varieties, except for the following two important differences:

- the linear map C_q (q-Cartan matrix) is not injective;
- it is not known if the smooth cyclic quiver variety $\mathcal{M}^\epsilon(v,w)$ is connected or not.

The smooth cyclic quiver variety $\mathcal{M}^\epsilon(v,w)$ is pure dimensional; cf. [Nak01, (4.1.6)]. For any v, choose a set $\{\alpha_v\}$ such that it parameterizes the connected component of $\mathcal{M}^\epsilon(v,w)$. For any l-dominant pair (v,w), since the restriction of π on the regular stratum ${\mathcal{M}_0}^{\epsilon\mathrm{reg}}(v,w)$ is a homeomorphism, the set $\{\alpha_v\}$ naturally parameterizes the connected components of this regular stratum:

$$ {\mathcal{M}_0}^{\epsilon\mathrm{reg}}(v,w) = \bigsqcup_{\alpha_v} {\mathcal{M}_0}^{\epsilon\mathrm{reg};\alpha_v}(v,w). \tag{8} $$

Let $1_{\mathcal{M}^\epsilon(v,w)}$ denote the perverse sheaf associated with the trivial local system of rank 1 on $\mathcal{M}^\epsilon(v,w)$. Denote the perverse sheaf $\pi_!(1_{\mathcal{M}^\epsilon(v,w)})$ by $\pi(v,w)$.

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Using the transverse slice theorem (cf. [Nak01]), we can simplify the decomposition of $\pi(v,w)$ as follows (cf. the proof of Theorem 8.6 in [Nak04]):

$$\pi(v,w) = \sum_{v':w-C_qv'\geqslant 0, v'\leqslant v} a_{v,v';w}(t)\mathcal{L}(v',w), \tag{9}$$

where we denote $\mathcal{F}[d]^{\oplus m}$ by $mt^d\mathcal{F}$ for any sheaf $\mathcal{F}$ and $m\in\mathbb{N}$, $d\in Z$, and we define

$$\mathcal{L}(v',w) = IC(\mathcal{M}_0{}^{\epsilon\mathrm{reg}}(v',w)) = \bigoplus_{\alpha_{v'}} IC(\mathcal{M}_0{}^{\epsilon\mathrm{reg};\alpha_{v'}}(v',w)). \tag{10}$$

Notice that we have $a_{v,v';w}(t)\in\mathbb{N}[t^{\pm}]$, $a_{v,v';w}(t^{-1}) = a_{v,v';w}(t)$, and $a_{v,v;w}=1$. We do not know if $\mathcal{L}(v',w)$ is a simple perverse sheaf or not.

For any decomposition $w = w^1 + w^2$, we have the restriction functor between the derived category of constructible sheaves

$$\widetilde{\mathrm{Res}}^w_{w^1,w^2} : \mathcal{D}_c(\mathcal{M}_0{}^\epsilon(w)) \to \mathcal{D}_c(\mathcal{M}_0{}^\epsilon(w^1)\times\mathcal{M}_0{}^\epsilon(w^2)).$$

By [VV03], $\widetilde{\mathrm{Res}}^w_{w^1,w^2}(\pi(v,w))$ equals

$$\bigoplus_{v^1+v^2=v} \pi(v^1,w^1)\boxtimes\pi(v^2,w^2)[d((v^2,w^2),(v^1,w^1)) - d((v^1,w^1),(v^2,w^2))], \tag{11}$$

where the bilinear form $d(\,,\,)$ is given by

$$d((v^1,w^1),(v^2,w^2)) = (w^1 - C_qv^1)\cdot\sigma^*v^2 + v^1\cdot\sigma^*w^2. \tag{12}$$

For each w, the Grothendieck group K_w is defined as the free abelian group generated by the perverse sheaves $\mathcal{L}(v,w)$ appearing in (9). It has two $\mathbb{Z}[t^{\pm}]$-bases: $\{\pi(v,w)|w-C_qv\geqslant 0, \mathcal{M}_0{}^{\epsilon\mathrm{reg}}(v,w)\neq\emptyset\}$ and $\{\mathcal{L}(v,w)\}$. Then its dual $R_w = \mathsf{Hom}_{\mathbb{Z}[t^{\pm}]}(K_w,\mathbb{Z}[t^{\pm}])$ has the corresponding dual bases $\{\chi(v,w)|w-C_qv\geqslant 0, \mathcal{M}_0{}^{\epsilon\mathrm{reg}}(v,w)\neq\emptyset\}$ and $\{\mathbf{L}(v,w)\}$. Notice that, throughout this paper, we only define $\mathcal{L}(v,w)$, $\mathbf{L}(v,w)$ for the l-dominant pairs (v,w) such that $\mathcal{M}_0{}^{\epsilon\mathrm{reg}}(v,w)\neq\emptyset$.

The restriction functors induce an $\mathbb{N}^{\widehat{I}}$-graded coassociative comultiplication on the $\mathbb{N}^{\widehat{I}}$-graded Grothendieck group $\bigoplus_w K_w$, which we denote by $\widetilde{\mathrm{Res}}$.

2.3 Quiver varieties and quiver representations

Let $\mathsf{Rep}(Q)$ denote the category of left $\mathbb{C}Q$-modules. Let $\mathcal{D}^b(Q)$ denote the bounded derived category of $\mathsf{Rep}(Q)$ with the shift functor Σ. In $\mathcal{D}^b(Q)$, we have Auslander–Reiten triangles. Also, let ν denote the derived tensor with the bimodule $\mathsf{Hom}_{\mathbb{C}Q}(\mathbb{C}Q,\mathbb{C})$. Then we have

$$D\,\mathsf{Hom}_{\mathcal{D}^b(Q)}(x,y) = \mathsf{Hom}_{\mathcal{D}^b(Q)}(y,\nu x) \quad \forall x,y\in\mathcal{D}^b(Q).$$

By abuse of notation, we use τ to denote the Auslander–Reiten translation, which is defined as $\Sigma^{-1}\nu$.

Let $\mathsf{Ind}\mathcal{D}^b(Q)$ be a full subcategory of $\mathcal{D}^b(Q)$ whose objects form a set of representatives of the isoclasses of the indecomposable objects of $\mathcal{D}^b(Q)$ such that it is stable under τ and Σ. Its subcategory $\mathsf{Ind}\,\mathsf{Rep}(Q)$ is naturally defined.

Assume that q is not a root of unity; then we can choose a natural identification of $\sigma\widehat{I}$ with (the objects of) $\mathsf{Ind}\mathcal{D}^b(Q)$ such that it commutes with τ.

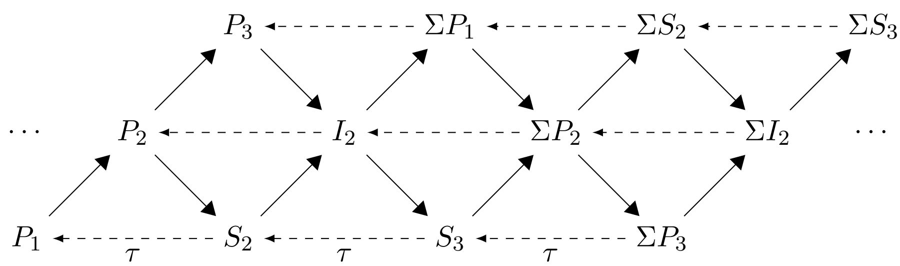

FIGURE 4. Auslander–Reiten quiver of $\mathsf{Ind}\mathcal{D}^b(Q)$.

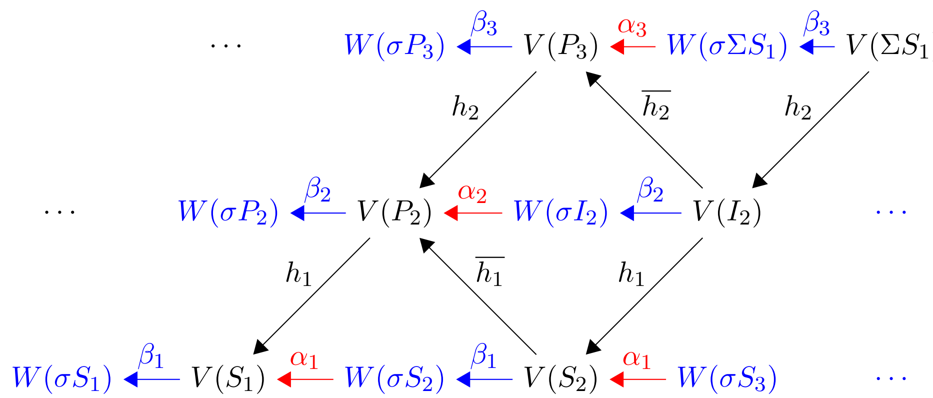

FIGURE 5. (Colour online) Vector space $\mathsf{Rep}^q(Q; v, w)$.

Define

$$\begin{aligned} W^+ &= \bigoplus_{x \in \mathsf{Ind}\,\mathsf{Rep}(Q)} \mathbb{N}e_{\sigma x}, \\ V^+ &= \bigoplus_{x \in \mathsf{Ind}\,\mathsf{Rep}(Q),\ x \text{ is not injective}} \mathbb{N}e_x, \\ W^S &= \bigoplus_{x \in \{S_i, i \in I\}} \mathbb{N}e_{\sigma x}. \end{aligned} \tag{13}$$

Example 2.3.1 (Quiver type A_3). Let us continue Example 2.2.1. The vertices in $\sigma\widehat{I}$ take the form (i, q^{i+2d}), $d \in \mathbb{Z}$; cf. Figure 2. On the other hand, the Auslander–Reiten quiver of $\mathsf{Ind}\mathcal{D}^b(Q)$ is given in Figure 4. Notice that the projective $\mathbb{C}Q$-module P_1 is also the simple module S_1.

So, we can identify $\sigma\widehat{I}$ with the objects of $\mathsf{Ind}\mathcal{D}^b(Q)$ by sending the vertex (i, q^{i+2d}) to the object $\tau^{-d}P_i$. Then Figure 3 becomes Figure 5. It follows that the dimension vectors in W^+ concentrate at the vertices σx, $x \in \mathsf{Ind}\,\mathbb{C}Q$-mod, those in V^+ concentrate at S_1, P_2, S_2, and those in W^S at σS_i, $i = 1, 2, 3$.

Recall that a pair (v, w) is called *l-dominant* if $w - C_q v \geqslant 0$. The vector spaces W^+, V^+, W^S are defined in (13). We shall use the following combinatorial property of the l-dominant pairs.

THEOREM 2.3.2 [LP13]. *Assume that q is not a root of unity. Then, for any $\widetilde{w} \in W^+$, there exists a unique l-dominant pair $(v, w) \in V^+ \times W^S$ such that $w - C_q v = \widetilde{w}$.*

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3. Grothendieck rings arising from cyclic quiver varieties

3.1 Constructions and main results

We follow the conventions in § 2. Let h denote the Coxeter number of the Dynkin type of the quiver Q. We make the following convention from now on:

Fix ϵ to be a $(2h)$th primitive root of unity and, by default, take q to be ϵ.

It follows that the automorphism τ^h on $I \times \langle\epsilon\rangle$ equals 1. The subset $\sigma\widehat{I}$ of $I \times \langle\epsilon\rangle$ has the cardinality nh.

Let us choose the natural covering map π from the set $\mathsf{Ind}\mathcal{D}^b(Q)$ to $\sigma\widehat{I}$ which sends the ith projective P_i to $(i, \epsilon\xi_i)$ and commutes with τ, namely, $\tau\pi(M) = \pi(\tau M)$.

Choose a section $M_?$ of this covering map π such that $\pi M_x = x$ for any $x \in \sigma\widehat{I}$. We further require that the image of $\sigma\widehat{I}$ under $M_?$ is contained in $(\mathsf{Ind}\,\mathsf{Rep}(Q)) \sqcup (\Sigma(\mathsf{Ind}\,\mathsf{Rep}(Q)))$. When the context is clear, we simply denote M_x by x and omit the notation of the covering map π.

Notice that the image of the section map $M_?$ is not closed under τ nor Σ.

Example 3.1.1. In Example 2.2.1, we can take $q = \epsilon$ to be a primitive 8th root of unity. Then the vertices $\sigma\widehat{I}$ take the form (i, q^{i+2d}), $i \in I$, $d \in \{0, 1, 2, 3\}$. We can construct the section map from $\sigma\widehat{I}$ to $\mathsf{Ind}\mathcal{D}^b(Q)$ which sends (i, q^{i+2d}) to $\tau^{-d}P_i$ (these are the objects already drawn in Figure 4).

The shift functor Σ induces an automorphism Σ on the set $\mathsf{Ind}\mathcal{D}^b(Q)$. It is inherited by $\sigma\widehat{I}$. We extend this automorphism Σ to $I \times \langle\epsilon\rangle$ by requiring $\Sigma\sigma = \sigma\Sigma$. It follows that $\Sigma^2 = 1$.

Let W^+, V^+, W^S be defined as in (13). We also define

$$\begin{aligned} W^- &= \Sigma^* W^+, \\ V^- &= \Sigma^* V^-, \\ W^{\Sigma S} &= \Sigma^* W^S. \end{aligned} \tag{14}$$

For any $i \in I$, we define

$$\begin{aligned} w^{f_i} &= e_{\sigma S_i} + e_{\sigma\Sigma S_i}, \\ W^0 &= \bigoplus_{i \in I} \mathbb{N} w^{f_i}, \\ v^{f_i} &= \sum_{x \in \sigma\widehat{I}} \dim \mathsf{Hom}_{\mathcal{D}^b(Q)}(S_i, M_x) e_x, \\ v^{\Sigma f_i} &= \Sigma^* v^{f_i}, \\ V^0 &= \bigoplus_{i \in I} (\mathbb{N} v^{f_i} \oplus \mathbb{N} v^{\Sigma f_i}). \end{aligned} \tag{15}$$

Following § 2.2, we consider the Grothendieck group $K = \bigoplus_{w \in W^S \oplus W^{\Sigma S}} K_w$. Its $\mathbb{N}^{\widehat{I}}$-graded dual $R = \bigoplus_{w \in W^S \oplus W^{\Sigma S}} R_w$ has the multiplication $\widetilde{\otimes}$ induced by the comultiplication $\widetilde{\mathrm{Res}}$ of K.

It follows from [VV03] that we have

$$\mathbf{L}(v^1, w^1) \,\widetilde{\otimes}\, \mathbf{L}(v^2, w^2) = \sum_v c^v_{v^1, v^2}(t) \mathbf{L}(v, w^1 + w^2) \tag{16}$$

such that $c^v_{v^1,v^2}(t) \in \mathbb{N}[t^{\pm}]$, $c^v_{v^1,v^2} = 0$ whenever $v < v^1 + v^2$, and $c^{v^1+v^2}_{v^1,v^2} = t^{d((v^2,w^2),(v^1,w^1)) - d((v^1,w^1),(v^2,w^2))}$. The term $c^{v_1+v_2}_{v^1,v^2}(t)\mathbf{L}(v_1 + v_2, w^1 + w^2)$ is called the leading term of the right-hand side of (16).

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PROPOSITION 3.1.2 [HL15, Theorem 7.3]. (1) *$R^+ = \bigoplus_{w\in W^S} R_w$ is the $\mathbb{Z}[t^{\pm}]$-algebra generated by $\{\mathbf{L}(0, e_{\sigma S_i}), i \in I\}$ with respect to the product $\widetilde{\otimes}$.*

(2) *$R^- = \bigoplus_{w\in W^{\Sigma S}} R_w$ is the $\mathbb{Z}[t^{\pm}]$-algebra generated by $\{\mathbf{L}(0, e_{\sigma\Sigma S_i}), i \in I\}$ with respect to the product $\widetilde{\otimes}$.*

We also define R^0 to be the algebra generated by

$$\{\mathbf{L}(v^{f_i}, w^{f_i}), \mathbf{L}(v^{\Sigma f_i}, w^{f_i}), i \in I\}. \tag{17}$$

We will call $\mathbf{L}(0, e_{\sigma S_i})$, $\mathbf{L}(v^{f_i}, w^{f_i})$, $\mathbf{L}(v^{\Sigma f_i}, w^{f_i})$, $\mathbf{L}(0, e_{\sigma\Sigma S_i})$, $i \in I$, the *Chevalley generators* of the Grothendieck ring R.

Remark 3.1.3. The generators $\mathbf{L}(0, e_{\sigma S_i})$, $\mathbf{L}(0, e_{\sigma\Sigma S_i})$ should be compared with the generators $y_{i,0}$, $y_{i,1}$ in [HL15, Theorem 7.3] for derived categories, respectively. We shall show that the relation (R1) in [HL15, Theorem 7.3] holds for our generators. But the relation (R2) does not hold in our case. See Example 3.2.3 for more details.

Let us use $(\,)_{t^{1/2}}$ and $(\,)_{\mathbb{Q}(t^{1/2})}$ to denote the extensions $(\,) \otimes \mathbb{Z}(t^{1/2})$ and $(\,) \otimes \mathbb{Q}(t^{1/2})$, respectively.

Let Φ be the linear map from $\mathbb{N}^{\widehat{I}}$ to the Grothendieck group $K_0(\mathsf{Rep}(Q)) \oplus K_0(\Sigma(\mathsf{Rep}(Q)))$ such that $\Phi(e_{\sigma x}) = x$. For any elements $x = (x^1, x^2)$, $y = (y^1, y^2) \in K_0(\mathsf{Rep}(Q)) \oplus K_0((\Sigma\,\mathsf{Rep}(Q)))$, define the following bilinear forms as combinations of the Euler forms:

$$\langle x, y\rangle_a = \langle x^1, y^1\rangle - \langle y^1, x^1\rangle + \langle x^2, y^2\rangle - \langle y^2, x^2\rangle, \tag{18}$$
$$(x, y) = \langle x^1, y^1\rangle + \langle y^1, x^1\rangle + \langle x^2, y^2\rangle + \langle y^2, x^2\rangle. \tag{19}$$

Following the convention in § 2.2, for any $w = w^1 + w^2$, $\widetilde{\mathrm{Res}}^{w}_{w^1,w^2}$ is a homomorphism from $(K_w)_{t^{1/2}}$ to $(K_w)_{t^{1/2}} \bigotimes_{\mathbb{Z}[t^{\pm(1/2)}]} (K_w)_{t^{1/2}}$. Define[4] its deformation $\mathrm{Res}^{w}_{w^1,w^2}$ to be $\widetilde{\mathrm{Res}}^{w}_{w^1,w^2}\, t^{-(1/2)\langle\Phi(w^1),\Phi(w^2)\rangle_a}$. Then we obtain a (coassociative) comultiplication Res on $K_{t^{1/2}}$ and correspondingly a multiplication $\otimes$ on $R_{t^{1/2}}$. We compare it with the twisted products in [Her04a, HL15, Nak01] in Example 3.2.3.

For any $w \in W^+$, $\Phi(w)$ can be viewed as a $\mathbb{C}Q$-module. We define $\deg\Phi(w)$ to be the total dimension of $\Phi(w)$ and the bilinear form

$$N(\Phi(w)) = (\Phi(w), \Phi(w)) - \deg\Phi(w). \tag{20}$$

Let $B^*_K = \{B^*_K(w) | w \in W^+\}$ denote the *dual canonical basis* of $U_t(\mathfrak{n}^+)$ with respect to Kashiwara's linear form $(\,,)_K$. Define the rescaled dual canonical basis $\widetilde{B}^*_K$ to be $\{\widetilde{B}^*_K(w) | w \in W^+\}$ such that $\widetilde{B}^*_K(w) = t^{(1/2)N(\Phi(w))} B^*_K(w)$.

The following was the main result of [HL15] for graded quiver varieties with a generic choice of q.

THEOREM 3.1.4 (One-half quantum group [HL15, Theorem 6.1]). (1) *There exists an algebra isomorphism $\widetilde{\kappa}$ from the Grothendieck ring $(R^+_{\mathbb{Q}(t^{1/2})}, \otimes)$ to the one-half quantum group $\mathbf{U}_t(\mathfrak{n}^+)_{\mathbb{Q}(t^{1/2})}$ such that*

$$\widetilde{\kappa}\mathbf{L}(0, e_{\sigma S_i}) = E_i \quad \forall i \in I.$$

(2) *This isomorphism identifies the basis $\{\mathbf{L}(v, w), w \in W^S\}$ with the rescaled dual canonical basis $\widetilde{B}^*_K$ such that $\widetilde{\kappa}(\mathbf{L}(v, w)) = \widetilde{B}^*_K(w - C_q v)$.*

[4] This choice of the degree arises from the comparison of the equations in Proposition 4.4.1 with the defining relations of $\widetilde{\mathbf{U}}_t(\mathfrak{g})$. It is also an anti-symmetrized version of the twist used by [Bri13].

THEOREM 3.1.5 (Triangular decomposition). *The ring* $(R, \otimes)$ *(respectively* $(R, \widetilde{\otimes})$*) decomposes into the tensor product of its subalgebras:*

$$R = R^{+} \bigotimes_{\mathbb{Z}[t^{\pm}]} R^{0} \bigotimes_{\mathbb{Z}[t^{\pm}]} R^{-}.$$

THEOREM 3.1.6. (i) *There exists an algebra isomorphism* κ *from* $(R_{\mathbb{Q}(t^{1/2})}, \otimes)$ *to* $\widetilde{\mathbf{U}}_t(\mathfrak{g})_{\mathbb{Q}(t^{1/2})}$ *such that we have* $\widetilde{\mathbf{U}}_t(\mathfrak{n}^+)_{\mathbb{Q}(t^{1/2})} = \kappa R^{+}_{\mathbb{Q}(t^{1/2})}$, $\widetilde{\mathbf{U}}_t(\mathfrak{h})_{\mathbb{Q}(t^{1/2})} = \kappa R^{0}_{\mathbb{Q}(t^{1/2})}$, $\widetilde{\mathbf{U}}_t(\mathfrak{n}^-)_{\mathbb{Q}(t^{1/2})} = \kappa R^{-}_{\mathbb{Q}(t^{1/2})}$, *and, for any* $i \in I$,

$$\begin{aligned}
\kappa(t^{1/2}) &= t^{1/2},\\
\kappa\mathbf{L}(0, e_{\sigma S_i}) &= \frac{1-t^2}{t}E_i,\\
\kappa\mathbf{L}(v^{\Sigma f_i}, w^{f_i}) &= K_i,\\
\kappa\mathbf{L}(v^{f_i}, w^{f_i}) &= K'_i,\\
\kappa\mathbf{L}(0, e_{\sigma\Sigma S_i}) &= \frac{t^2-1}{t}F_i.
\end{aligned}$$

(ii) *Let* I *be the ideal of* $(R, \otimes)$ *generated by the center elements* $\mathbf{L}(v^{f_i} + v^{\Sigma f_i}, 2w^{f_i}) - 1$, $i \in I$*. Then the map* κ *induces an isomorphism between the quotient ring* R/I *and the quantum group* $\mathbf{U}_t(\mathfrak{g})$.

Finally, we consider the dual canonical basis of $\mathbf{U}_t(\mathfrak{n}^+)$ with respect to Lusztig's bilinear form $(\ ,\ )_L$, which is denoted by

$$B^*_L = \{B^*_L(w) | w \in W^+\}.$$

Define the rescaled dual canonical basis to be $\widetilde{B}^*_L = \{\widetilde{B}^*_L(w) | w \in W^+\}$ such that

$$\widetilde{B}^*_L(w) = t^{(1/2)N(\Phi(w)) - \mathsf{deg}\,\Phi(w)} B^*_L(w). \tag{21}$$

It is not obvious to see that on the Grothendieck ring R^+, our twisted product $\otimes$ agrees with the non-commutative multiplication $*$ defined in [HL15], which we will show in the last section. Once we see that they coincide on the subalgebra R^+, [HL15, Theorem 6.1] is translated as the following.

THEOREM 3.1.7 [HL15, Theorem 6.1]. *The isomorphism* κ *identifies* $\{\mathbf{L}(v, w), w \in W^S\}$ *with* $\widetilde{B}^*_L$ *such that* $\kappa\mathbf{L}(v, w) = \widetilde{B}^*_L(w - C_q v)$.

3.2 Examples

Example 3.2.1 (Type $\mathfrak{sl}_2$). Assume that the quiver Q consists of a single point. Then h equals 2. Also, ϵ is a 4th primitive root of unity. The vector space $\mathsf{Rep}^{\epsilon}(Q; v, w)$ for cyclic quiver varieties is given by Figure 6.

The Chevalley generators of the Grothendieck ring $(R, \otimes)$ are given by

$$\mathbf{L}(0, e_{\sigma S}), \quad \mathbf{L}(e_S, e_{\sigma S + \sigma\Sigma S}), \quad \mathbf{L}(e_{\Sigma S}, e_{\sigma S + \sigma\Sigma S}), \quad \mathbf{L}(0, e_{\sigma\Sigma S}).$$

Theorem 3.1.6 identifies $R_{\mathbb{Q}(t^{1/2})}$ with $\widetilde{\mathbf{U}}_t(\mathfrak{sl}_2)_{\mathbb{Q}(t^{1/2})}$ and the above generators with E, K', K, F, respectively.

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$$\begin{tikzcd}
{\color{blue}\text{height} = 1} & {\color{blue}\epsilon} & {\color{blue}\epsilon^2} & {\color{blue}\epsilon^3} & {\color{blue}\epsilon^4 = 1} \\
{\color{blue}W(\sigma S)} & V(S) \arrow[l, blue, "\beta"'] & {\color{blue}W(\sigma\Sigma S)} \arrow[l, red, "\alpha"'] & V(\Sigma S) \arrow[l, blue, "\beta"'] & {\color{blue}W(\sigma S)} \arrow[l, red, "\alpha"']
\end{tikzcd}$$

FIGURE 6. (Colour online) $\mathsf{Rep}^{\epsilon}(Q; v, w)$ for cyclic quiver varieties of type $\mathfrak{sl}_2$.

$$\begin{tikzcd}
{\color{blue}\text{height} = 1} & {\color{blue}\epsilon} & {\color{blue}\epsilon^2} & {\color{blue}\epsilon^3} & {\color{blue}\epsilon^4} & {\color{blue}\epsilon^5} & {\color{blue}\epsilon^6 = 1} \\
V(\Sigma S_2) & {\color{blue}W(\sigma P_2)} \arrow[l, red, "\alpha_2"'] & V(P_2) \arrow[l, blue, "\beta_2"'] \arrow[dl, "h"] & {\color{blue}W(\sigma\Sigma S_1)} \arrow[l, red, "\alpha_2"'] & V(\Sigma S_1) \arrow[l, blue, "\beta_2"'] \arrow[dl, "h"] & {\color{blue}W(\sigma\Sigma S_2)} \arrow[l, red, "\alpha_2"'] & V(\Sigma S_2) \arrow[l, blue, "\beta_2"'] \arrow[dl, "h"] \\
{\color{blue}W(\sigma S_1)} & V(S_1) \arrow[l, blue, "\beta_1"'] \arrow[ul, "\overline{h}"'] & {\color{blue}W(\sigma S_2)} \arrow[l, red, "\alpha_1"'] & V(S_2) \arrow[l, blue, "\beta_1"'] \arrow[ul, "\overline{h}"'] & {\color{blue}W(\sigma\Sigma P_2)} \arrow[l, red, "\alpha_1"'] & V(\Sigma P_2) \arrow[l, blue, "\beta_1"'] \arrow[ul, "\overline{h}"'] & {\color{blue}W(\sigma S_1)} \arrow[l, red, "\alpha_1"']
\end{tikzcd}$$

FIGURE 7. (Colour online) $\mathsf{Rep}^{\epsilon}(Q; v, w)$ for cyclic quiver varieties of type $\mathfrak{sl}_3$.

Example 3.2.2 (Type $\mathfrak{sl}_3$). Assume that the quiver Q takes the form $(2 \xrightarrow{h} 1)$. Then h equals 3. Also, ϵ is a 6th root of unity. The vector space $\mathsf{Rep}^{\epsilon}(Q; v, w)$ for cyclic quiver varieties is given by Figure 7.

The Chevalley generators of the Grothendieck ring $(R, \otimes)$ are given by

$$\begin{gathered}
\mathbf{L}(0, e_{\sigma S_i}), \quad i = 1, 2, \\
\mathbf{L}(e_{S_1} + e_{P_2}, e_{\sigma S_1 + \sigma\Sigma S_1}), \quad \mathbf{L}(e_{S_2} + e_{\Sigma S_1}, e_{\sigma S_2 + \sigma\Sigma S_2}), \\
\mathbf{L}(e_{\Sigma S_1} + e_{\Sigma P_2}, e_{\sigma S_1 + \sigma\Sigma S_1}), \quad \mathbf{L}(e_{\Sigma S_2} + e_{S_1}, e_{\sigma S_2 + \sigma\Sigma S_2}), \\
\mathbf{L}(0, e_{\sigma\Sigma S_i}), \quad i = 1, 2.
\end{gathered}$$

Theorem 3.1.6 identifies $R_{\mathbb{Q}(t^{1/2})}$ with $\widetilde{\mathbf{U}}_t(\mathfrak{sl}_3)_{\mathbb{Q}(t^{1/2})}$ and the above generators with E_i, K'_1, K'_2, K_1, K_2, F_i, respectively.

Example 3.2.3. We continue Example 3.2.2 and compare various twisted products.

Let us take Chevalley generators from the positive part and the negative part of the quantum group, respectively. Our twisted product $\otimes$ satisfies

$$\mathbf{L}(0, e_{\sigma S_1}) \otimes \mathbf{L}(0, e_{\sigma\Sigma S_2}) = \mathbf{L}(0, e_{\sigma\Sigma S_2}) \otimes \mathbf{L}(0, e_{\sigma S_1}).$$

On the other hand, the twisted product in [HL15, Theorem 7.3(R2)] and [Her04a], if defined over the pairs (v, w), would demand the following relation:

$$\mathbf{L}(0, e_{\sigma S_1}) \otimes \mathbf{L}(0, e_{\sigma\Sigma S_2}) = t^{-(\alpha_1, \alpha_2)} \mathbf{L}(0, e_{\sigma\Sigma S_2}) \otimes \mathbf{L}(0, e_{\sigma S_1}).$$

Therefore, $\otimes$ is not the same as the product used in [HL15, Her04a].

Also, $\otimes$ is not the twisted product in [Nak04] either. Recall that our geometrical restriction functor is twisted by the Euler form in (18). But [Nak04] twisted the geometrical restriction functor by a different bilinear form d_W for generic q, which takes different values. For q a root of unity, [Nak04] (cf. also [Her04b, Theorem 3.5]) used the different twisted product $\widetilde{\otimes}$ associated

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with the anti-symmetrized version of the bilinear form $d(\,,\,)$, which gives us

$$\begin{aligned} &d((0,e_{\sigma S_2}),(0,e_{\sigma S_1})) - d((0,e_{\sigma S_1}),(0,e_{\sigma S_2})) = 0,\\ &\qquad\qquad \mathbf{L}(0,e_{\sigma S_1})\,\widetilde{\otimes}\,\mathbf{L}(0,e_{\sigma S_2}) = t^0\mathbf{L}(0,e_{\sigma S_2}+e_{\sigma S_1}) + \text{other terms.}\end{aligned}$$

On the other hand, our product $\otimes$ would twist the above leading t-power by $t^{-(1/2)\langle S_1,S_2\rangle}$.

In fact, our twisted product $\otimes$ is defined for the pair (v,w), where $v\in\mathbb{N}^{\sigma\widehat{I}}$, $w\in\mathbb{N}^{\widehat{I}}$, while the twisted products in [Nak04] (for generic q) and [HL15, Her04a] are defined over the dimension vectors w. In order to compare our $\otimes$ with the latter two products, we have to reduce the pair (v,w) to the dimension $w-C_qv$. This would demand the Cartan elements $\mathbf{L}(e_{S_1}+e_{P_2},e_{\sigma S_1+\sigma\Sigma S_1})$, $\mathbf{L}(e_{S_2}+e_{\Sigma S_1},e_{\sigma S_2+\sigma\Sigma S_2})$, $\mathbf{L}(e_{\Sigma S_1}+e_{\Sigma P_2},e_{\sigma S_1+\sigma\Sigma S_1})$, $\mathbf{L}(e_{\Sigma S_2}+e_{S_1},e_{\sigma S_2+\sigma\Sigma S_2})$ to be center with respect to $\otimes$, which is not true. The author does not know any non-trivial twisted product defined over cyclic quiver varieties such that these Cartan elements become center elements. The incompatibility of our product $\otimes$ and the twisted products in [Nak04] (for generic q) and [HL15, Her04a] could be expected, because abelian categories of 2-periodic complexes are not subcategories of derived categories.

Nevertheless, the restriction of the twisted product $\otimes$ on $\mathbf{L}(0,e_{\sigma S_i})$, $\mathbf{L}(0,e_{\sigma S_j})$, $i,j=1,2$, agrees with that of [HL15, Theorem 7.3(R1)] for the corresponding elements; cf. § 5.

4. Proofs

For simplicity, we shall often denote $\mathsf{Hom}_{\mathcal{D}^b(Q)}(\,,\,)$ by $\mathsf{Hom}(\,,\,)$.

4.1 *l*-dominant pairs

Lemma 4.1.1. *For any $x\in\mathsf{Ind}\,\mathsf{Rep}(Q)$, $y\in\sigma\widehat{I}$, we have*

$$\mathsf{Hom}_{\mathcal{D}^b(Q)}(x,\tau M_y) = \mathsf{Hom}_{\mathcal{D}^b(Q)}(x,M_{\tau y}).$$

Proof. Notice that $\tau^h=1$ and $\sigma\widehat{I}$ is identified with $\mathsf{Ind}\,\mathsf{Rep}(Q)\sqcup\Sigma\mathsf{Ind}\,\mathsf{Rep}(Q)$. The statement obviously holds if M_y is not a projective $\mathbb{C}Q$-module. On the other hand, assume that M_y is a projective $\mathbb{C}Q$-module. Then $\Sigma^{-1}M_{\tau y}$ is an injective $\mathbb{C}Q$-module and both sides vanish. □

Lemma 4.1.2. *For any $i\in I$, $w^{f_i}-C_qv^{f_i}$ vanishes.*

Proof. For any $x\in\sigma\widehat{I}$, we have an almost split triangle in $\mathcal{D}^b(Q)$

$$\tau M_x\to E\to M_x\to\Sigma\tau M_x.$$

For simplicity, we denote $\mathsf{Hom}_{\mathcal{D}^b(Q)}(\,,\,)$ by $\mathsf{Hom}(\,,\,)$. Applying the functor $\mathsf{Hom}(S_i,\,)$ to this triangle, we get a long exact sequence

$$\begin{aligned}&\mathsf{Hom}(S_i,\Sigma^{-1}M_x)\xrightarrow{\omega^1}\mathsf{Hom}(S_i,\tau M_x)\\ &\qquad\qquad\to\mathsf{Hom}(S_i,E)\to\mathsf{Hom}(S_i,M_x)\xrightarrow{\omega^2}\mathsf{Hom}(S_i,\Sigma\tau M_x).\end{aligned}$$

By Lemma 4.1.1 and (15), the coordinate of $C_qv^{f_i}$ at the vertex σx is

$$(C_qv^{f_i})_{\sigma x} = \dim\mathsf{Hom}(S_i,\tau M_x) - \dim\mathsf{Hom}(S_i,E) + \dim\mathsf{Hom}(S_i,M).$$

(i) Assume $x\in\mathsf{Ind}\,\mathsf{Rep}(Q)$. Then $\mathsf{Hom}(S_i,\Sigma^{-1}M_x)$ vanishes.

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If $x \neq S_i$, we get $\omega^2 = 0$ by the universal property of Auslander–Reiten triangles and consequently

$$(C_q v^{f_i})_{\sigma x} = \dim \mathsf{Hom}(S_i, \tau M_x) - \dim \mathsf{Hom}(S_i, E) + \dim \mathsf{Hom}(S_i, M) = 0.$$

For $x = S_i$, we get $\mathsf{Ker}\, w^2 = 0$ and

$$(C_q v^{f_i})_{\sigma S_i} = \dim \mathsf{Hom}(S_i, S_i) = 1.$$

(ii) Assume $x \in \Sigma \mathsf{Ind}\, \mathsf{Rep}(Q)$. Then $\mathsf{Hom}(S_i, \Sigma\tau M_x)$ vanishes.

If $x \neq \Sigma S_i$, we get $\omega^1 = 0$ by the universal property of Auslander–Reiten triangles and consequently

$$(C_q v^{f_i})_{\sigma x} = \dim \mathsf{Hom}(S_i, \tau M_x) - \dim \mathsf{Hom}(S_i, E) + \dim \mathsf{Hom}(S_i, M) = 0.$$

For $x = \Sigma S_i$, we get $\mathsf{Cok}\, w^1 = 0$ and consequently

$$(C_q v^{f_i})_{\sigma \Sigma S_i} = \dim \mathsf{Hom}(S_i, \tau \Sigma S_i) = 1. \qquad \square$$

4.2 Proof of the one-half quantum group

We prove Proposition 4.2.1 in this subsection, which tells us that the study of cyclic quiver varieties $\mathcal{M}_0{}^{\epsilon}(w)$, $w \in W^+$, can be reduced to the study of the graded quiver varieties $\mathcal{M}_0{}^{\bullet}(w)$ for a generic choice of q. More precisely, we show that these cyclic quiver varieties are free of 'wrapping paths'; cf. Example 4.2.2.

Proposition 4.2.1 allows us to translate the results obtained in [HL15, LP13] for the latter varieties into Proposition 3.1.2 and Theorem 3.1.4. For completeness, we give a sketch of the proofs.

PROPOSITION 4.2.1. *For any $w \in W^+$, the cyclic quiver variety $\mathcal{M}_0{}^{\epsilon}(w)$ is isomorphic to the graded quiver variety $\mathcal{M}_0{}^{\bullet}(w)$.*

We have studied the graded quiver varieties $\mathcal{M}_0{}^{\bullet}(w)$ for $w \in w^+$. By Proposition 4.2.1, their results can be used for the cyclic quiver variety $\mathcal{M}_0{}^{\epsilon}(w)$, $w \in W^+$.

We give an example to show how a 'wrapping path' vanishes.

Example 4.2.2. Let us look at Figure 7. For any given $w \in W^+$, take any composition of irreducible morphisms which only passes through the vertices $x \in \sigma\widehat{I}$ or σS_i, $i \in I$, with the ending points of the type σS_i.

For example, we can take a composition p such that the sequence of the vertices it passes through is $(\sigma S_2, S_1, \sigma S_1, \Sigma P_2, \Sigma S_1, S_2, \sigma S_2)$. Then p horizontally wraps the figure, in the sense that the heights of these vertices occupy the whole cyclic group $\langle \epsilon \rangle$.

Take the factor p' of p corresponding to the subsequence $(\Sigma P_2, \Sigma S_1, S_2)$. Because of the relation $\mu = 0$ and since w is concentrated on σS_i, $i \in I$, p' corresponds to a morphism from ΣP_2 to S_2 in $(\mathcal{D}^b(Q))^{op}$. But such a morphism must vanish. Therefore, p' and p vanish.

Proof of Proposition 4.2.1. We shall use the notions of Nakajima categories in the sense of [KS13]. Let $\mathcal{R}$ be the mesh category associated with a generic $q \in \mathbb{C}^*$ and $\mathcal{R}^{\epsilon}$ the mesh category associated with ϵ. Let $\mathcal{S}$ denote the singular Nakajima category which is generated by the objects σx, $x \in \mathsf{Ind}\,\mathsf{Rep}(Q)$, in the mesh category $\mathcal{R}$. Similarly, define the singular Nakajima category $\mathcal{S}^{\epsilon}$ as a subcategory of $\mathcal{R}^{\epsilon}$.

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By our convention, an $\mathcal{S}$-module is a functor from $\mathcal{S}$ to the category of complex vector spaces. Then the variety of w-dimensional $\mathcal{S}$-modules, denoted by $\mathsf{Rep}(\mathcal{S}, w)$, is isomorphic to $\mathcal{M}_0{}^\bullet(w)$, cf. [KS13, LP13], and $\mathcal{M}_0{}^\epsilon(w)$ is a closed subvariety in $\mathsf{Rep}(\mathcal{S}^\epsilon, w)$. Notice that we can naturally embed $\mathcal{M}_0{}^\bullet(w)$ into $\mathcal{M}_0{}^\epsilon(w)$. Therefore, to verify the proposition, it suffices to show that $\mathcal{S}$ and $\mathcal{S}^\epsilon$ are equivalent. Its proof consists of the following two steps.

(i) For any two modules $x, y \in \mathsf{Ind}\,\mathsf{Rep}(Q)$, let p be any composition of irreducible morphisms in $\mathcal{R}^\epsilon$ such that p starts from σx, ends at σy, and does not pass any object $\sigma z \in \mathcal{R}^\epsilon$ with $z \in \Sigma\mathsf{Ind}\,\mathsf{Rep}(Q)$ in its definition. Let $(\underline{i}, \underline{a}) = ((i_0, a_0), \ldots, (i_r, a_r))$ denote the sequence of the objects that p passes through, where $(i_0, a_0) = \sigma x$, $(i_r, a_r) = \sigma y$, $r \in \mathbb{N}$. Notice that our convention of $\mathcal{R}^\epsilon$ implies $a_{t+1} = a_t * \epsilon^{-1}$ for all $0 \leqslant t \leqslant r-1$. By abuse of notation, let $\underline{a}$ also denote the set $\{a_t | 0 \leqslant t \leqslant r\}$.

Assume that the sequence $(\underline{i}, \underline{a})$ contains some object outside $\mathsf{Ind}\,\mathsf{Rep}(Q)$. We want to show that the morphism p factors through some object in $\sigma\Sigma\mathsf{Ind}\,\mathsf{Rep}(Q)$.

First, notice that the sequence $(\underline{i}, \underline{a})$ must contain a consecutive subsequence $(\underline{i}', \underline{a}')$ from $\Sigma x'$ to y', where x', y' are some indecomposable injective $\mathsf{Rep}(Q)$. We can require $(\underline{i}', \underline{a}')$ to be small in the sense that $\underline{a}' \neq \langle\epsilon\rangle$. The factor of p associated with the small subsequence $(\underline{i}', \underline{a}')$ is denoted by p'.

Define the subcategory $\mathcal{X}^\epsilon$ of $\mathcal{R}^\epsilon$ such that its set of objects is $\{I_i, i \in I\} \sqcup \Sigma\mathsf{Ind}\,\mathsf{Rep}(Q) \sqcup \sigma\Sigma\mathsf{Ind}\,\mathsf{Rep}(Q)$ and its morphisms are generated by the irreducible morphism among these objects in the mesh category $\mathcal{R}^\epsilon$. Define the subcategory $\mathcal{X}$ of $\mathcal{R}$ similarly. By comparing the mesh relations, we see that the two subcategories are equivalent. Associate to these categories their quotients $\underline{\mathcal{X}}$ and $\underline{\mathcal{X}^\epsilon}$ by sending all the morphisms factoring through $\sigma\Sigma\mathsf{Ind}\,\mathsf{Rep}(Q)$ to 0. Then the quotient categories are still equivalent.

Notice that $\underline{\mathcal{X}}$ is equivalent to a subcategory of $(\mathcal{D}^b(Q))^{op}$. Therefore, all morphisms in $\underline{\mathcal{X}}$ from $\Sigma x'$ to y', $x', y' \in \{I_i, i \in I\}$, vanish. Because the subsequence $(\underline{i}', \underline{a}')$ is small, the morphism p' is well defined on $\mathcal{X}^\epsilon$. It follows that $p' = 0$ in $\underline{\mathcal{X}^\epsilon}$. Therefore, in the category $\mathcal{R}^\epsilon$, p' and p factor through the objects of $\sigma\Sigma\mathsf{Ind}\,\mathsf{Rep}(Q)$.

(ii) By (i), we deduce that the singular category $\mathcal{S}^\epsilon$ is the subcategory of $\mathcal{R}^\epsilon$ whose set of objects is $\sigma\widehat{I}$ and whose morphisms are linear combinations of compositions of the irreducible maps among the elements in $\sigma\widehat{I} \cup \mathsf{ind}\,\mathbb{C}Q$. By comparing the mesh relations, we see that $\mathcal{S}^\epsilon$ is equivalent to $\mathcal{S}$. $\square$

Proposition 4.2.3 [HL15, LP13]. *For any $w \in W^S$, $\mathcal{M}_0{}^\bullet(w)$ is isomorphic to $\mathsf{Rep}(Q, \sum_i w_{\sigma S_i} e_i)$. Moreover, the non-empty regular strata are in bijection with the orbits of $\mathsf{Rep}(Q, \sum_i w_{\sigma S_i} e_i)$ and, consequently, in bijection with the dual canonical basis elements of $U_t(\mathfrak{n}^+)$ with the homogeneous degree $\sum_i w_{\sigma S_i}\alpha_i$.*

Proof of Proposition 3.1.2 and Theorem 3.1.4. (i) We first prove Proposition 3.1.2(1) and Theorem 3.1.4(1).

Notice that R^+ is generated by $\mathbf{L}(0, e_{\sigma S_i})$, $i \in I$. We will show that these generators satisfy the quantum Serre relations in Proposition 4.4.5. Then the identification between the Chevalley generators induces a surjective map from $\mathbf{U}_t(\mathfrak{n}^+)_{\mathbb{Q}(t^{1/2})}$ to $R^+_{\mathbb{Q}(t^{1/2})}$. It remains to check that the two W^S-graded algebras have the same graded dimension, which follows from Propositions 4.2.1 and 4.2.3.

(ii) The proof of Proposition 3.1.2(2) is the same as in (i).

(iii) The claim of Theorem 3.1.4(2) is a consequence of Theorem 3.1.4(1). More details can be found in the proof of [HL15, Theorem 6.1(2)]. $\square$

Remark 4.2.4. One can also obtain Theorem 3.1.4 by identifying $\otimes$ on the subalgebra R^+ with the multiplication of the deformed Grothendieck ring in [Her04a, HL15] and applying the result of [HL15]. The identification will be discussed in the last section.

4.3 Proof of the triangular decomposition

Denote the extension $(\,)\otimes_{\mathbb{N}}\mathbb{Z}$ by $(\,)_{\mathbb{Z}}$.

For any $i, j \in I$, we have

$$\begin{aligned}(v^{f_i})_{I_j} &= (v^{\Sigma f_i})_{\Sigma I_j} = \delta_{ij},\\ (v^{f_i})_{\Sigma I_j} &= (v^{\Sigma f_i})_{I_j} = 0.\end{aligned} \tag{22}$$

The following lemma follows as a consequence.

LEMMA 4.3.1. *$\mathbb{N}^{\sigma\widehat{I}}$ is a subset of $V^+_{\mathbb{Z}} \oplus V^0 \oplus V^-_{\mathbb{Z}}$.*

Proof. For any $v \in \mathbb{N}^{\sigma\widehat{I}}$, we define $v^0 = \sum b_i v^{f_i} + \sum b'_i v^{\Sigma f_i}$ such that $b_i = v(I_i)$ and $b'_i = v(\Sigma I_i)$. Define v^+ to be the restriction of $v - v^0$ on $V^+_{\mathbb{Z}}$ and v^- the restriction of $v - v^0$ on $V^-_{\mathbb{Z}}$. Equation (22) guarantees that $v = v^+ + v^0 + v^-$ is our desired decomposition. □

Denote the projections of $\mathbb{N}^{\sigma\widehat{I}}$ to the three summands in Lemma 4.3.1 by pr^+, pr^0, pr^-, respectively. The following result is essentially known by [LP13].

PROPOSITION 4.3.2. *For any $w \in W^S_{\mathbb{Z}}$, $v \in V^+_{\mathbb{Z}}$, if $w - C_q v \geqslant 0$, then $v \in V^+$, $w \in W^S$.*

Proof. To verify the statement, by Proposition 4.2.1, we can work in the case where q is not a root of unity instead. We then prove it by using Theorem 2.3.2 and [LP13, Theorem 3.14].

Let $\widehat{A}$ denote the repetitive algebra of $A = \mathbb{C}Q$. By using Syzygy functors in $\mathsf{mod}\,\widehat{A}$ (the category of left $\widehat{A}$-modules), we can identify the sets W^S, V^+ with subsets of $\mathbb{N}^{\psi(\mathsf{proj}\,\widehat{A})}$, $\mathbb{N}^{\psi(\mathsf{Ind}\,\mathsf{mod}\,\widehat{A}-\mathsf{proj}\,\widehat{A})}$ studied in [LP13, § 3.1]; cf. [LP13, Remark 3.17]. From now on, we work in the context of [LP13].

Denote $\widetilde{w} = w - C_q v$. By Theorem 2.3.2, there exists a unique l-dominant pair (v', w'), $v' \in V^+$, $w' \in W^S$, such that $w' - C_q v' = \widetilde{w}$.

To any $\widehat{A}$-module N of dimension w^N, we associate the module $\underline{N}$ defined in [LP13, Lemma 3.12]. Its dimension will be denoted by $(v^N, w^N) \in \mathbb{N}^{\psi(\mathsf{Ind}\,\mathsf{mod}\,\widehat{A}-\mathsf{proj}\,\widehat{A})} \times \mathbb{N}^{\psi(\mathsf{proj}\,\widehat{A})}$. Moreover, the pair (v^N, w^N) is l-dominant. Notice that we always have $w^P - C_q v^P = 0$ for any projective P in $\mathsf{proj}\,\widehat{A}$. Let us take some projective P with its dimension big enough such that $v + v^P \geqslant 0$, $w + w^P \geqslant 0$. Then $(v + v^P, w + w^P)$ is an l-dominant pair. By [LP13, Theorem 3.14], it determines the isoclass of an $\widehat{A}$ module N of dimension w^N such that $(v^N, w^N) = (v + v^P, w + w^P)$.

Since both w' and w^N are contained in $\mathbb{N}^{\psi(\mathsf{proj}\,\widehat{A})}$ and $w' - C_q v' = w^N - C_q v^N$, by [LP13, § 4.3], there exist some projective $\widehat{A}$-modules P^1, P^2 such that $(v' + v^{P^1}, w' + w^{P^1}) = (v^N + v^{P^2}, w^N + w^{P^2})$. Then we have $(v', w') = (v + v^P + v^{P^2} - v^{P^1}, w + w^P + w^{P^2} - w^{P^1})$. Notice that $w^P + w^{P^2} - w^{P^1}$ is not contained in $W^S \otimes \mathbb{Z}$ unless it vanishes (in other words, $P \oplus P^2 = P^1$). Because w' and w are contained in $W^S \otimes \mathbb{Z}$, it follows that $w^P + w^{P^2} - w^{P^1}$ vanishes. Therefore, we obtain $(v', w') = (v, w)$. □

PROPOSITION 4.3.3. *For any $w \in W^S \oplus W^{\Sigma S}$, $v \in \mathbb{N}^{\sigma\widehat{I}}$, assume that the pair (v, w) is l-dominant; then we have a unique decomposition of (v, w) into l-dominant pairs (v^+, w^+), (v^0, w^0), (v^-, w^-) such that $v^+ \in V^+$, $v^0 \in V^0$, $v^- \in V^-$, $w^+ \in W^S$, $w^0 \in W^0$, $w^- \in W^{\Sigma S}$, $v^+ + v^0 + v^- = v$, $w^+ + w^0 + w^- = w$, and $w^0 - C_q v^0 = 0$.*

Proof. Denote the projections of $\mathbb{Z}^{\widehat{I}}$ onto $W^+_{\mathbb{Z}}$ and $W^-_{\mathbb{Z}}$ by π^+ and π^-, respectively.

We first take $v^0 = \mathsf{pr}^0 V^0$ and denote the natural decomposition of v^0 in V^0 by $v^0 = \sum_i b_i v^{f_i} + \sum_i b'_i v^{\Sigma f_i}$. Further define $w^0 = \sum_i (b_i + b'_i) w^{f_i}$. Define $v^+ = \mathsf{pr}^+ v$, $v^- = \mathsf{pr}^- v$.

We have $\pi^+ C_q v^- = 0$. Therefore, $C_q v^+$ equals $\pi^+ C_q(v - v^0)$. Because $w^0 - C_q v^0 = 0$, we have $\pi^+(w - w^0) - C_q v^+ = \pi^+(w - w^0) - \pi^+ C_q(v - v^0) = \pi^+(w - C_q v) \geqslant 0$. By Proposition 4.3.2, $(v^+, \pi^+(w - w^0))$ is an l-dominant pair with $v^+ \in V^+$, $\pi^+(w - w^0) \in W^S$. Similarly, we obtain the l-dominant pair $(v^-, \pi^-(w - w^0))$ with $v^- \in V^-$, $\pi^-(w - w^0) \in W^{\Sigma S}$.

Define $w^+ = \pi^+(w - w^0)$ and $w^- = \pi^-(w - w^0)$. Then the decomposition $(v, w) = (v^+, w^+) + (v^0, w^0) + (v^-, w^-)$ satisfies the conditions we impose.

Finally, let us prove the uniqueness. Lemma 4.3.1 implies that the decomposition $v = v^+ + v^0 + v^-$ is unique. Then w^0 is determined by v^0. It follows that the decomposition $w = w^+ + w^0 + w^-$ is unique. □

Proof of Theorem 3.1.5. As a consequence of Proposition 4.3.3, cf. also Proposition 2.2.2, for any $w \in W^S \oplus W^{\Sigma S}$, there exist finitely many v such that (v, w) is l-dominant. Combining Proposition 4.3.3 and (16), we get Theorem 3.1.5 by induction on v. □

4.4 Proof of the main results

We explicitly calculate some relations of the generators of $(R, \widetilde{\otimes})$.

Proposition 4.4.1. *For any $i, j \in I$, we have*

$$\mathbf{L}(0, e_{\sigma S_i}) \,\widetilde{\otimes}\, \mathbf{L}(v^{f_j}, w^{f_j}) = t^{2\langle S_i, S_j\rangle} \mathbf{L}(v^{f_j}, w^{f_j}) \,\widetilde{\otimes}\, \mathbf{L}(0, e_{\sigma S_i}), \tag{23}$$
$$\mathbf{L}(0, e_{\sigma S_i}) \,\widetilde{\otimes}\, \mathbf{L}(v^{\Sigma f_j}, w^{f_j}) = t^{-2\langle S_j, S_i\rangle} \mathbf{L}(v^{\Sigma f_j}, w^{f_j}) \,\widetilde{\otimes}\, \mathbf{L}(0, e_{\sigma S_i},) \tag{24}$$
$$\mathbf{L}(0, e_{\sigma \Sigma S_i}) \,\widetilde{\otimes}\, \mathbf{L}(v^{f_j}, w^{f_j}) = t^{-2\langle S_j, S_i\rangle} \mathbf{L}(v^{f_j}, w^{f_j}) \,\widetilde{\otimes}\, \mathbf{L}(0, e_{\sigma \Sigma S_i}), \tag{25}$$
$$\mathbf{L}(0, e_{\sigma \Sigma S_i}) \,\widetilde{\otimes}\, \mathbf{L}(v^{\Sigma f_j}, w^{f_j}) = t^{2\langle S_i, S_j\rangle} \mathbf{L}(v^{\Sigma f_j}, w^{f_j}) \,\widetilde{\otimes}\, \mathbf{L}(0, e_{\sigma \Sigma S_i}). \tag{26}$$

Proof. We shall further prove that for each relation, either side consists of only the leading term[5] when it decomposes via (16).

(i) We start by verifying the first relation.

By Proposition 4.3.3, any l-dominant pair $(v, e_{\sigma S_i} + w^{f_j})$ decomposes into the sum of three l-dominant pairs (v^+, w^+), (v^0, w^0), (v^-, w^-). Applying (16) to the left-hand side of the first relation, we see that the term $\mathbf{L}(v, e_{\sigma S_i} + w^{f_j})$ has non-zero coefficients only if $v \geqslant v^{f_j}$. So, we obtain $v^0 \geqslant v^{f_j}$ and, consequently, $w^0 \geqslant w^{f_j}$. It follows that the only possible decomposition of w is $w^+ = e_{\sigma S_i}$, $w^0 = w^{f_j}$, $w^- = 0$. Consequently, v has only one possible decomposition: $v^+ = v^- = 0$, $v^0 = v^{f_j}$. Therefore, both sides are just multiples of the leading term $\mathbf{L}(v^{f_j}, e_{\sigma S_i} + w^{f_j})$.

The claim follows from the calculation of the t-power for the coefficients of the leading terms:

$$\begin{aligned} d((0, e_{\sigma S_i}), (v^{f_j}, w^{f_j})) &= e_{\sigma S_i} * \sigma^* v^{f_j} = v^{f_j}(\sigma^2 S_i) = \dim \mathsf{Hom}_{\mathcal{D}^b(Q)}(S_j, \tau S_i) \\ &= \dim \mathsf{Hom}_{\mathcal{D}^b(Q)}(S_i, \Sigma S_j), \\ d((v^{f_j}, w^{f_j}), (0, e_{\sigma S_i})) &= v^{f_j} * \sigma^* e_{\sigma S_i} = v^{f_j} * e_{S_i} \\ &= \dim \mathsf{Hom}_{\mathcal{D}^b(Q)}(S_j, S_i) = \dim \mathsf{Hom}_{\mathcal{D}^b(Q)}(S_i, S_j). \end{aligned}$$

(ii) The verification of the second relation is similar.

By Proposition 4.3.3, any l-dominant pair $(v, e_{\sigma S_i} + w^{f_j})$ decomposes into the sum of three l-dominant pairs (v^+, w^+), (v^0, w^0), (v^-, w^-). Applying (16) to the left-hand side of the second

[5] This situation is usually called *special* or *affine-minuscule*.

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relation, we see that the term $\mathbf{L}(v, e_{\sigma S_i} + w^{f_j})$ has non-zero coefficients only if $v \geqslant v^{\Sigma f_j}$. So, we obtain $v^0 \geqslant v^{\Sigma f_j}$ and, consequently, $w^0 \geqslant w^{f_j}$. It follows that the only possible decomposition of w is $w^+ = e_{\sigma S_i}$, $w^0 = w^{f_j}$, $w^- = 0$. Consequently, v has only one possible decomposition: $v^+ = v^- = 0$, $v^0 = v^{\Sigma f_j}$. Therefore, both sides are just multiples of the leading term $\mathbf{L}(v^{\Sigma f_j}, e_{\sigma S_i} + w^{f_j})$.

The claim follows from the calculation of the t-power for the coefficients of the leading terms:

$$\begin{aligned} d((0, e_{\sigma S_i}), (v^{\Sigma f_j}, w^{f_j})) &= v^{f_j}(\Sigma\tau S_i) \\ &= \dim \mathsf{Hom}_{\mathcal{D}^b(Q)}(S_i, S_j) = \dim \mathsf{Hom}_{\mathcal{D}^b(Q)}(S_i, S_j), \\ d((v^{\Sigma f_j}, w^{f_j}), (0, e_{\sigma S_i})) &= \Sigma^* v^{f_j} * \sigma^* e_{\sigma S_i} \\ &= \dim \mathsf{Hom}_{\mathcal{D}^b(Q)}(S_j, \Sigma S_i). \end{aligned}$$

(iii), (iv) The automorphism Σ^* on the dimension vectors v, w induces isomorphisms of cyclic quiver varieties, which are compatible with the (twisted) restriction functors $\widetilde{\mathrm{Res}}^w_{w^1,w^2}$, $\mathrm{Res}^w_{w^1,w^2}$, as well as the bilinear form $d(\,,)$. Therefore, the first two relations imply

$$\begin{aligned} \mathbf{L}(0, \Sigma^* e_{\sigma S_i}) \,\widetilde{\otimes}\, \mathbf{L}(\Sigma^* v^{f_j}, \Sigma^* w^{f_j}) &= t^{2\langle S_i, S_j\rangle} \mathbf{L}(\Sigma^* v^{f_j}, \Sigma^* w^{f_j}) \,\widetilde{\otimes}\, \mathbf{L}(0, \Sigma^* e_{\sigma S_i}), \\ \mathbf{L}(0, \Sigma^* e_{\sigma S_i}) \,\widetilde{\otimes}\, \mathbf{L}(\Sigma^* v^{\Sigma f_j}, \Sigma^* w^{f_j}) &= t^{-2\langle S_j, S_i\rangle} \mathbf{L}(\Sigma^* v^{\Sigma f_j}, \Sigma^* w^{f_j}) \,\widetilde{\otimes}\, \mathbf{L}(0, \Sigma^* e_{\sigma S_i}). \end{aligned}$$

These are just the fourth and third relations, respectively. □

Proposition 4.4.2. *In $(R, \widetilde{\otimes})$, for any $i, j \in I$, we have*

$$[\mathbf{L}(0, e_{\sigma S_i}), \mathbf{L}(0, e_{\sigma\Sigma S_j})] = \delta_{ij}(t - t^{-1})(\mathbf{L}(v^{f_i}, w^{f_i}) - \mathbf{L}(v^{\Sigma f_i}, w^{\Sigma f_i})). \tag{27}$$

Proof. (i) Assume $i \neq j$. We deduce from Proposition 4.3.3 that the only l-dominant pair $(v, e_{\sigma S_i} + e_{\sigma\Sigma S_j})$ is given by $v = 0$. Let us calculate the bilinear forms:

$$\begin{aligned} d((0, e_{\sigma S_i}), (0, e_{\sigma\Sigma S_j})) &= 0, \\ d((0, e_{\sigma\Sigma S_j}), (0, e_{\sigma S_i})) &= 0. \end{aligned}$$

The statement follows.

(ii) Assume $i = j$. By Proposition 4.3.3, the only l-dominant pairs (v, w^{f_i}) are given by $v = 0, v^{f_i}, v^{\Sigma f_i}$. Then we have

$$\begin{aligned} \pi(0, e_{\sigma S_i}) &= \mathcal{L}(0, e_{\sigma S_i}), \\ \pi(0, e_{\sigma\Sigma S_i}) &= \mathcal{L}(0, e_{\sigma\Sigma S_i}), \\ \pi(0, w^{f_i}) &= \mathcal{L}(0, w^{f_i}), \\ \pi(v^{f_i}, w^{f_i}) &= \mathcal{L}(v^{f_i}, w^{f_i}) + a_{v^{f_i},0;w^{f_i}} \mathcal{L}(0, w^{f_i}), \qquad (28) \\ \pi(v^{\Sigma f_i}, w^{f_i}) &= \mathcal{L}(v^{\Sigma f_i}, w^{f_i}) + a_{v^{\Sigma f_i},0;w^{f_i}} \mathcal{L}(0, w^{f_i}). \qquad (29) \end{aligned}$$

Notice that, by the definition of the GIT quotient and the dimension vector v^{f_i}, the GIT quotient $\mathcal{M}^\epsilon(v^{f_i}, e_{\sigma S_i})$ is a point; cf. Example 4.4.3. Therefore, it is isomorphic to the variety $\mathcal{M}_0{}^\epsilon(e_{\sigma S_i}) = \mathcal{M}_0{}^\epsilon(0, e_{\sigma S_i})$. So, we get $\pi(v^{f_i}, e_{\sigma S_i}) = 1_{\{0\}} = \mathcal{L}(0, e_{\sigma S_i})$. Similarly, we obtain

$$\begin{aligned} \pi(v^{\Sigma f_i}, e_{\sigma S_i}) &= \pi(v^{f_i}, e_{\sigma S_i}) = \mathcal{L}(0, e_{\sigma S_i}), \\ \pi(v^{\Sigma f_i}, e_{\sigma\Sigma S_i}) &= \pi(v^{f_i}, e_{\sigma\Sigma S_i}) = \mathcal{L}(0, e_{\sigma\Sigma S_i}). \end{aligned}$$

Notice that, for any $v \in V^+$, $\mathcal{M}^\epsilon(v, e_{\sigma\Sigma S_i})$ is empty unless $v = 0$. Therefore, by applying the restriction functor to the left-hand side of (28), we obtain

$$\begin{aligned}
&\widetilde{\mathrm{Res}}^{w^{f_i}}_{e_{\sigma S_i},e_{\sigma\Sigma S_i}}\pi(v^{f_i},w^{f_i})\\
&\quad=\bigoplus_{v^1,v^2}\pi(v^1,e_{\sigma S_i})\boxtimes\pi(v^2,e_{\sigma\Sigma S_i})[d((v^2,e_{\sigma\Sigma S_i}),(v^1,e_{\sigma S_i}))-d((v^1,e_{\sigma S_i}),(v^2,e_{\sigma\Sigma S_i}))]\\
&\quad=\pi(v^{f_i},e_{\sigma S_i})\boxtimes\pi(0,e_{\sigma\Sigma S_i})[1]\\
&\quad=\mathcal{L}(0,e_{\sigma S_i})\boxtimes\mathcal{L}(0,e_{\sigma\Sigma S_i})[1].
\end{aligned}$$

In other words, the external tensor $\mathcal{L}(0,e_{\sigma S_i})\boxtimes\mathcal{L}(0,e_{\sigma\Sigma S_i})$ will have coefficient t when we apply the restriction functor to the left-hand side of (28).

The following relation is obvious by definition:

$$\begin{aligned}
\widetilde{\mathrm{Res}}^{w^{f_i}}_{e_{\sigma S_i},e_{\sigma\Sigma S_i}}\pi(0,w^{f_i})&=\pi(0,e_{\sigma S_i})\boxtimes\pi(0,e_{\sigma\Sigma S_i})\\
&=\mathcal{L}(0,e_{\sigma S_i})\boxtimes\mathcal{L}(0,e_{\sigma\Sigma S_i}).
\end{aligned}$$

Therefore, by applying the restriction functor to the right-hand side of (28), the second term will contribute an external tensor $\mathcal{L}(0,e_{\sigma S_i})\boxtimes\mathcal{L}(0,e_{\sigma\Sigma S_i})$ with the bar-invariant coefficient $a_{v^{f_i},0;w^{f_i}}$. Because the coefficients appearing under the restriction functor are non-negative, in order for this external product to have coefficient t as in the left-hand side of (28), we must have [6] $a_{v^{f_i},0;w^{f_i}}=0$.

Similarly, we have $a_{v^{\Sigma f_i},0;w^{f_i}}=0$ and

$$\widetilde{\mathrm{Res}}^{w^{f_i}}_{e_{\sigma S_i},e_{\sigma\Sigma S_i}}\pi(v^{\Sigma f_i},w^{f_i})=\mathcal{L}(0,e_{\sigma S_i})\boxtimes\mathcal{L}(0,e_{\sigma\Sigma S_i})[-1].$$

By taking the dual of the restriction functor, we obtain the following equation:

$$\mathbf{L}(0,e_{\sigma S_i})\,\widetilde{\otimes}\,\mathbf{L}(0,e_{\sigma\Sigma S_i})=\mathbf{L}(0,w^{f_i})+t\mathbf{L}(v^{f_i},w^{f_i})+t^{-1}\mathbf{L}(v^{\Sigma f_i},w^{f_i}).\tag{30}$$

Similarly, by using the isomorphisms of cyclic quiver varieties induced by the automorphism Σ^* on the dimension vectors v,w, we obtain

$$\mathbf{L}(0,e_{\sigma\Sigma S_i})\,\widetilde{\otimes}\,\mathbf{L}(0,e_{\sigma S_i})=\mathbf{L}(0,w^{f_i})+t\mathbf{L}(v^{\Sigma f_i},w^{f_i})+t^{-1}\mathbf{L}(v^{f_i},w^{f_i}).\tag{31}$$

The proposition follows. □

Example 4.4.3. Let us continue Example 3.2.1 and verify Proposition 4.4.2 for the case sl_2.

In this case, since $I=\{1\}$, we drop the subscript i for simplicity.

We first consider the decompositions of perverse sheaves (and their shifts)

$$\pi(v^f,w^f)=\mathcal{L}(v^f,w^f)+a_{v^f,0;w^f}\mathcal{L}(0,w^f),\tag{32}$$
$$\pi(v^{\Sigma f},w^f)=\mathcal{L}(v^{\Sigma f},w^f)+a_{v^{\Sigma f},0;w^f}\mathcal{L}(0,w^f).\tag{33}$$

In fact, we can compute the coefficients directly as follows. The smooth quiver variety $\mathcal{M}^\epsilon(v^f,w^f)$ is the $\mathbb{C}^*$-quotient of the variety $\{(\beta,\alpha)|\mathbb{C}\xleftarrow{\beta}\mathbb{C}\xleftarrow{\alpha}\mathbb{C},\ker\beta=0\}$, where the torus $\mathbb{C}^*$ naturally acts on β. Therefore, it is simply the vector space $\mathbb{C}$. The quiver variety $\mathcal{M}_0{}^\epsilon(v^f,w^f)$ is simply the vector space $\mathbb{C}$. The projection map from $\mathcal{M}^\epsilon(v^f,w^f)$ to $\mathcal{M}_0{}^\epsilon(v^f,w^f)$ sending (β,α) to $\beta\alpha$ is an isomorphism. Therefore, the coefficient $a_{v^f,0;w^f}$ vanishes and we have

$$\pi(v^f,w^f)=\mathcal{L}(v^f,w^f).$$

[6] It might be possible to verify this statement by studying the fiber of $\mathcal{M}^\epsilon(v^{f_i},w^{f_i})$ over the origin of $\mathcal{M}_0{}^\epsilon(w^{f_i})$; cf. Example 4.4.3.

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The automorphism Σ^* on the dimension vectors v, w induces isomorphisms of quiver varieties. So, we similarly have $a_{v^{\Sigma f},0;w^f} = 0$ and

$$\pi(v^{\Sigma f}, w^f) = \mathcal{L}(v^{\Sigma f}, w^f).$$

Also notice that, because the GIT quotient $\mathcal{M}^\epsilon(v^f, e_{\sigma S})$ is simply a point, we have $\pi(v^f, e_{\sigma S}) = 1_{\{0\}} = \mathbf{L}(0, e_{\sigma S})$. Similarly, $\pi(v^{\Sigma f}, e_{\sigma S}) = 1_{\{0\}} = \mathbf{L}(0, e_{\sigma S})$.

Therefore, we obtain

$$\begin{aligned}
&\widetilde{\mathrm{Res}}^{w^f}_{e_{\sigma S}, e_{\sigma\Sigma S}} \pi(v^f, w^f) \\
&\quad = \bigoplus_{v^1, v^2} \pi(v^1, e_{\sigma S}) \boxtimes \pi(v^2, e_{\sigma\Sigma S})[d((v^2, e_{\sigma\Sigma S}), (v^1, e_{\sigma S})) - d((v^1, e_{\sigma S}), (v^2, e_{\sigma\Sigma S}))] \\
&\quad = \pi(v^f, e_{\sigma S}) \boxtimes \pi(0, e_{\sigma\Sigma S})[1] \\
&\quad = \mathcal{L}(0, e_{\sigma S}) \boxtimes \mathcal{L}(0, e_{\sigma\Sigma S})[1]
\end{aligned}$$

and, similarly,

$$\widetilde{\mathrm{Res}}^{w^f}_{e_{\sigma S}, e_{\sigma\Sigma S}} \pi(v^{\Sigma f}, w^f) = \mathcal{L}(0, e_{\sigma S}) \boxtimes \mathcal{L}(0, e_{\sigma\Sigma S})[-1].$$

By isomorphisms of quiver varieties induced by the automorphism Σ^*, the above relations imply

$$\begin{aligned}
\widetilde{\mathrm{Res}}^{w^f}_{e_{\sigma\Sigma S}, e_{\sigma S}} \pi(v^{\Sigma f}, w^f) &= \mathcal{L}(0, e_{\sigma\Sigma S}) \boxtimes \mathcal{L}(0, e_{\sigma S})[1], \\
\widetilde{\mathrm{Res}}^{w^f}_{e_{\sigma\Sigma S}, e_{\sigma S}} \pi(v^f, w^f) &= \mathcal{L}(0, e_{\sigma\Sigma S}) \boxtimes \mathcal{L}(0, e_{\sigma S})[-1].
\end{aligned}$$

The following equations are obvious by definition.

$$\begin{aligned}
\widetilde{\mathrm{Res}}^{w^f}_{e_{\sigma S}, e_{\sigma\Sigma S}} \pi(0, w^f) &= \mathcal{L}(0, e_{\sigma S}) \boxtimes \mathcal{L}(0, e_{\sigma\Sigma S}), \\
\widetilde{\mathrm{Res}}^{w^f}_{e_{\sigma\Sigma S}, e_{\sigma S}} \pi(0, w^f) &= \mathcal{L}(0, e_{\sigma\Sigma S}) \boxtimes \mathcal{L}(0, e_{\sigma S}).
\end{aligned}$$

Equations (30) and (31) are obtained by taking the dual of the restriction functors $\widetilde{\mathrm{Res}}^{w^f}_{e_{\sigma S}, e_{\sigma\Sigma S}}$ and $\widetilde{\mathrm{Res}}^{w^f}_{e_{\sigma\Sigma S}, e_{\sigma S}}$.

PROPOSITION 4.4.4. *For any $i, j \in I$, we have*

$$\mathbf{L}(v^{f_i}, w^{f_i}) \,\widetilde{\otimes}\, \mathbf{L}(v^{f_j}, w^{f_j}) = t^{\langle S_i, S_j\rangle - \langle S_j, S_i\rangle} \mathbf{L}(v^{f_i} + v^{f_j}, w^{f_i} + w^{f_j}), \tag{34}$$

$$\mathbf{L}(v^{f_i}, w^{f_i}) \,\widetilde{\otimes}\, \mathbf{L}(v^{\Sigma f_j}, w^{f_j}) = t^{\langle S_i, S_j\rangle - \langle S_j, S_i\rangle} \mathbf{L}(v^{f_i} + v^{\Sigma f_j}, w^{f_i} + w^{f_j}), \tag{35}$$

$$\mathbf{L}(v^{\Sigma f_i}, w^{f_i}) \,\widetilde{\otimes}\, \mathbf{L}(v^{\Sigma f_j}, w^{f_j}) = t^{\langle S_i, S_j\rangle - \langle S_j, S_i\rangle} \mathbf{L}(v^{\Sigma f_i} + v^{\Sigma f_j}, w^{f_i} + w^{f_j}). \tag{36}$$

Proof. We deduce from 4.3.3 that there exists no l-dominant pair $(v, w^{f_i} + w^{f_j})$ such that $v > v^{f_i} + v^{f_j}$ or $v > v^{f_i} + v^{\Sigma f_j}$ or $v > v^{\Sigma f_i} + v^{f_j}$ or $v > v^{\Sigma f_i} + v^{\Sigma f_j}$. Therefore, the products in the statements consist only of the leading terms in (16).

We only check the first product. The verifications for the other products are similar. It is straightforward to check that

$$\begin{aligned}
d((v^{f_i}, w^{f_i}), (v^{f_j}, w^{f_j})) &= v^{f_i} \cdot \sigma^* w^{f_j} = v^{f_i} \cdot e_{S_j} + v^{f_i} \cdot e_{\Sigma S_j} \\
&= \dim \mathsf{Hom}(S_i, S_j) + \dim \mathsf{Ext}(S_i, S_j)
\end{aligned}$$

and similarly

$$d((v^{f_j}, w^{f_j}), (v^{f_i}, w^{f_i})) = \dim \mathsf{Hom}(S_j, S_i) + \dim \mathsf{Ext}(S_j, S_i).$$

The statement follows. □

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The following relations have already been proved in [HL15]. We compute them for completeness.

PROPOSITION 4.4.5. (i) *For any $i, j \in I$ such that $\mathsf{Ext}^1_{\mathbb{C}Q}(S_j, S_i) = \mathbb{C}$, we have*

$$(\mathbf{L}(0, e_{\sigma S_i})^{\otimes 2}) \otimes \mathbf{L}(0, e_{\sigma S_j}) - (t + t^{-1})\mathbf{L}(0, e_{\sigma S_i}) \otimes \mathbf{L}(0, e_{\sigma S_j}) \otimes \mathbf{L}(0, e_{\sigma S_i}) \\ + \mathbf{L}(0, e_{\sigma S_j}) \otimes (\mathbf{L}(0, e_\sigma) S_i)^{\otimes 2} = 0.$$

(ii) *For any $i, j \in I$ such that $\mathsf{Ext}^1_{\mathbb{C}Q}(S_i, S_j) = \mathbb{C}$, we have*

$$(\mathbf{L}(0, e_{\sigma S_i})^{\otimes 2}) \otimes \mathbf{L}(0, e_{\sigma S_j}) - (t + t^{-1})\mathbf{L}(0, e_{\sigma S_i}) \otimes \mathbf{L}(0, e_{\sigma S_j}) \otimes \mathbf{L}(0, e_{\sigma S_i}) \\ + \mathbf{L}(0, e_{\sigma S_j}) \otimes (\mathbf{L}(0, e_\sigma) S_i)^{\otimes 2} = 0.$$

(iii) *In $(R, \otimes)$, for any $i, j \in I$ such that $\mathsf{Ext}^1_{\mathbb{C}Q}(S_j, S_i) = 0$, we have*

$$[\mathbf{L}(0, e_{\sigma S_i}), \mathbf{L}(0, e_{\sigma S_j})] = 0.$$

Proof. (i), (ii) Notice that $i \neq j$. Denote $\delta_{\tau S_j, S_i}$ by δ and $\langle S_i, S_j\rangle_a$ by χ. Then the situations (i) and (ii) correspond to the cases $(\delta, \chi) = (1, 1)$ and $(\delta, \chi) = (0, -1)$, respectively. For any pairs (v^1, w^1), (v^2, w^2), we define $\langle (v^1, w^1), (v^2, w^2)\rangle_a$ to be $\langle w^1, w^2\rangle_a$. Let us denote

$$w' = e_{\sigma S_i} + e_{\sigma S_j}, \\ w = 2e_{\sigma S_i} + e_{\sigma S_j}.$$

We need the following coefficients, because the multiplication $\widetilde{\otimes}$ is replaced by the twisted multiplication $\otimes$:

$$A = t^{-(1/2)\langle (0, e_{\sigma S_i}), (0, e_{\sigma S_j})\rangle_a}, \\ B = t^{-(1/2)\langle (e_{S_i}, e_{\sigma S_i}), (0, e_{\sigma S_j})\rangle_a}, \\ C = t^{-(1/2)\langle (0, e_{\sigma S_i}), (0, w')\rangle_a}, \\ D = t^{-(1/2)\langle (e_{S_i}, e_{\sigma S_i}), (0, w')\rangle_a}, \\ E = t^{-(1/2)\langle (0, e_{\sigma S_i}), (e_{S_i}, w')\rangle_a}.$$

It follows that $A = B = C = D = E = t^{-(1/2)\chi}$.

First, compute the following bilinear forms:

$$d((e_{S_i}, e_{\sigma S_i}), (0, e_{\sigma S_j})) = 0, \\ d((0, e_{\sigma S_j}), (e_{S_i}, e_{\sigma S_i})) = \delta, \\ d((e_{S_i}, e_{\sigma S_i}), (0, w')) = 1, \\ d((0, w'), (e_{S_i}, e_{\sigma S_i})) = \delta, \\ d((0, e_{\sigma S_i}), (e_{S_i}, w')) = 0, \\ d((e_{S_i}, w'), (0, e_{\sigma S_i})) = 1.$$

Similar to the proof of Proposition 4.4.2, we have the following decompositions:

$$\mathbf{L}(0, e_{\sigma S_i}) \otimes \mathbf{L}(0, e_{\sigma S_j}) = A\mathbf{L}(0, w') + Bt^{\delta}\mathbf{L}(e_{S_i}, w'), \\ \mathbf{L}(0, e_{\sigma S_j}) \otimes \mathbf{L}(0, e_{\sigma S_i}) = A^{-1}\mathbf{L}(0, w') + B^{-1}t^{-\delta}\mathbf{L}(e_{S_i}, w'), \\ \mathbf{L}(0, e_{\sigma S_i}) \otimes \mathbf{L}(0, w') = C\mathbf{L}(0, w) + Dt^{\delta-1}\mathbf{L}(e_{S_i}, w), \\ \mathbf{L}(0, w') \otimes \mathbf{L}(0, e_{\sigma S_i}) = C^{-1}\mathbf{L}(0, w) + D^{-1}t^{1-\delta}\mathbf{L}(e_{S_i}, w),$$

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$$\mathbf{L}(0, e_{\sigma S_i}) \otimes \mathbf{L}(e_{S_i}, w') = Et\mathbf{L}(e_{S_i}, w),$$
$$\mathbf{L}(e_{S_i}, w') \otimes \mathbf{L}(0, e_{\sigma S_i}) = E^{-1}t^{-1}\mathbf{L}(e_{S_i}, w).$$

The proposition follows from direct calculation.

(iii) The statement is obvious. □

Proof of Theorem 3.1.6. (i) We replace $\widetilde{\otimes}$ by $\otimes$ in R. The relations in Propositions 4.4.1, 4.4.2, and 4.4.4 now become

$$\mathbf{L}(0, e_{\sigma S_i}) \otimes \mathbf{L}(v^{f_j}, w^{f_j}) = t^{a_{ij}}\mathbf{L}(v^{f_j}, w^{f_j}) \otimes \mathbf{L}(0, e_{\sigma S_i}), \tag{37}$$
$$\mathbf{L}(0, e_{\sigma S_i}) \otimes \mathbf{L}(v^{\Sigma f_j}, w^{f_j}) = t^{-a_{ji}}\mathbf{L}(v^{\Sigma f_j}, w^{f_j}) \otimes \mathbf{L}(0, e_{\sigma S_i}), \tag{38}$$
$$\mathbf{L}(0, e_{\sigma \Sigma S_i}) \otimes \mathbf{L}(v^{f_j}, w^{f_j}) = t^{-a_{ji}}\mathbf{L}(v^{f_j}, w^{f_j}) \otimes \mathbf{L}(0, e_{\sigma \Sigma S_i}), \tag{39}$$
$$\mathbf{L}(0, e_{\sigma \Sigma S_i}) \otimes \mathbf{L}(v^{\Sigma f_j}, w^{f_j}) = t^{a_{ij}}\mathbf{L}(v^{\Sigma f_j}, w^{f_j}) \otimes \mathbf{L}(0, e_{\sigma \Sigma S_i}), \tag{40}$$
$$[\mathbf{L}(0, e_{\sigma S_i}), \mathbf{L}(0, e_{\sigma \Sigma S_j})] = \delta_{ij}(t - t^{-1})(\mathbf{L}(v^{f_i}, w^{f_i}) - \mathbf{L}(v^{\Sigma f_i}, w^{\Sigma f_i})), \tag{41}$$
$$\mathbf{L}(v^{f_i}, w^{f_i}) \otimes \mathbf{L}(v^{f_j}, w^{f_j}) = \mathbf{L}(v^{f_i} + v^{f_j}, w^{f_i} + w^{f_j}), \tag{42}$$
$$\mathbf{L}(v^{f_i}, w^{f_i}) \otimes \mathbf{L}(v^{\Sigma f_j}, w^{f_j}) = \mathbf{L}(v^{f_i} + v^{\Sigma f_j}, w^{f_i} + w^{f_j}), \tag{43}$$
$$\mathbf{L}(v^{\Sigma f_i}, w^{f_i}) \otimes \mathbf{L}(v^{\Sigma f_j}, w^{f_j}) = \mathbf{L}(v^{\Sigma f_i} + v^{\Sigma f_j}, w^{f_i} + w^{f_j}). \tag{44}$$

Notice that the relations in Proposition 4.4.5 remain unchanged.

Comparing the above relations with those of the Chevalley generators of $\widetilde{\mathbf{U}}_t(\mathfrak{g})$, we can define a surjective algebra homomorphism ϕ from $\widetilde{\mathbf{U}}_t(\mathfrak{g})_{\mathbb{Q}(t^{1/2})}$ to the Grothendieck ring $(K^*_{\mathbb{Q}(t^{1/2})}, \otimes)$ such that

$$\begin{aligned}
\phi(t^{1/2}) &= t^{1/2},\\
\phi(E_i) &= \frac{-t}{t^2-1}\mathbf{L}(0, e_{\sigma S_i}),\\
\phi(K_i) &= \mathbf{L}(v^{\Sigma f_i}, w^{f_i}),\\
\phi(K_i') &= \mathbf{L}(v^{f_i}, w^{f_i}),\\
\phi(F_i) &= \frac{t}{t^2-1}\mathbf{L}(0, e_{\sigma \Sigma S_i}).
\end{aligned}$$

This map is an isomorphism by Theorems 3.1.4 and 3.1.5. We define $\kappa = \phi^{-1}$.

(ii) Notice that, for any $i \in I$, $\mathbf{L}(v^{f_i}, w^{f_i}) \otimes \mathbf{L}(v^{\Sigma f_i}, w^{f_i}) = \mathbf{L}(v^{f_i} + v^{\Sigma f_i}, 2w^{f_i})$ is a center element in $(R, \otimes)$. The statement follows from (i). □

Proof of Theorem 3.1.7. The claim follows from [HL15] by Proposition 5.2.4. □

5. Comparison of products

To conclude the paper, we show that the twisted product $\otimes$ of the Grothendieck ring R^+ agrees with the non-commutative multiplication $*$ defined in [HL15, Her04a] via the reduction from (v, w) to $w - C_q v$. Notice that the twisted products do not agree in general and, usually, such reduction is impossible because Cartan elements are not center elements; cf. Example 3.2.3.

Recall that the restriction of the twisted product $\otimes$ on R^+ is determined by the bilinear form $\mathcal{N}$:

$$\mathcal{N}(m^1, m^2) = d(m^2, m^1) + d(m^1, m^2) + \tfrac{1}{2}\langle \Phi(w^2), \Phi(w^1)\rangle_a \tag{45}$$

for any $m^1 = (v^1, w^1), m^2 = (v^2, w^2) \in \mathbb{N}^{\mathsf{Ind\,Rep}\,Q - \mathsf{Inj}\,Q} \times W^S$, where we use $\mathsf{Inj}\,Q$ to denote the injectives in $\mathsf{Ind\,Rep}\,Q$. On the other hand, the non-commutative multiplication in [HL15] for the

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corresponding Grothendieck ring is determined by the bilinear form $\mathscr{N}$ defined on $W^+ \times W^+$. For the rest of this section, we will show that these two products agree by proving Proposition 5.2.4.

Remark 3.3 of [HL15] should imply Proposition 5.2.4, which is the main result of this section. We give an alternative approach to this result by considering the 'lift' of $\widetilde{w} \in W^+$ into l-dominant pairs.

For any $N \in \mathsf{Ind}\,\mathsf{Rep}(Q)$, denote by $[N] = \sum N_i S_i$ the class of N in $K_0(\mathsf{Rep}\, Q)$. For simplicity, we denote $\mathsf{Hom}_{\mathcal{D}^b(Q)}(\,,\,)$ by $\mathsf{Hom}(\,,\,)$.

5.1 l-dominant pairs

For our purpose, we want to lift any l-dominant $(0, \widetilde{w})$, $\widetilde{w} \in W^+$, to an l-dominant pair $(v, w) \in V^+ \times W^S$, whose existence is guaranteed by [LP13].

Inspired by [LP13, Corollary 3.15(iii)], we associate to any $N \in \mathsf{Ind}\,\mathsf{Rep}(Q)$ the pair $\iota(N) = (\iota_V(N), \iota_W(N))$ defined by

$$\begin{aligned} \iota_W(N) &= \sum_i (N_i \cdot e_{\sigma S_i}), \\ \iota_V(N) &= \sum_{x \in (\mathsf{Ind}\,\mathsf{Rep}(Q) - \mathsf{Inj}\, Q)} (\dim \mathsf{Hom}(\tau^{-1}x, [N]) - \dim \mathsf{Hom}(\tau^{-1}x, N)) \cdot e_x. \end{aligned} \tag{46}$$

In fact, we can rewrite $\iota_V(e_{\sigma N}) = \sum_{x \in \mathsf{Ind}\,\mathsf{Rep}(Q)} (\dim \mathsf{Hom}(\tau^{-1}x, [N]) - \dim \mathsf{Hom}(\tau^{-1}x, N)) \cdot e_x$ by taking τ as the functor defined for $\mathcal{D}^b(Q)$.

Example 5.1.1. Let us take the example of Figure 7. Then we have

$$\begin{aligned} \iota(S_1) &= (0, e_{\sigma S_1}), \\ \iota(S_2) &= (0, e_{\sigma S_2}), \\ \iota(P_2) &= (e_{S_1}, e_{\sigma S_1} + e_{\sigma S_2}). \end{aligned}$$

Proposition 5.1.2. *The pair $\iota(N)$ is l-dominant and we have $\iota_W(N) - C_q \iota_V(N) = e_{\sigma N}$.*

Proof. The claim should be a translation of the result of [LP13] from repetitive algebras to representations of Q. We give a straightforward proof here.

To simplify the notation, let us denote the pair $\iota(N)$ by $\iota = (\iota_V, \iota_W)$. For any $x \in \mathsf{Ind}\,\mathsf{Rep}\, Q$, denote the AR-triangle in $\mathcal{D}^b(Q)$ by $\tau x \to E \to x$, where $E = \bigoplus_j E_j$, with each E_j an indecomposable in $\mathcal{D}^b(Q)$. Then the σx-component of $\iota_W - C_q \iota_V$ is given by

$$(\iota_W - C_q \iota_V)_{\sigma x} = (\iota_W)_{\sigma x} - (\iota_V)_{\tau x} - (\iota_V)_x + \sum_j (\iota_V)_{E_j}.$$

It suffices to verify the following equality:

$$(\iota_W - C_q \iota_V)_{\sigma x} = \delta_{x,N}.$$

We have

$$\begin{aligned} (\iota_W)_{\sigma x} &= \sum_i \delta_{x, S_i} N_i, \\ (\iota_V)_{\tau x} &= \dim \mathsf{Hom}(x, [N]) - \dim \mathsf{Hom}(x, N), \\ (\iota_V)_x &= \dim \mathsf{Hom}(\tau^{-1}x, [N]) - \dim \mathsf{Hom}(\tau^{-1}x, N), \\ \sum_j (\iota_V)_{E_j} &= \dim \mathsf{Hom}(\tau^{-1}E, [N]) - \sum_j \dim \mathsf{Hom}(\tau^{-1}E, N). \end{aligned}$$

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Applying the contravariant functor $\mathsf{Hom}(\ ,N)$ to the AR-triangle $x \to \tau^{-1}E \to \tau^{-1}x \to \Sigma x$, we obtain a long exact sequence

$$\begin{aligned}\mathsf{Hom}(x,\Sigma^{-1}N) \xrightarrow{w^1} \mathsf{Hom}(\tau^{-1}x,N) &\to \mathsf{Hom}(\tau^{-1}E,N)\\ &\xrightarrow{w^3} \mathsf{Hom}(x,N) \xrightarrow{w^2} \mathsf{Hom}(\tau^{-1}x,\Sigma N).\end{aligned}$$

Notice that $\mathsf{Hom}(x,\Sigma^{-1}N)=0$ and consequently $w^1=0$. By using the universal property of AR-triangles, we see that w^3 is surjective if $N\neq x$ and $\dim \mathsf{Cok}\, w^3=1$ if $N=x$. Therefore, we obtain

$$\dim \mathsf{Hom}(\tau^{-1}x,N) - \dim \mathsf{Hom}(\tau^{-1}E,N) + \dim \mathsf{Hom}(x,N) = \delta_{x,N}.$$

By applying the functors $\mathsf{Hom}(\ ,S_i)$ for all $i\in I$ to this AR-triangle, we obtain

$$\dim \mathsf{Hom}(\tau^{-1}x,[N]) - \dim \mathsf{Hom}(\tau^{-1}E,[N]) + \dim \mathsf{Hom}(x,[N]) = \sum_i \delta_{x,S_i} N_i.$$

Putting these results together, we obtain the desired equality. □

5.2 Comparison of bilinear forms

For any $M,N\in \mathsf{Ind}\,\mathsf{Rep}(Q)$, recall that the Euler form $\langle M,N\rangle = \dim \mathsf{Hom}(M,N) - \dim \mathsf{Hom}(M,\Sigma N)$ depends only on the class $[M]$, $[N]$. The symmetrized Euler form is given by $(M,N)=\langle M,N\rangle + \langle N,M\rangle$.

DEFINITION 5.2.1 (q-degree order). For any $(i,a),(j,b)\in \mathsf{Ind}\,\mathsf{Rep}(Q)$, we can write $a=q^{\xi(i)+A}$, $b=q^{\xi(j)+B}$ for some $0\leqslant A,B\leqslant 2h$ such that $\xi(i)+A-\xi(j)-B<h$. If $\xi(i)+A>\xi(j)+B$, we say that the q-degree of (i,a) is higher (or larger) than that of (j,b) and the q-degree of (j,b) is lower (or smaller) than that of (i,a).

Example 5.2.2. In Figure 7, the q-degree of P_2 is higher than that of S_1.

PROPOSITION 5.2.3. *For any different objects $M,N\in \mathsf{Ind}\,\mathsf{Rep}(Q)$, assume that the q-degree of M is not higher than that of N; then we have*

$$d(\iota(N),\iota(M)) - d(\iota(M),\iota(N)) + \tfrac{1}{2}\langle N,M\rangle_a = \tfrac{1}{2}(M,N). \tag{47}$$

Proof. By definition, we have

$$\begin{aligned}\tfrac{1}{2}(M,N) - \tfrac{1}{2}\langle N,M\rangle_a &= \langle M,N\rangle,\\ d(\iota(N),\iota(M)) &= e_{\sigma N}\cdot \sigma^*\iota_V(M) + \iota_V(N)\cdot \sigma^*\iota_W(M)\\ &= e_{\tau N}\cdot \iota_V(M)\cdot + \iota_V(N)\cdot \sum M_i e_{\sigma S_i},\\ d(\iota(M),\iota(N)) &= e_{\tau M}\cdot \iota_V(N) + \iota_V(M)\cdot \sum N_i e_{\sigma S_i}.\end{aligned}$$

So, we should check that

$$e_{\tau N}\cdot \iota_V(M) + \iota_V(N)\cdot \sum M_i e_{\sigma S_i} - e_{\tau M}\cdot \iota_V(N) - \iota_V(M)\cdot \sum N_i e_{\sigma S_i} = \langle M,N\rangle. \tag{48}$$

First, assume that N and M are not projective. By using the definition of ι_V, we have

$$\begin{aligned}\iota_V(M)\cdot e_{\tau N} &= \dim \mathsf{Hom}(N,[M]) - \dim \mathsf{Hom}(N,M),\\ \iota_V(N)\cdot \sum M_i e_{\sigma S_i} &= \dim \mathsf{Hom}(\tau^{-1}[M],[N]) - \dim \mathsf{Hom}(\tau^{-1}[M],N)\\ &= \dim \mathsf{Hom}([N],\Sigma[M]) - \dim \mathsf{Hom}(N,\Sigma[M]),\end{aligned}$$

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$$\begin{aligned}\iota_V(N)\cdot e_{\tau M} &= \dim \mathsf{Hom}(M,[N]) - \dim \mathsf{Hom}(M,N),\\ \iota_V(M)\cdot \sum N_i e_{\sigma S_i} &= \dim \mathsf{Hom}(\tau^{-1}[N],[M]) - \dim \mathsf{Hom}(\tau^{-1}[N],M)\\ &= \dim \mathsf{Hom}([M],\Sigma[N]) - \dim \mathsf{Hom}(M,\Sigma[N]).\end{aligned}$$

If N is projective, we have $e_{\tau N}\cdot\iota_V(M)=0$. On the other hand, $\dim \mathsf{Hom}(N,[M]) - \dim \mathsf{Hom}(N,M)$ vanishes. So, the above expression of $e_{\tau N}\cdot\iota_V(M)$ remains effective. Similarly, the above expression of $e_{\tau M}\cdot\iota_V(N)$ remains effective even if M is projective. So, we can remove the projectivity assumption on M and N.

Because the q-degree of M is no larger than that of N, we have $\mathsf{Hom}(N,M)=0$. The left-hand side of (48) becomes

$$\begin{aligned}&(\dim \mathsf{Hom}(N,[M]) - \dim \mathsf{Hom}(N,\Sigma[M]))\\ &\quad - (\dim \mathsf{Hom}(M,[N]) - \dim \mathsf{Hom}(M,\Sigma[N]))\\ &\quad + \dim \mathsf{Hom}([N],\Sigma[M]) - \dim([M],\Sigma[N]) + \dim \mathsf{Hom}(M,N)\\ &= \langle N,[M]\rangle - \langle M,[N]\rangle + \dim \mathsf{Hom}([N],\Sigma[M]) - \dim([M],\Sigma[N])\\ &\quad + \dim \mathsf{Hom}(M,N).\end{aligned}$$

We can replace N and M by $[N]$ and $[M]$ respectively in the last expression. Then, by using definition of $\langle\,,\rangle$, the last expression becomes

$$\dim \mathsf{Hom}([N],[M]) - \dim \mathsf{Hom}([M],[N]) + \dim \mathsf{Hom}(M,N) = \dim \mathsf{Hom}(M,N).$$

As the last step, $\dim \mathsf{Hom}(M,N)=\langle M,N\rangle$ because the q-degree of M is no larger than that of N. $\square$

PROPOSITION 5.2.4. *For any dominant pairs $m^1=(v^1,w^1)$, $m^2=(v^2,w^2)$ in $\mathbb{N}^{\mathsf{Ind\,Rep}(Q)-\mathsf{Inj}\,Q}\times W^S$, we have*

$$\mathcal{N}(m^1,m^2)=\tfrac{1}{2}\mathscr{N}(w-C_q v^1, w-C_q v^2), \tag{49}$$

where the form $\mathscr{N}$ defined on $W^+\times W^+$ is the bilinear form in [HL15, (5)].

Proof. By [HL15, Proposition 3.2], the right-hand side of (47) is just $\frac{1}{2}\mathscr{N}(e_{\sigma M},e_{\sigma N})$. Therefore, we have $\mathcal{N}=\frac{1}{2}\mathscr{N}$ in the situation of Proposition 5.2.3. Then the claim holds true in general because $\mathcal{N}$ and $\mathscr{N}$ are anti-symmetrized bilinear forms. $\square$

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