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ON A CONJECTURE CONCERNING THE NUMBER OF SOLUTIONS TO $a^x + b^y = c^z$

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Abstract

Let a, b, c be fixed coprime positive integers with $\min\{a, b, c\} > 1$. Let $N(a, b, c)$ denote the number of positive integer solutions (x, y, z) of the equation $a^x + b^y = c^z$. We show that if (a, b, c) is a triple of distinct primes for which $N(a, b, c) > 1$ and (a, b, c) is not one of the six known such triples, then $c > 10^{18}$, and there are exactly two solutions (x_1, y_1, z_1), (x_2, y_2, z_2) with $2 \mid x_1$, $2 \mid y_1$, $z_1 = 1$, $2 \nmid y_2$, $z_2 > 1$, and, taking $a < b$, we must have $a = 2$, $b \equiv 1 \bmod 12$, $c \equiv 5 \bmod 12$, with (a, b, c) satisfying further strong restrictions. These results support a conjecture put forward by Scott and Styer ['Number of solutions to $a^x + b^y = c^z$', *Publ. Math. Debrecen* **88** (2016), 131–138].

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1. Introduction

Let $\mathbb{N}$ be the set of all positive integers. Let a, b, c be fixed coprime positive integers with $\min\{a, b, c\} > 1$. The equation

$$a^x + b^y = c^z, \quad x, y, z \in \mathbb{N} \tag{1.1}$$

has been studied deeply with abundant results (see [14], see also [2]). In 1933, Mahler [15] used his p-adic analogue of the Diophantine approximation method of Thue–Siegel to prove that (1.1) has only finitely many solutions (x, y, z). However, his method is ineffective. Let $N(a, b, c)$ denote the number of solutions (x, y, z) of (1.1). An effective upper bound for $N(a, b, c)$ was first given by Gel'fond [8]. A straightforward application of an upper bound on the number of solutions of binary S-unit equations due to Beukers and Schlickewei [5] gives $N(a, b, c) \le 2^{36}$. The following more accurate upper bounds for $N(a, b, c)$ have been obtained in recent years.

(i) (Scott and Styer [22]) If $2 \nmid c$, then $N(a, b, c) \le 2$.
(ii) (Hu and Le [9]) If $\max\{a, b, c\} > 5 \cdot 10^{27}$, then $N(a, b, c) \le 3$.
(iii) (Hu and Le [10]) If $2 \mid c$ and $\max\{a, b, c\} > 10^{62}$, then $N(a, b, c) \le 2$.

(iv) (Miyazaki and Pink [16]) If $2 \mid c$, $a < b$ and $\max\{a, b, c\} \le 10^{62}$, then $N(a, b, c) \le 2$ except for $N(3, 5, 2) = 3$.

Nevertheless, the problem of establishing $N(a, b, c) \le 1$ with a finite number of exceptions remains open. This open question is addressed by the following conjecture in [22]. Assuming without loss of generality that a, b and c are not perfect powers, the conjecture may be formulated as follows.

CONJECTURE 1.1. For $a < b$, we have $N(a, b, c) \le 1$, except for:

(i) $N(2, 2^r - 1, 2^r + 1) = 2$, $(x, y, z) = (1, 1, 1)$ and $(r + 2, 2, 2)$, where r is a positive integer with $r \ge 2$;
(ii) $N(2, 3, 11) = 2$, $(x, y, z) = (1, 2, 1)$ and $(3, 1, 1)$;
(iii) $N(2, 3, 35) = 2$, $(x, y, z) = (3, 3, 1)$ and $(5, 1, 1)$;
(vi) $N(2, 3, 259) = 2$, $(x, y, z) = (4, 5, 1)$ and $(8, 1, 1)$;
(v) $N(2, 5, 3) = 2$, $(x, y, z) = (1, 2, 3)$ and $(2, 1, 2)$;
(vi) $N(2, 5, 133) = 2$, $(x, y, z) = (3, 3, 1)$ and $(7, 1, 1)$;
(vii) $N(2, 7, 3) = 2$, $(x, y, z) = (1, 1, 2)$ and $(5, 2, 4)$;
(viii) $N(2, 89, 91) = 2$, $(x, y, z) = (1, 1, 1)$ and $(13, 1, 2)$;
(xi) $N(2, 91, 8283) = 2$, $(x, y, z) = (1, 2, 1)$ and $(13, 1, 1)$;
(x) $N(3, 5, 2) = 3$, $(x, y, z) = (1, 1, 3)$, $(1, 3, 7)$, and $(3, 1, 5)$;
(xi) $N(3, 10, 13) = 2$, $(x, y, z) = (1, 1, 1)$ and $(7, 1, 3)$;
(xii) $N(3, 13, 2) = 2$, $(x, y, z) = (1, 1, 4)$ and $(5, 1, 8)$;
(xiii) $N(3, 13, 2200) = 2$, $(x, y, z) = (1, 3, 1)$ and $(7, 1, 1)$.

Later in this paper, in referring to the solutions in cases (i) through (xiii) above, it will be helpful to have established the following result.

LEMMA 1.2. *The values of $N(a, b, c)$ in Conjecture 1.1 are exact: there are no further solutions in any of the thirteen cases.*

PROOF. For cases (i) through (xii), this follows from Theorem 1 of [22] and Theorem 6 of [19]. For case (xiii), consideration modulo 16 shows that $z > 1$ requires $4 \mid x - y$, which contradicts consideration modulo 5. So $z = 1$, and there are only two solutions. □

Although the results on (1.1) known so far support Conjecture 1.1, it is generally far from being resolved. This difficult problem is made more approachable by taking c prime. More than thirty years ago, the first author [12] discussed the upper bound for $N(a, b, c)$ when a, b, c are distinct primes. Many authors have used this approach to the problem. Later, [22] removed the difficulty caused by taking c composite when c is odd, and Hu and Le [10] and Miyazaki and Pink [16] handled even composite c; these later results established $N(a, b, c) \le 2$ with the single exceptional case $(a, b, c) = (3, 5, 2)$.

Establishing $N(a, b, c) \le 1$ involves many exceptional cases and is, in general, much more difficult, suggesting perhaps that the old approach of considering only prime bases may be a practical way to begin considering this problem.

Our first result requires c prime with congruence restrictions on a and b not necessarily prime.

THEOREM 1.3. *If* $a \equiv 2 \bmod 3$, $b \not\equiv 0 \bmod 3$ *and* $c > 3$ *is an odd prime, then* $N(a, b, c) \le 1$, *except for the following possibility:* (1.1) *has exactly two solutions* (x_1, y_1, z_1) *and* (x_2, y_2, z_2), *and these solutions satisfy* $2 \mid x_1$, $2 \mid y_1$, $2 \nmid z_1$ *and* $2 \nmid y_2$, *with* $c \equiv 5 \bmod 12$.

Theorem 1.3 is used to establish the following result.

THEOREM 1.4. *If* a, b, c *are distinct primes with* $a < b$ *and* $N(a, b, c) > 1$, *then* $c \equiv 5 \bmod 12$, *and there must be exactly two solutions* (x_1, y_1, z_1) *and* (x_2, y_2, z_2), *with* $2 \mid x_1$, $2 \mid y_1$, $z_1 = 1$ *and* $2 \nmid y_2$, *except for* $(a, b, c) = (2, 3, 5)$, $(2, 3, 11)$, $(2, 5, 3)$, $(2, 7, 3)$, $(3, 5, 2)$ *and* $(3, 13, 2)$.

Further restrictions are given by the following result.

THEOREM 1.5. *If* a, b, c *are distinct primes with* $a < b$ *and* $N(a, b, c) > 1$, *then, if* $(a, b, c) \ne (2, 3, 5)$, $(2, 3, 11)$, $(2, 5, 3)$, $(2, 7, 3)$, $(3, 5, 2)$ *or* $(3, 13, 2)$, *we must have* $a = 2$ *with* $b \equiv 1 \bmod 12$ *and* $c \equiv 5 \bmod 12$; *further, if* $b \equiv 1 \bmod 24$, *then* $c \equiv 17 \bmod 24$.

COROLLARY 1.6. *If* a, b, c *are distinct primes with* $a < b$ *and* $N(a, b, c) > 1$, *then, if* $(a, b, c) \ne (2, 3, 5)$, $(2, 3, 11)$, $(2, 5, 3)$, $(2, 7, 3)$, $(3, 5, 2)$ *or* $(3, 13, 2)$, *we must have* $N(a, b, c) = 2$ *and, letting the two solutions be* (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *as in Theorem* 1.4, *we must have* $z_2 > 1$.

In a later section, we will use the following version of Conjecture 1.1 in which a, b and c are restricted to prime values.

CONJECTURE 1.7. For a, b and c distinct primes with $a < b$, we have $N(a, b, c) \le 1$, except for:

(i) $N(2, 3, 5) = 2$, $(x, y, z) = (1, 1, 1)$ and $(4, 2, 2)$;
(ii) $N(2, 3, 11) = 2$, $(x, y, z) = (1, 2, 1)$ and $(3, 1, 1)$;
(iii) $N(2, 5, 3) = 2$, $(x, y, z) = (1, 2, 3)$ and $(2, 1, 2)$;
(iv) $N(2, 7, 3) = 2$, $(x, y, z) = (1, 1, 2)$ and $(5, 2, 4)$;
(v) $N(3, 5, 2) = 3$, $(x, y, z) = (1, 1, 3)$, $(1, 3, 7)$ and $(3, 1, 5)$;
(vi) $N(3, 13, 2) = 2$, $(x, y, z) = (1, 1, 4)$ and $(5, 1, 8)$.

In the final section of this paper, we will explain a method by which we have shown the following result.

THEOREM 1.8. *Any counterexample to Conjecture* 1.7 *must have*

$$b > 10^9, \quad c > 10^{18}.$$

This result is in marked contrast to results which can be obtained without assuming a, b and c are prime: in that more general case, the lower bound on c when (a, b, c) gives a counterexample to Conjecture 1.1 is still quite low (the latest such results are

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given by Miyazaki and Pink [17] in considering Conjecture 1.1 for the special case in which either a or b is congruent to ± 1 modulo c). Theorem 1.8 improves Lemma 2.9 in Section 2 which follows.

2. Preliminaries

We now divide all solutions (x, y, z) of (1.1) into four classes according to the parities of x and y: $2 \mid x$ and $2 \mid y$; $2 \nmid x$ and $2 \mid y$; $2 \mid x$ and $2 \nmid y$; or $2 \nmid x$ and $2 \nmid y$. We will call these classes the *parity classes* of (1.1).

LEMMA 2.1. *If c is an odd prime, then, for a given parity class, there is at most one solution (x, y, z) to (1.1), except for $(a, b, c) = (3, 10, 13)$ or $(10, 3, 13)$.*

PROOF. Since c is an odd prime, using the notation of [22], we see that for any given parity class of (1.1), there is only one ideal factorisation in C_D. Therefore, by Lemma 2 of [22], we obtain the lemma immediately. □

LEMMA 2.2 [20, Lemma 2]. *The equation*

$$3^x + 2^y = n^z$$

has no solutions in positive integers (x, y, z, n) with $z > 1$ except for $3^2 + 2^4 = 5^2$.

LEMMA 2.3. *Let X, n be positive integers with $2 \nmid X$ and $n > 1$. Then $|X^2 - 2^n| > 2^{0.26n}$, except for $X^2 - 2^n = 1$ or -7.*

PROOF. This lemma is a special case of Corollary 1.7 of [1] with $y = 2$. □

LEMMA 2.4 [7, 13]. *The equation*

$$X^2 + 2^m = Y^n, \quad X, Y, m, n \in \mathbb{N}, \gcd(X, Y) = 1, n > 2,$$

has only the solutions $(X, Y, m, n) = (5, 3, 1, 3)$, $(11, 5, 2, 3)$ and $(7, 3, 5, 4)$.

LEMMA 2.5 [4, Theorem 8.4]. *The equation*

$$X^2 - 2^m = Y^n, \quad X, Y, m, n \in \mathbb{N}, \gcd(X, Y) = 1, m > 1, n > 2, \tag{2.1}$$

has only the solution $(X, Y, m, n) = (71, 17, 7, 3)$.

LEMMA 2.6 [20, Theorem 6]. *Let A, B be distinct odd positive primes. For a given positive integer k, the equation*

$$A^m - B^n = 2^k, \quad m, n \in \mathbb{N}, \tag{2.2}$$

has at most one solution in positive integers (m, n).

LEMMA 2.7 [19, Theorem 6]. *If $a < b$ and $c = 2$, then $N(a, b, 2) \leq 1$, except for $N(3, 5, 2) = 3$ and $N(3, 13, 2) = 2$.*

LEMMA 2.8 [22]. *If $2 \nmid c$, then $N(a, b, c) \leq 2$.*

LEMMA 2.9 [6]. *If a, b, c are distinct primes with $\max\{a, b, c\} < 100$, then Conjecture 1.7 is true.*

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LEMMA 2.10 [21, Lemma 4.2]. *The equation*

$$(1+\sqrt{-D})^r = m \pm \sqrt{-D}$$

has no solutions with integer $r > 1$ when D is a positive integer congruent to 2 mod 4 and m is any integer, except for $D = 2$, $r = 3$.

Further, when $D \equiv 0 \bmod 4$ is a positive integer such that $1 + D$ is prime or a prime power, (2.5) has no solutions with integer $r > 1$ except for $D = 4$, $r = 3$.

LEMMA 2.11. *If the equation*

$$3^m - 2^n = 3^x - 2^y = d$$

has a solution in positive integers (m, n, x, y) with $m \neq x$, then $d = 1$, -5 or -13.

PROOF. This follows easily from a conjecture of Pillai [18] first proven by Stroeker and Tijdeman [23] using lower bonds on linear forms in logarithms. It is also an immediate consequence of the elementary corollary to Theorem 4 of [19]. □

LEMMA 2.12. *If $N(a, b, c) > 1$ when $a \equiv 2 \bmod 3$, $b \not\equiv 0 \bmod 3$ and c is an odd prime, then (1.1) has exactly two solutions (x_1, y_1, z_1) and (x_2, y_2, z_2), where $2 \nmid y_1 - y_2$.*

PROOF. If $a \equiv 2 \bmod 3$, $b \not\equiv 0 \bmod 3$ and c is an odd prime, then the parity of y determines the parity class. Since Lemma 2.1 shows that there is at most one solution per parity class, we see that $N(a, b, c) > 1$ implies that there are exactly two solutions (x_1, y_1, z_1) and (x_2, y_2, z_2) and $2 \nmid y_1 - y_2$. □

LEMMA 2.13 [3, Theorem 1.1]. *Let c and b be positive integers. Then there exists at most one pair of positive integers (z, y) for which*

$$0 < |c^z - b^y| < \tfrac{1}{4}\max\{c^{z/2}, b^{y/2}\}.$$

3. Proofs of Theorems 1.3 and 1.4

PROOF OF THEOREM 1.3. Assume $N(a, b, c) > 1$ with $a \equiv 2 \bmod 3$, $b \not\equiv 0 \bmod 3$ and $c > 3$ an odd prime. By Lemma 2.12, there must be exactly two solutions (x_1, y_1, z_1) and (x_2, y_2, z_2), with $2 \nmid y_1 - y_2$. Take y_1 even. Then, since $c > 3$, consideration modulo 3 shows that $2 \mid x_1$, $2 \mid y_1$ and $2 \nmid z_1$, which requires $c \equiv 2 \bmod 3$ and $c \equiv 1 \bmod 4$. Thus, $c \equiv 5 \bmod 12$. □

PROOF OF THEOREM 1.4. Assume that a, b and c are distinct primes with $a < b$. If $N(a, b, c) > 1$, we can take $a = 2$ ($c \neq 2$ by Lemma 2.7, except for $(a, b, c) = (3, 5, 2)$ and $(3, 13, 2)$, given as exceptions in the statement of the theorem).

Additionally, we can also take $b \neq 3$: using Lemma 2.2, we find that if the equation $2^x + 3^y = c^z$ has two solutions (x_1, y_1, z_1) and (x_2, y_2, z_2), we either must have $c = 5$ with

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$\{z_1, z_2\} = \{1, 2\}$ or we must have $z_1 = z_2 = 1$, in which case we have $3^{y_1} - 2^{x_2} = 3^{y_2} - 2^{x_1} = d$, where $d = 1, -5$ or -13 by Lemma 2.11; recalling Lemma 1.2 and using the solutions given in cases (x) and (iii) in Conjecture 1.1, we see that $d = -5$ corresponds to $c = 35$, while using case (xii) with case (iv) shows that $d = -13$ corresponds to $c = 259$; since neither of these values of c is prime, we must have $d = 1$, and it is a familiar elementary result that we must have $(x_1, y_1, x_2, y_2) = (1, 2, 3, 1)$ giving $c = 11$.

Thus we find that the only cases with $a = 2$, $b = 3$ and $N(a, b, c) > 1$ are $(a, b, c) = (2, 3, 5)$ and $(2, 3, 11)$, given as exceptions in the statement of the theorem.

Now assume $a = 2$ and $c = 3$, and assume (1.1) has two solutions (x_1, y_1, z_1) and (x_2, y_2, z_2). By Lemma 2.12, we can assume $2 \mid y_1$. If z_1 is even, then $3^{z_1/2} - b^{y_1/2} = 2$ and $3^{z_1/2} + b^{y_1/2} = 2^{x_1-1}$, so that $3^{z_1/2} = 2^{x_1-2} + 1$; it is a familiar elementary result that we must have either $(x_1, z_1) = (3, 2)$ (giving $b^{y_1/2} = 1$) or $(x_1, z_1) = (5, 4)$ (giving $b^{y_1/2} = 7$). Since $b > 1$, we find that $(a, b, c) = (2, 7, 3)$ is the only possibility when z_1 is even.

Additionally, if $2 \nmid z_1$ when $a = 2$ and $c = 3$, then, since $2 \mid y_1$, we have $x_1 = 1$, and we can factor in $\mathbb{Q}(\sqrt{-2})$:

$$\pm b^{y_1/2} \pm \sqrt{-2} = (1 + \sqrt{-2})^{z_1},$$

where the '±' are independent. Since clearly $z_1 > 1$, we can use Lemma 2.10 to see that $z_1 = 3$ and $(a, b, c) = (2, 5, 3)$.

Thus, recalling Lemma 1.2, we find that the only cases with $a = 2$, $c = 3$ and $N(a, b, c) > 1$ are $(a, b, c) = (2, 7, 3)$ and $(2, 5, 3)$, given as exceptions in the statement of the theorem.

So, excluding the six exceptions given in the statement of the theorem, we can assume $a = 2$, $b \neq 3$ and $c \neq 3$ when $N(a, b, c) > 1$ for a triple of primes (a, b, c). Now, Theorem 1.4 follows immediately from Theorem 1.3, except for the conclusion $z_1 = 1$. To see that $z_1 = 1$, note that, since $2 \nmid z_1$, it suffices to use Lemma 2.4, noting $(a, b, c) = (2, 5, 3)$ and $(2, 7, 3)$ have been excepted, and handling $(2, 11, 5)$ using Lemma 2.1 with consideration modulo 3 and modulo 5. □

4. Proofs of Theorem 1.5 and Corollary 1.6

PROOF OF THEOREM 1.5. Assume a, b and c are distinct primes with $a < b$, $(a, b, c) \neq (2, 3, 5)$, $(2, 3, 11)$, $(2, 5, 3)$, $(2, 7, 3)$, $(3, 5, 2)$ or $(3, 13, 2)$, and $N(a, b, c) > 1$. From Theorem 1.4,

$$c \equiv 5 \bmod 12 \tag{4.1}$$

and

$$2^{x_1} + b^{y_1} = c, \quad 2 \mid x_1, 2 \mid y_1. \tag{4.2}$$

Assume first

$$b \equiv 1 \bmod 3. \tag{4.3}$$

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We have

$$2^{x_2} + b^{y_2} = c^{z_2}, \quad 2 \mid x_2, 2 \nmid y_2, 2 \nmid z_2. \tag{4.4}$$

Since $2 \mid x_2$ and $2 \nmid y_2$, by (4.1) and (4.4), we get $b \equiv b^{y_2} \equiv c^{z_2} - 2^{x_2} \equiv 1 - 0 \equiv 1 \bmod 4$. Hence, by (4.3),

$$b \equiv 1 \bmod 12. \tag{4.5}$$

If $c \equiv 5 \bmod 8$ and $b \equiv 1 \bmod 8$, then from (4.2) and (4.4), we get $x_1 = 2$ and $x_2 = 2$, respectively. This implies that (2.2) has two solutions $(m, n) = (1, y_1)$ and (z_2, y_2) for $(A, B) = (c, b)$ and $k = 2$. By Lemma 2.6, this is impossible. Hence, by (4.1) and (4.5), $(b, c) \not\equiv (1, 5) \bmod 24$ and

$$(b, c) \equiv (1, 17), (13, 5) \text{ or } (13, 17) \bmod 24. \tag{4.6}$$

Now assume $b \equiv 2 \bmod 3$. Then we have (4.1), (4.2) and

$$2^{x_2} + b^{y_2} = c^{z_2}, \quad 2 \nmid x_2, 2 \nmid y_2, \tag{4.7}$$

$$b \equiv 2 \bmod 3, \tag{4.8}$$

and

$$c^{z_2} \equiv 1 \bmod 3. \tag{4.9}$$

Further, by (4.1) and (4.9),

$$2 \mid z_2. \tag{4.10}$$

When $x_2 > 1$ and $y_2 > 1$, we see from (4.7) and (4.10) that (2.1) has a solution $(X, Y, m, n) = (c^{z_2/2}, b, x_2, y_2)$. Hence, by Lemma 2.5, $(b, c) = (17, 71)$, for which a solution (x_1, y_1, z_1) is impossible since $z_1 = 1$. So we have either $x_2 = 1$ or $y_2 = 1$.

If $y_2 = 1$, then from (4.7) and (4.10),

$$2^{x_2} + b = c^{z_2}, \quad 2 \nmid x_2, 2 \mid z_2. \tag{4.11}$$

Note that $x_2 > 1$ since $c^{z_2} > c$.

Apply Lemma 2.3 to (4.11) to obtain

$$b = (c^{z_2/2})^2 - 2^{x_2} > 2^{0.26x_2}. \tag{4.12}$$

Further, by (4.2), (4.11) and (4.12),

$$2^{x_2} + b = c^{z_2} \geq c^2 = (2^{x_1} + b^{y_1})^2 > b^{2y_1} \geq b^4 > 2^{1.04x_2}, \tag{4.13}$$

whence we obtain

$$b > 2^{x_2}(2^{0.04x_2} - 1). \tag{4.14}$$

However, by (4.2), we have $b < \sqrt{c}$. By Lemma 2.9, we can assume $\max\{b, c\} > 100$, so by (4.11), $2^{x_2} = c^{z_2} - b > c^2 - \sqrt{c} \geq 101^2 - \sqrt{101} > 10190$, whence we get $x_2 \geq 15$. So we have $2^{0.04x_2} - 1 \geq 2^{0.6} - 1 > 1/2$. By (4.14),

$$b > 2^{x_2 - 1}. \tag{4.15}$$

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Therefore, by (4.11), (4.13) and (4.15), we have $3b > 2^{x_2} + b = c^{z_2} \geq c^2 > b^4$, which is a contradiction. So we obtain $y_2 > 1$. This implies

$$x_2 = 1. \tag{4.16}$$

By (4.16), (4.1), (4.7) and (4.8),

$$b = 2^r h - 1, \quad c = 2^s k + 1, \quad r > 1, s > 1, 2 \nmid h, 2 \nmid k, 3 \mid h.$$

Recalling (4.16), (4.7) and (4.10), we have

$$2^{x_2} + b^{y_2} = 2 + (2^r h - 1)^{y_2} = 2^r h_1 + 1 = c^{z_2} = 2^{s + v_2(z_2)} k_1 + 1, \quad 2 \nmid h_1, 2 \nmid k_1, 3 \mid h_1, \tag{4.17}$$

where $v_2(n)$ is the greatest integer t such that $2^t \mid n$. From (4.17), we see that $r > s$, so that in (4.2), we must have $x_1 = s$.

Now we apply Lemma 2.13. Clearly $2 < c^{z_2/2}/4$ (recall Lemma 2.9), so that applying Lemma 2.13 to (4.2), we find that we must have

$$2^{x_1} = 2^s \geq \frac{c^{1/2}}{4} > \frac{b^{y_1/2}}{4} \geq \frac{b}{4} \geq \frac{3 \cdot 2^r - 1}{4} \geq \frac{3 \cdot 2^{s+1} - 1}{4} = 3 \cdot 2^{s-1} - \frac{1}{4}$$

so that

$$2 \geq 3 - \frac{1}{2^{s+1}},$$

which is false for all positive s.

Thus, we have $b \not\equiv 2 \bmod 3$. So $b \equiv 1 \bmod 3$ and (4.6) holds. □

PROOF OF COROLLARY 1.6. First, $N(a, b, c) = 2$ follows from Theorem 1.4. If $z_1 = z_2$, then, recalling Lemma 2.7 and noting that $c \neq 35, 133$ or 259, we must have $b^{y_2} - 2^{x_1} = b^{y_1} - 2^{x_2} > 0$, so that Lemma 2.1 gives $2 \nmid x_1 - x_2$. Consideration modulo 3 shows that this requires $2 \nmid y_1 - y_2$ and $b \equiv 2 \bmod 3$, contradicting Theorem 1.5. So $z_1 \neq z_2$, and by Theorem 1.4, $z_2 > 1$. □

5. The unlikelihood of counterexamples to Conjecture 1.7

We outline the algorithm used to justify Theorem 1.8.

For a given prime value of b and some small prime p (or small prime power), we will consider all solutions $(x_1, y_1, x_2, y_2, z_2)$ to $2^{x_1} + b^{y_1} \equiv c \bmod p$ and $2^{x_2} + b^{y_2} \equiv c^{z_2} \bmod p$. Note that the exponents are defined modulo $p - 1$.

When $b \equiv 13 \bmod 24$ and $c \equiv 5 \bmod 24$, we must have $x_1 = 2$. From above, $2 \mid y_1$, $2 \mid x_2$, $2 \nmid y_2$, $2 \nmid z_2 > 1$, and z_2 divides the class number of $\mathbb{Q}(-b)$.

For a given $b \equiv 13 \bmod 24$, we find all $z_2 > 1$ with z_2 odd and dividing the class number. Fix b and z_2. For a given prime p, we consider each $y_1 \bmod p - 1$ with y_1 even. Define $c \equiv 2^2 + b^{y_1} \bmod p$; for each value of x_2 even and y_2 odd modulo $p - 1$, we see if $2^{x_2} + b^{y_2} \equiv c^{z_2} \bmod p$. If there is a solution, we add (y_1, x_2, y_2) to a list of all possible solutions modulo this prime p. We now consider another small prime (or prime power) q. For each possible solution modulo p, we now see if there is a solution modulo q. If we are fortunate, there are no solutions modulo q that are consistent with a solution modulo p, in which case this choice of b and z_2 cannot have any solutions. Otherwise, we add another prime r and see if any solutions are consistent modulo r. Often for a given b, z_2 has no consistent solutions after checking a few primes (or prime powers). In rare instances, the program required up to fourteen primes or prime powers to eliminate all possible solutions for a given b and z_2, but we never needed primes exceeding 241.

For $b \equiv 13 \bmod 24$ and $c \equiv 17 \bmod 24$, we must have $x_2 = 2$. The procedure is similar except that we now consider tuples (x_1, y_1, y_2). For $b \equiv 1 \bmod 24$, we cannot specify x_1 and x_2 so there are many more cases to check, but the same essential algorithm can be used.

We used Maple® for some preprocessing, then used Sage® (in which we could access the Pari® class number command) for the calculations. Total calculation time was approximately 100 hours. See the last author's website for programs and details.

In summary, we showed that for primes $b < 10^9$, there are no solutions $(x_1, y_1, z_1, x_2, y_2, z_2)$ outside those listed in Conjecture 1.7. Since $c > b^2$, we have $c > 10^{18}$. This concludes the demonstration of Theorem 1.8.

In another direction, it is interesting to note that the case $(b, c) \equiv (13, 17) \bmod 24$ requires the equation

$$4 + b^{y_2} = c^{z_2}, \quad 2 \nmid y_2 > 1, 2 \nmid z_2 > 1. \tag{5.1}$$

Note that if $y_2 = 1$, then $c = 2^{x_1} + b^{y_1} > 4 + b = c^{z_2}$, which is impossible; note also that $z_2 = 1$ is impossible by Corollary 1.6.

The conditions in (5.1) are extremely unlikely even without the extra consideration of an additional solution (x_1, y_1, z_1).

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