Let $M$ be the mid point of $PQ$.

\[ AM^2 + PM^2 = AO^2 + r^2, \]
\[ \therefore AM^2 + r^2 - OM^2 = AO^2 + r^2, \]
\[ \therefore \angle AOM = 90^\circ \]

But \[ \angle OMP = 90^\circ \]
\[ \therefore PQ \text{ is parallel to } OA. \]

I venture to say that 99 per cent of casual readers will see nothing wrong about this. But what if $M$ is at the centre?

(The student of elementary geometrical conics will easily prove that if, more generally, $AP^2 + AQ^2 = c^2$, then $PQ$ envelopes a parabola with focus at $O$. In the special case the parabolic envelope breaks down into a couple of points, one at infinity, the other the centre).

John Dougall.

The solution of "homogeneous" quadratics.

\[ 2x^2 - 5xy + 4y^2 = 4 \]
\[ 3y^2 - x^2 = 3 \] \hspace{1cm} (1)

A common method is to put $y = mx$.

\[ x^2(2 - 5m + 4m^2) = 4 \]
\[ x^2(3m^2 - 1) = 3 \] \hspace{1cm} (2)

By division \[ \frac{x^2(2 - 5m + 4m^2)}{x^2(3m^2 - 1)} = \frac{4}{3} \] \hspace{1cm} (3)

or \[ \frac{2 - 5m + 4m^2}{3m^2 - 1} = \frac{4}{3} \] \hspace{1cm} (4)

\[ m = \frac{2}{3}, \]
and from either of (1), $x = \pm 3$, $y = \pm 2$.

The point is that we have missed the obvious solutions $x = 0$, $y = \pm 1$. We dropped them at the passage from (3) to (4). In fact from (3) we can only infer (4), if $x$ is not zero, so that we should say, either $x = 0$, or

\[ \frac{2 - 5m + 4m^2}{3m^2 - 1} = \frac{4}{3}, \]
and then try $x = 0$ in (1).
MATHEMATICAL NOTES.

Instead of putting \( y = mx \), it is therefore perhaps preferable to form a homogeneous equation from the two given equations, by multiplying them by 3 and 4 and subtracting.

Thus

\[
6x^2 - 15xy + 12y^2 = 12y^2 - 4x^2,
\]

or

\[x(2x - 3y) = 0,\]

\[x = 0, \text{ or } 2x - 3y = 0.\]

The example illustrates the danger of cancelling a common factor from numerator and denominator of a fraction without considering the possibility of that factor being zero.

It may also be found useful (as the Editor remarks to me) as the basis of a lesson on infinite roots of an equation. The equation for \( m \), which we expect to be a quadratic, turns out to be of the first degree. The second value of \( \frac{y}{x} \) is here \( \frac{1}{0} \) or infinity.

It may even happen that both values of \( m \) are infinite. A boy with \( y = mx \) as his only resource would be rather nonplussed with the example

\[
x^2 + xy + y^2 = 1,
\]

\[
2x^2 + 3xy + 3y^2 = 3.
\]

Infinite roots appear in another way in this class of equations, namely, in the case when the quadratic functions in the two given equations have a common factor.

Take

\[(x + y)(x - y) = 3, \quad (1)\]

\[(x + y)(2x + y) = 15. \quad (2)\]

Here

\[(x + y)(2x + y) = 5(x + y)(x - y),\]

\[x + y = 0 \quad \text{or} \quad x = 2y.\]

If we put \( x + y = 0 \) in (1) we get \( 0 = 3 \), and we say that the equations have no solution which makes \( x + y = 0 \).

But if we use the \( y = mx \) method, we find \( m = -1 \) or \( \frac{1}{2} \).

Then from (1), \( x^2(1 - m^2) = 3, \)

\[x^2 = \frac{3}{1 - m^2}.\]

If \( m = \frac{1}{2} \), this gives \( x = \pm 2, y = \pm 1 \); but if \( m = -1 \), \( x = \pm \infty, \)

\[y = \mp \infty.\]

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