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A COUNTEREXAMPLE TO A CONJECTURE OF D. B. FUKS

BY
LUC DEMERS

Introduction. In [3] D. B. Fuks defined a duality of functors in the category $\mathscr{H}$ of weak homotopy types. In general this duality is more difficult to work with than the duality of functors of the category $\mathscr{T}$ of pointed Kelley spaces [2]. It happens however that all so-called strong functors [2] F of $\mathscr{T}$ induce functors $\bar{F}$ of $\mathscr{H}$, and if we denote the duality operators of $\mathscr{H}$ and $\mathscr{T}$ by $\mathscr{D}$ and D respectively, then there are many cases where $(\overline{DF})=\mathscr{D}(\bar{F})$.

This has lead Fuks to make the following conjecture: a functor F of the category $\mathscr{H}$ is reflexive (i.e. $F\simeq\mathscr{D}^2F$) if and only if there exists a functor G of $\mathscr{T}$ such that $F=\bar{G}$ and $\mathscr{D}F=\overline{DG}$.

Not only would this conjecture enable us to compute $\mathscr{D}F$ in the most interesting cases, but it would imply the following strong corollary.

COROLLARY. *Let G_1 and G_2 be reflexive functors of the category $\mathscr{T}$, and let $f\colon G_1\to G_2$ be a natural transformation such that for any C.W. complex A, $f_A\colon G_1(A)\to G_2(A)$ is a weak homotopy equivalence. Then for any C.W. complex A,*

$$(Df)_A\colon DG_2(A)\to DG_1(A)$$

is also a weak homotopy equivalence.

Unfortunately, we will provide a counterexample to this corollary, which will prove that Fuks' conjecture is false.

1. **The Counterexample.** Consider the functor $QX=$space of paths in X which start or end at the base point.

In other words, QX is the pull-back of the diagram

$$\begin{array}{ccc} & & (I',X) \\ & & \downarrow \\ X\vee X & \to & X\times X \end{array}$$

where $I'=$ disjoint union of the unit interval [0,1] with a point $*$ serving as the base point. The horizontal map is the inclusion, and the vertical map is defined as $\lambda\rightsquigarrow(\lambda(0),\lambda(1))$.

In [4, p. 210] there is defined a natural transformation

$$m\colon\Sigma\Omega\to Q$$

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with the property that for any 1-connected C.W. complex X, $m_X: \Sigma\Omega X \to QX$ is a homotopy equivalence.

This implies in particular that

$$(m * \Sigma^2)_X: \Sigma\Omega\Sigma^2 X \to Q\Sigma^2 X$$

is a homotopy equivalence for any C.W. complex X.

Now if we assume that Q is a reflexive functor (which will be proved later), Fuks' conjecture would imply that

$$((Dm) * \Omega^2)_X: DQ \circ \Omega^2 X \to \Omega\Sigma\Omega^2 X$$

is a homotopy equivalence for all C.W. complexes X.

In particular, if we take $X = K(Z, 3)$, we have $\Omega^2 X \simeq S^1 = K(Z, 1)$, so that $DQ(S^1) \simeq \Omega\Sigma(S^1)$. But from the explicit computation of DQ, it will be clear that $DQ(S^1)$ is a finite complex, while the homology of $\Omega\Sigma(S^1)$ is infinite, by a theorem of Bott and Samelson [1]. Hence $DQ(S^1)$ and $\Omega\Sigma(S^1)$ cannot have the same homotopy type.

We will now introduce a new functor which will be proved to be the dual of Q.

2. **The Functor** T. QX was defined as the pull-back of the natural transformations

$$X \vee X \to X \times X$$

and

$$(I', X) \to X \times X.$$

Now if we take the dual of this situtation, we get two transformations:

$$X \vee X \to X \times X$$

and

$$X \vee X \to X \wedge I'$$

where the first map is the same as the first one above, and the second is defined as

$$(x, *) \rightsquigarrow (x, 0)$$

$$(*, x) \rightsquigarrow (x, 1)$$

We define then TX to be the push-out of the diagram

$$\begin{array}{ccc} X \vee X & \to & X \times X \\ \downarrow & & \\ X \wedge I' & & \end{array}$$

Since the duality operator D transforms direct limits into inverse limits, it is clear that $DT \simeq Q$. Because of this, there is a natural transformation $\sigma: T \to DQ$ corresponding to the canonical transformation $T \to D^2 T$. Explicitly, σ is defined as follows:

From the definition of Q as a pull-back, we have two natural transformations

$$\omega_X: QX \to X \vee X \qquad (\omega_X(\lambda) = (\lambda(0), \lambda(1)))$$

and

$$\tau_X: QX \to (I', X) \qquad (\tau_X(\lambda) = \lambda).$$

If we denote the wedge functor by W (i.e. $W(X) = X \vee X$), then there is a natural equivalence $\alpha_X: X \times X \to D(W)(X)$ (see [2]) given by the formula

$$\alpha_X(x_0, x_1)_Y(y, *) = (x_0, y) \in X \wedge Y$$

$$\alpha_X(x_0, x_1)_Y(*, y) = (x_1, y) \in X \wedge Y.$$

Thus we have a natural map $X \times X \to DQX$ obtained as the composition

$$X \times X \xrightarrow{\alpha_X} D(W)(X) \xrightarrow{(D\omega)_X} DQ(X).$$

Explicitly,

$$\begin{aligned} [(D\omega)_X \circ \alpha_X(x_0, x_1)]_Y(\lambda) &= \alpha_X(x_0, x_1)_Y \omega_Y(\lambda) \\ &= \alpha_X(x_0, x_1)(\lambda(0), \lambda(1)) \\ &= (x_0, \lambda(0)) \quad \text{if } \lambda(1) = * \\ &= (x_1, \lambda(1)) \quad \text{if } \lambda(0) = *. \end{aligned}$$

On the other hand, by taking the dual of τ, we obtain a map $(D\tau)_X: X \wedge I' \to DQ(X)$ given by the formula $(D\tau)_X(x, t)_Y(\lambda) = (x, \lambda(t)) \in X \wedge Y$, where $\lambda \in QY$. It is easy to see that the two maps

$$X \times X \to DQX$$

and

$$X \wedge I' \to DQX$$

agree on $X \vee X$, so that they combine to give us a map

$$\sigma_X: TX \to DQX.$$

3. **The Functor $\bar{T}$ and the Map $\eta: DQ \to \bar{T}$.** Instead of finding directly an inverse for σ, we will introduce an auxiliary functor $\bar{T}$, define $\eta: DQ \to \bar{T}$ and show first that $\eta \circ \sigma$ is a natural equivalence of functors.

As usual, I will stand for the unit interval [0, 1] with 0 as the base-point; we will let J denote the unit interval [0, 1] with 1 as the base point.

$\rho: I \to S^1$ and $\pi: J \to S^1$ will denote the usual identification maps. $\bar{T}X$ is then defined as the pull-back of the diagram

$$\begin{array}{ccc} & & X \wedge I \\ & & \downarrow X \wedge \rho \\ X \wedge J & \xrightarrow{X \wedge \pi} & \Sigma X \end{array}$$

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Define $\mu_X: DQX \to X \wedge J$ as $\mu(S) = T_J(\mathrm{id}_J)$, where $S: Q \to \Sigma_X$ is an element of DQX, and id_J denotes the identity map of J, and

$$\nu_X: DQX \to X \wedge I$$

as

$$\nu_X(S) = S_I(\mathrm{id}_I).$$

These maps are continuous, since they are the composition of the following continuous maps:

$$(Q, \Sigma_X) \xrightarrow[\text{at } J]{\text{evaluation}} (QJ,\ X \wedge J) \xrightarrow[\text{at id}_J]{\text{evaluation}} X \wedge J$$

and

$$(Q, \Sigma_X) \xrightarrow[\text{at } I]{\text{evaluation}} (QI,\ X \wedge I) \xrightarrow[\text{at id}_I]{\text{evaluation}} X \wedge I.$$

Moreover, because of the naturality of $S: Q \to \Sigma_X$, we have

$$(X \wedge \rho) \circ \nu_X(S) = (X \wedge \pi) \circ \mu_X(S) = S_{S^1}(\rho).$$

Thus the maps μ_X and ν_X induce a unique $\eta_X: DQX \to \bar{T}X$.

We will now prove that $\eta \circ \sigma$ is a natural equivalence.

4. **The Natural Equivalence** $\eta \circ \sigma: T \to \bar{T}$. From the definition of T as a push-out, we have that

$$TX = X \wedge I' \bigcup_{X \vee X} X \times X.$$

Let $(x, t) \in X \wedge I'$. We have

$$\mu_X \circ \sigma_X(x, t) = \sigma_X(x, t)_J\,(\mathrm{id}_J) = (x, t) \in X \wedge J$$

$$\nu_X \circ \sigma_X(x, t) = \sigma_X(x, t)_I\,(\mathrm{id}_I) = (x, t) \in X \wedge I.$$

Hence

$$\eta_X \circ \sigma_X(x, t) = ((x, t), (x, t)) \in X \wedge J \times X \wedge I.$$

On the other hand,

$$\mu_X \circ \sigma_X(x_0, x_1) = \sigma_X(x_0, x_1)(\mathrm{id}_J) = (x_0, 0) \in X \wedge J$$

$$\nu_X \circ \sigma_X(x_0, x_1) = \sigma_X(x_0, x_1)_I(\mathrm{id}_I) = (x_1, 1) \in X \wedge I.$$

Thus $(\eta \circ \sigma)_X: TX \to \bar{T}X$ is the map defined as

$$\left.\begin{aligned} (\eta\sigma)_X(x, t) &= ((x, t), (x, t)) \\ (\eta\sigma)_X(x_0, x_1) &= ((x_0, 0), (x_1, 1)) \end{aligned}\right\} \in X \wedge J \times X \wedge I.$$

It is clear that $(\eta\sigma)_X$ is both an injection and a surjection, so that it remains to show that its inverse is continuous. To that end, we define two subspaces A and B of $\bar{T}X$ with the properties that

(1) A and B are closed in $\bar{T}X$
(2) $\bar{T}X = A \cup B$
(3) $(\eta\sigma)^{-1}$ is continuous on both A and B.

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$$A = \{((x, t), (x, t)) \in X \wedge J \times X \wedge I\}$$
$$B = \{((x_0, 0), (x_1, 1)) \in X \wedge J \times X \wedge I\}$$

Note first that under the homeomorphism

$$X \wedge J \times X \wedge I \to X \wedge I \times X \wedge I$$

sending $((x, t), (y, s))$ to $((x, 1-t), (y, s))$, A and B are mapped respectively onto

$$A^1 = \{((x, 1-t), (x, t)) \in X \wedge I \times X \wedge I\}$$

and

$$B^1 = \{(x, 1) \times (x_1, 1) \in X \wedge I \times X \wedge I\}$$

Let $q\colon X \times I \to X \wedge I$ be the indentification map, and consider

$$q \times q\colon (X \times I) \times (X \times I) \to (X \wedge I) \times (X \wedge I).$$

The inverse image of A^1 under $q \times q$ is the union of the four subspaces

$\{((x, 1-t), (x, t)) \in (X \times I) \times (X \times I)\}$

$\{((x, 0), (y, 1)) \in (X \times I) \times (X \times I)\}$

$\{(x, 1), (y, 0) \in (X \times I) \times (X \times I)\}$

$(q \times q)^{-1}(*)$

These are all closed subsets of $(X \times I) \times (X \times I)$. If we can show that $q \times q$ is an identification map, then we will have proved that A^1 is closed.

But in the category of nonpointed Kelley spaces, the product with a fixed space is left adjoint to a hom functor and hence commutes with direct limits. Thus we have two push-out diagrams:

$$\begin{array}{ccc} (X \times \{0\}) \times (X \times I) & \longrightarrow & (X \times I) \times (X \times I) \\ \downarrow & & \downarrow{\scriptstyle q'} \\ * \times (X \times I) & \longrightarrow & (X \wedge I) \times (X \times I) \end{array}$$

and

$$\begin{array}{ccc} (X \wedge I) \times (X \times \{0\}) & \longrightarrow & (X \wedge I) \times (X \times I) \\ \downarrow & & \downarrow{\scriptstyle q''} \\ (X \wedge I) \times * & \longrightarrow & (X \wedge I) \times (X \wedge I) \end{array}$$

Since q' and q'' are both identification maps, so is their composition $q'' \circ q' = q \times q$. Thus A' is closed. As for B', it is the product of two closed subsets of $X \wedge I$. It remains only to show that $(\eta \circ \sigma)^{-1}$ is continuous on both A and B.

For A, we have the following commutative diagram

$$\begin{array}{ccc} X \wedge I' & \xrightarrow{(\eta\sigma)} & (X \wedge J) \times (X \wedge I) \\ \uparrow{\scriptstyle p} & & \uparrow{\scriptstyle r \times q} \\ X \times I & \xrightarrow{\nabla} & (X \times I) \times (X \times I) \end{array}$$

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where p, q, and r are identification maps and ∇ is the map

$$\nabla(x, t) = ((x, 1-t), (x, t)).$$

Since ∇ is a homeomorphism into, $(r\times q)\circ\nabla$ is an identification map onto its image, which is A. Hence $\eta\sigma$ is a homeomorphism of $X \wedge I'$ onto A, because p is also an identification map.

As for B, it is clear that $\eta\sigma$: $X\times X\to B$ is a homeomorphism, because B has the topology of a product.

This insures the continuity of $(\eta\sigma)^{-1}$ over the whole of $\overline{T}X$.

5. **σ And η are Natural Equivalence.** We have just shown that $\eta\circ\sigma$ is an isomorphism of functors. But η is a monomorphism. Indeed, let R, S: $Q\to\Sigma_X$ be two elements of DQX. We have

$$\eta_X(R) = (R_J(\mathrm{id}_J), R_I(\mathrm{id}_I))$$
$$\eta_X(S) = (S_J(\mathrm{id}_J), S_I(\mathrm{id}_I)).$$

Suppose that $\eta_X(R)=\eta_X(S)$, and let Y be any space, and $\lambda\in QY$.

If λ is a map $I\to Y$, then the naturality of R implies that the following diagram is commutative:

$$\begin{array}{ccc} Q(I) & \xrightarrow{R_I} & X\wedge I \\ {\scriptstyle Q(\lambda)}\downarrow & & \downarrow{\scriptstyle X\wedge\lambda} \\ Q(Y) & \xrightarrow{R_Y} & X\wedge Y \end{array}$$

i.e. $X\wedge\lambda(R_I(\mathrm{id}_I))=R_Y(\lambda)$.

If $\eta_X(S)=\eta_X(R)$, we have $R_I(\mathrm{id}_I)=S_I(\mathrm{id}_I)$, so that $R_Y(\lambda)=S_Y(\lambda)$. A similar thing happens if λ is a map $J\to Y$, so that in all cases, $R_Y(\lambda)=S_Y(\lambda)$, i.e. $R=S$.

Thus η_X is a monomorphism. This and the fact that $\eta_X\sigma_X$ is a homeomorphism imply that both η_X and σ_X are homeomorphisms, i.e. σ and η are natural equivalences.

Hence $T\simeq DQ$, which proves that T, (and hence Q) is reflexion. It is clear that if X is a finite C.W. complex, TX is also a finite C.W. complex (cf. §1).

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University of Ottawa,
Ottawa, Ontario