A CHARACTERIZATION OF MINIMAL PRIME IDEALS

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Abstract. Let P be a prime ideal of a ring R, $O(P)=\{a \in R \mid aRs=0$, for some $s \in R\backslash P\}$ and $\overline{O}(P)=\{x \in R \mid x^n \in O(P)$, for some positive integer $n\}$. Several authors have obtained sheaf representations of rings whose stalks are of the form $R/O(P)$. Also in a commutative ring a minimal prime ideal has been characterized as a prime ideal P such that $P=\overline{O}(P)$. In this paper we derive various conditions which ensure that a prime ideal $P=\overline{O}(P)$. The property that $P=\overline{O}(P)$ is then used to obtain conditions which determine when $R/O(P)$ has a *unique* minimal prime ideal. Various generalizations of $O(P)$ and $\overline{O}(P)$ are considered. Examples are provided to illustrate and delimit our results.

0. Introduction. Throughout this paper R denotes an associative ring not necessarily with unity, $\mathbf{P}(R)$ its prime radical, $\mathbf{N}(R)$ its set of all nilpotent elements and $\mathbf{N}_r(R)$ its nil radical. R is called a *2-primal ring* if $\mathbf{P}(R)=\mathbf{N}(R)$. We refer to **[4]**, **[5]**, **[6]**, **[8]**, **[9]**, **[17]**, and **[19]** for more details on 2-primal rings.

A proper ideal P of R is called *completely prime* (*completely semiprime*) if $xy \in P$ ($x^2 \in P$) implies $x \in P$ or $y \in P$ ($x \in P$). Andrunakievic and Rjabuhin **[1]** and , independently, Stewart **[18]** have shown that a reduced ring R (i.e., $\mathbf{N}(R)=0$) is a subdirect product of integral domains. Thus a proper ideal is completely semiprime if and only if it is an intersection of completely prime ideals.

All prime ideals are taken to be proper ideals. Let X be a nonempty subset of R, then $<X>_R$, $\ell(X)$ and $r(X)$ denote the ideal of R generated by X, the left annihilator of X in R, and the right annihilator of X in R, respectively. Let P be a prime ideal. The following definitions are fundamental to the remainder of our discussion:

$$O_P=\{a \in R \mid as=0,\ \text{for some } s \in R\backslash P\}=\bigcup_{s\in R\backslash P}\ell(s),$$

$$\overline{O}_P=\{x \in R \mid x^n \in O_P,\ \text{for some positive integer } n\},$$

$$N_P=\{y \in R \mid ys \in \mathbf{P}(R),\ \text{for some } s \in R\backslash P\},$$

$$O(P)=\{a \in R \mid aRs=0,\ \text{for some } s \in R\backslash P\}=\bigcup_{s\in R\backslash P}\ell(Rs),$$

$$\overline{O}(P)=\{x \in R \mid x^n \in O(P),\ \text{for some positive integer } n\},$$

and

$$N(P)=\{y \in R \mid yRs \subseteq \mathbf{P}(R),\ \text{for some } s \in R\backslash P\}.$$

Observe that $O(P) \subseteq N(P) \subseteq P$. Furthermore if P is a completely prime ideal, then $\overline{O}_P \subseteq P$ and $N_P \subseteq P$.

EXAMPLE 0.1. Let F be a field and

$$R=\begin{pmatrix} F & F \\ 0 & F \end{pmatrix},\quad P_1=\begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix},\quad \text{and}\quad P_2=\begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}.$$

Then P_1 and P_2 are all the prime ideals of R and satisfy the following properties: (i) $O_{P_1} = \left\{\begin{pmatrix} x & xy \\ 0 & 0 \end{pmatrix} \mid x, y \in F\right\}$, which is not an ideal of R and $O_{P_2} = P_2$; (ii) $\overline{O}_{P_1} = P_1$ and $\overline{O}_{P_2} = P_2$; (iii) $N_{P_1} = P_1$ and $N_{P_2} = P_2$; (iv) $O(P_1) = 0$ and $O(P_2) = P_2$; (v) $\overline{O}(P_1) = \mathbf{P}(R) \neq P_1$ and $\overline{O}(P_2) = P_2$; and (vi) $N(P_1) = P_1$ and $N(P_2) = P_2$.

Various authors **[10]**, **[12]**, **[13]**, **[14]**, **[17]** and **[20]** have obtained sheaf representations of rings whose stalks are of the form $R/O(P)$. In Lemma 3.1 of **[11]** Kist characterized a minimal prime ideal of a commutative ring R as a prime ideal P such that $P = \overline{O}_P$ (our terminology). It is clear that for a prime ideal P in a commutative ring, $O_P = O(P)$ and $\overline{O}(P) = \overline{O}_P = N(P) = N_P$. Also observe that in a reduced ring, $O(P) = O_P = \overline{O}(P) = \overline{O}_P = N(P) = N_P$. Next Shin in Corollary 1.10 of **[17]** generalized Kist's result by proving that in a 2-primal ring a prime ideal P is a minimal prime ideal if and only if $P = N(P)$.

In this paper we provide examples (Examples 2.6 and 2.8) of 2-primal rings and minimal prime ideals P such that $\overline{O}_P \neq N(P)$ and $\overline{O}(P) \neq N(P)$. Since $\overline{O}(P)/O(P)$ is the set of all nilpotent elements in the ring $R/O(P)$, the condition that $P = \overline{O}(P)$ gives us important information about the ring $R/O(P)$. We derive various conditions for noncommutative rings which allow us to characterize a minimal prime ideal P as one for which $P = \overline{O}_P$ or $P = \overline{O}(P)$ (Theorem 2.3). In particular, we show that the ring

$$R = \begin{pmatrix} A_1 & A_{12} & \cdots & A_{1n} \\ 0 & A_2 & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{pmatrix},$$

where each A_i is a ring with unity and A_{ij} is a left A_i-right A_j-bimodule for $i < j$, is a 2-primal strongly π-regular ring if and only if each A_i is a 2-primal strongly π-regular ring (Theorem 2.9). Therefore we can see immediately that every $n \times n$ upper triangular matrix ring over a 2-primal strongly π-regular ring with unity is a 2-primal strongly π-regular ring. As a consequence of this result and Theorem 2.8 of **[6]**, we obtain that $P = \overline{O}_P$ for every prime ideal P in such rings. As we illustrated in Example 0.1, the conditions $P = \overline{O}(P)$ and $P = \overline{O}_P$ are distinct. Thus our results generalize Kist's characterization and are distinct from Shin's results. Moreover we obtain conditions which ensure that $P/O(P)$ is the unique minimal prime ideal of $R/O(P)$ (Theorem 3.3).

1. Preliminaries and basic results.

We start with the following result.

PROPOSITION 1.1. *Let P be a prime ideal of R.*

(i) $O(P) = \sum_{s \in R \setminus P} \ell(Rs) = Sa \in R \mid (aR) \nsubseteq PS$.

(ii) $\mathbf{N}(R) \subseteq \overline{O}(P)$ *and if* $\overline{O}(P)$ *or* $\overline{O}_P$ *is an ideal then it is a completely semiprime ideal.*

(iii) *If* $\overline{O}(P)$ *or* $\overline{O}_P$ *is an ideal, then every minimal prime ideal belonging to* $\overline{O}(P)$ *or* $\overline{O}_P$*, respectively, is completely prime.*

(iv) $\mathbf{N}(R/O(P)) = \overline{O}(P)/O(P)$.

(v) $O(P) \subseteq O_P$, *(hence* $\overline{O}(P) \subseteq \overline{O}_P$*), and* $N(P) \subseteq N_P$.

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Proof. Parts (ii) and (iv) are immediate consequences of definitions.

(i) Let $x, y \in O(P)$. There exist $s_1, s_2 \in R\backslash P$ such that $xRs_1 = 0 = yRs_2$. Then there exists $b \in R$ such that $s_1bs_2 \in R\backslash P$. Hence $(x-y)R(s_1bs_2) = 0$. So $O(P)$ is an additive subgroup of R. The remainder of this part of the proof is straightforward.

(iii) This part follows from Proposition 1.6 of [**4**].

(v) Let $x \in O(P)$. There exists $s \in R\backslash P$ such that $xRs = 0$. In this case, there exists $b \in R$ such that $bs \in R\backslash P$, otherwise $Rs \subseteq P$ so $R^2s \subseteq P$, hence $R \subseteq P$, a contradiction. Now since $xbs = 0$ with $bs \in R\backslash P$, it follows that $x \in O_P$. Similarly $N(P) \subseteq N_P$.

We use Spec(R) and mSpec(R) to denote the set of prime ideals of R and the set of minimal prime ideals of R, respectively.

Proposition 1.2. *Let R be a 2-primal ring. Then:*

(i) $\overline{O}_P \subseteq N_P \subseteq P$;

(ii) $\overline{O}(P) \subseteq N(P) \subseteq P$;

(iii) $\bigcap_{P\in \mathrm{mSpec}(R)} \overline{O}(P) = \bigcap_{P\in \mathrm{Spec}(R)} \overline{O}_P = \mathbf{N}(R) = \mathbf{P}(R)$.

Proof. (i) Let $a \in \overline{O}_P$. Then there exists $b \in R\backslash P$ and a positive integer n such that $a^nb = 0 \in \mathbf{N}(R)$. Since $\mathbf{N}(R)$ is completely semiprime, $ab \in \mathbf{N}(R)$. Then $a \in N_P$.

Now assume that $c \in N_P$. Then there exists $d \in R\backslash P$ and a positive integer m such that $(cd)^m = 0$. There exists a minimal prime ideal Q of R such that $Q \subseteq P$. By Proposition 1.11 of [**17**], Q is completely prime. Hence $(cd)^m = 0 \in Q$ implies $c \in Q \subseteq P$. Thus $N_P \subseteq P$.

(ii) Let $x \in \overline{O}(P)$. Then $x^nRb = 0 \in \mathbf{P}(R)$ for some positive integer n and $b \in R\backslash P$. Since $\mathbf{P}(R)$ is completely semiprime, $xRb \subseteq \mathbf{P}(R)$. Hence $x \in N(P)$. So $\overline{O}(P) \subseteq N(P)$. Clearly, $N(P) \subseteq P$.

(iii) Using parts (i) and (ii), we have

$$\mathbf{N}(R) \subseteq \bigcap_{P\in \mathrm{mSpec}(R)} \overline{O}(P) \subseteq \bigcap_{P\in \mathrm{mSpec}(R)} \overline{O}_P \subseteq \bigcap_{P\in \mathrm{mSpec}(R)} P = \mathbf{P}(R) = \mathbf{N}(R).$$

Also

$$\mathbf{N}(R) \subseteq \bigcap_{P\in \mathrm{Spec}(R)} \overline{O}_P \subseteq \bigcap_{P\in \mathrm{Spec}(R)} P = \mathbf{P}(R).$$

The following definitions are critical to our characterizations of minimal prime ideals.

Definition 1.3. Let $x, y \in R$ and n a positive integer. We say R satisfies the

(i) *(CZ1) condition* if whenever $(xy)^n = 0$ then $x^my^m = 0$, for some positive integer m,

(ii) *(CZ2) condition* if whenever $(xy)^n = 0$ then $x^mRy^m = 0$, for some positive integer m.

Observe that any local ring with nil Jacobson radical satisfies condition (CZ2).

Using the following definitions, Shin [**17**] was able to generalize various sheaf representations of Hofmann [**10**], Koh [**12**] and [**13**], and Lambek [**14**].

Definition 1.4. (i) A ring R is called *almost symmetric* if it satisfies the following two conditions:

(SI) $r(x)$ is an ideal for each $x \in R$;

(SII) for any $a, b, c \in R$, if $a(bc)^n = 0$ for a positive integer n, then $ab^mc^m = 0$ for some positive integer m.

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(ii) A ring R is called *pseudo symmetric* if it satisfies the following two conditions:

(PSI) R/I is 2-primal whenever $I = 0$ or $I = r(aR)$ for some $a \in R$;

(PSII) for any $a, b, c \in R$, if $aR(bc)^n = 0$ for a positive integer n, then $a(RbR)^m c^m = 0$ for some positive integer m.

Lambek [**14**] calls a ring R with unity *symmetric* provided $abc = 0$ implies $acb = 0$ for any $a, b, c \in R$. Note that commutative rings and reduced rings are symmetric. Let $S = Z + Zi + Zj + Zk$ be the ring of integer quaternions and Z_4 the ring of integers modulo 4. Then the ring $S \oplus Z_4$ is symmetric, but it is neither commutative nor reduced. Symmetric rings are almost symmetric, but there is an example of an almost symmetric ring which is not symmetric in Example 5.1(a) of [**17**]. By Proposition 1.6 of [**17**] almost symmetric rings are pseudo symmetric. But there is a pseudo symmetric ring in Example 5.1(c) of [**17**] which is not almost symmetric. Shin [**17**, pp. 44–45] observed that rings with the (SI) condition are 2-primal and $O_P = O(P)$. In particular, almost symmetric rings are 2-primal and satisfy $O_P = O(P)$.

LEMMA 1.5. (i) *If R satisfies condition* (PSII), *then R satisfies condition* (CZ2).

(ii) *If R satisfies either condition* (SII) *or condition* (CZ2), *then R satisfies condition* (CZ1).

(iii) *If R has a unity and satisfies condition* (CZ1), *then every idempotent is central.*

(iv) *If R satisfies condition* (SI), *then conditions* (CZ1) *and* (CZ2) *are equivalent.*

(v) *If R satisfies condition* (PSII) *and has a right unity, then R is a 2-primal ring.*

Proof. Assume $x, y \in R$ such that $(xy)^n = 0$.

(i) Then $xR(xy)^n = 0$. Hence $x(RxR)^k y^k = 0$. So $x^m R y^m = 0$ for some m.

(ii) If R satisfies condition (SII), then $x(xy)^n = 0$ implies $x^{m+1}y^m = 0$. Therefore it follows that $x^{m+1}y^{m+1} = 0$. If R satisfies (CZ2), then $x^m R y^m = 0$. So $x^{m+1}y^{m+1} = x^m(xy)y^m = 0$.

(iii) This part is a direct consequence of Lemma 2.3 in [**6**].

(iv) This part is straightforward.

(v) Let $x \in R$ with $x^n = 0$. Then $xR(xc)^n = 0$, where c is a right unity for R. Hence there is an m such that $x(RxR)^m = 0$. So $x \in \mathbf{P}(R)$. Therefore R is a 2-primal ring.

DEFINITION 1.6. A ring R is called a *permutation identity ring* if for some $n \geq 2$, there exists a permutation $\sigma \neq 1$, on n symbols, such that $x_1 x_2 \cdots x_n = x_{\sigma 1} x_{\sigma 2} \cdots x_{\sigma n}$, for each $x_1, x_2, \ldots, x_n \in R$.

Observe that commutative rings satisfy the permutation identity given by $\sigma = (12)$ (for $n = 2$). The permutation identity rings given by the permutation $\sigma = (23)$ (for $n = 3$), in general, do not have unity but satisfy Lambek's condition in [**14**] ($abc = 0$ implies $acb = 0$). So these rings are almost symmetric. The ring of 4×4 strictly upper triangular matrices over a ring with unity and not of characteristic two is a permutation identity ring (take $\sigma = (23)$ for $n = 4$) but it does not satisfy condition (SI). Thus it is *not* almost symmetric. However our next result shows that every permutation identity ring is pseudo symmetric. Thus the class of permutation identity rings provides a large class (see [**3**], [**15**], and [**16**] for numerous examples and constructions) of pseudo symmetric rings.

PROPOSITION 1.7. *If R is a permutation identity ring, then R is pseudo symmetric.*

Proof. The class of permutation identity rings is closed under homomorphic images, and it is contained in the class of 2-primal rings by Corollary 2.10 of [**3**]. Hence a permutation identity ring satisfies condition (PSI). Now let $a, b, c \in R$ such that $aR(bc)^n = 0$ for some positive integer n. From [**3**, p. 127] or [**15**], there exists a positive integer k such that $uxyv = uyxv$ for $u, v \in R^k$ and $x, y \in R$. Then for $x_i, y_i \in R$ with $i = 1, \ldots, k+n$, it follows that $a(\prod_{i=1}^{k+n} x_i b y_i)c^{n+k} = a(\prod_{i=1}^{k} x_i b y_i)(\prod_{i=k+1}^{k+n} x_i b y_i)c^n c^k = a(\prod_{i=1}^{k} x_i b y_i)(\prod_{i=k+1}^{k+n} x_i y_i)(bc)^n c^k = 0$. Therefore $a(RbR)^{n+k}c^{n+k} = 0$, so R satisfies condition (PSII). Consequently, R is pseudo symmetric.

In the following example, essentially Example 5.4(d) of [**17**], we form the Dorroh extension of a permutation identity ring to exhibit a 2-primal ring which *does not* satisfy condition (CZ1). By Lemma 1.5(i) and (ii) R satisfies neither condition (SII) nor (PSII). Hence the Dorroh extension of a permutation identity ring is *not* pseudo symmetric.

EXAMPLE 1.8. Observe that if C is a commutative ring, then T is a permutation identity ring with $\sigma = (12)$ (for $n = 3$), where

$$T = \begin{pmatrix} C & C \\ 0 & 0 \end{pmatrix}.$$

Also the Dorroh extension of a permutation identity ring is 2-primal by Corollary 2.10 of [**3**] and Proposition 2.4(ii) of [**4**]. Now let Z_2 be the field of two elements and let R be the Dorroh extension of the ring $\begin{pmatrix} Z_2 & Z_2 \\ 0 & 0 \end{pmatrix}$ by the ring Z of integers (i.e., the ring with unity formed from $\begin{pmatrix} Z_2 & Z_2 \\ 0 & 0 \end{pmatrix} \times Z$ with componentwise addition and with multiplication given by $(x, k)(y, m) = (xy + mx + ky, km)$). Then $(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 1)$ is a noncentral idempotent. By Lemma 1.5(iii), R does not satisfy condition (CZ1).

Observe that in general the set $\overline{O}(P)$ is not an ideal (e.g., any simple ring R with unity such that $N(R) \neq 0$). However our next result provides a remedy for this behaviour.

PROPOSITION 1.9. *If R satisfies condition* (SI), *then $\overline{O}_P = \overline{O}(P)$ and $\overline{O}(P)$ is an ideal for all prime ideals P of R.*

Proof. As noted above, Shin observed that $O_P = O(P)$ when R satisfies condition (SI) [**17**, p. 45]. To show that $\overline{O}(P)$ is an ideal, let $x, y \in \overline{O}(P)$. Then there exist positive integers m, n and $s, t \in R\backslash P$ such that $x^m Rs = 0 = y^n Rt$. Observe that the condition (SI) is equivalent to the condition: for any $a, b \in R$, $ab = 0$ implies $aRb = 0$ by Lemma 1.2 of [**17**]. Using this equivalence, a routine argument shows that there is a positive integer k such that $(x - y)^k Rsbt = 0$, where $b \in R$ such that $sbt \in R\backslash P$. Again using the equivalence of the condition (SI) with the above mentioned property, yields $R[\overline{O}(P)] \subseteq \overline{O}(P)$ and $[\overline{O}(P)]R \subseteq \overline{O}(P)$.

2. Minimal prime ideals. In this section we use the definitions and properties introduced in Section 1 to obtain characterizations of a minimal prime ideal P in terms of O_P and $O(P)$.

The following lemma provides some characteristics of the conditions $P = \overline{O}(P)$ or $P = \overline{O}_P$.

LEMMA 2.1. *Let P be a prime ideal of R.*

(i) *If* $P = \overline{O}(P)$, *then* $P = \overline{O}_P$.

(ii) *If* $P = \overline{O}_P$, *then P is completely prime and is minimal among completely prime ideals of R. In particular, if* $P = \overline{O}_P$ *for every minimal prime ideal P of R, then R is 2-primal.*

(iii) *If* $\overline{O}_P = P$ *and R is a 2-primal ring, then P is a minimal prime ideal of R which is completely prime.*

(iv) $P = \overline{O}(P)$ *if and only if* $P/O(P) = \mathbf{N}(R/O(P))$.

(v) *If* $R/O(P)$ *is 2-primal, then* $P = \overline{O}(P)$ *if and only if* $P/O(P)$ *is the unique minimal prime ideal of* $R/O(P)$ *(i.e.,* $P/O(P) = \mathbf{P}(R/O(P))$*).*

Proof. (i) From Proposition 1.1(ii) and Proposition 1.6 of [**4**], P is completely prime. Hence $\overline{O}_P \subseteq P$ and so $\overline{O}_P = P$.

(ii) By Proposition 1.1(iii), P is completely prime. If Q is a completely prime ideal of R such that $Q \subseteq P$, then $Q \subseteq P = \overline{O}_P \subseteq \overline{O}_Q \subseteq Q$. The remainder of proof of this part follows from Proposition 1.11 in [**17**].

(iii) If Q is a prime ideal such that $Q \subseteq P$, then $P = \overline{O}_P \subseteq \overline{O}_Q$. By Proposition 1.2, $\overline{O}_Q \subseteq Q$. Hence $P = Q$. So P is a minimal prime ideal. Hence it is completely prime by [**15**].

Part (iv) follows directly from the definition of $\overline{O}(P)$, and part (v) follows from part (iv) and the definition of a 2-primal ring.

PROPOSITION 2.2. *Let P be a prime ideal of R.*

(i) *If P is not left essential in R, then* $P = O(P)$.

(ii) *If* P^n *is not left essential in R for some positive integer n, then* $P \subseteq \overline{O}(P)$.

Proof. (i) Assume P is not left essential in R. Then there exists a nonzero left ideal L such that $P \cap L = 0$. Hence $PL = 0$. So $P \subseteq O(P)$. Hence $O(P) = P$.

(ii) The proof of this part is similar to part (i).

In the following result (our main theorem) we provide weak commutativity type conditions which allow us to characterize minimal prime ideals P of R in terms of $\overline{O}_P$ and $\overline{O}(P)$. This result generalizes Kist's characterization of a minimal prime ideal in a commutative ring. (See Lemma 3.1 of [**11**].)

THEOREM 2.3. *Let R be a 2-primal ring and P a prime ideal of R.*

(i) *If R satisfies condition* (CZ1), *then P is a minimal prime ideal of R if and only if* $P = \overline{O}_P$.

(ii) *If R satisfies condition* (CZ2), *then P is a minimal prime ideal of R if and only if* $P = \overline{O}(P)$.

Proof. Assume that P is a minimal prime ideal. Using the 2-primal condition we will develop a property of P that allows us to use conditions (CZ1) and (CZ2). By [**17**] P is completely prime and so $S = R\backslash P$ is multiplicatively closed. By Proposition 1.2(i), $\overline{O}_P \subseteq P$. Assume that $a \in P$. If $a = 0$, then $a \in \overline{O}_P$. Now suppose that $a \neq 0$. Let F be the multiplicative system generated by $S \cup \{a\}$. We assert that $0 \in F$. Assume to the contrary that $0 \notin F$. Partial order the collection of ideals disjoint with F by inclusion. By Zorn's lemma, we

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get an ideal M which is maximal disjoint with F. Then M is a prime ideal and M is properly contained in P, a contradiction. Hence $0 \in F$, so that

$$0 = a^{n_1} s_1 \cdots a^{n_k} s_k,$$

where $s_i \in S$ and we may assume that the integers $n_1, n_2, \ldots, n_k$ are positive.

Since $\mathrm{N}(R)$ is completely semiprime, Lemma 7 of **[5]** yields $a^n s_1 \cdots s_k \in \mathrm{N}(R)$, where $n = n_1 + \cdots + n_k$. Let $s = s_1 \cdots s_k$. Since P is completely prime, $s \in S$ and so $s^m \in S$ for any positive integer m. Again, since $\mathrm{N}(R)$ is completely semiprime, it follows that $as \in \mathrm{N}(R)$ and so there is n such that $(as)^n = 0$.

(i) By condition (CZ1), there exists m such that $a^m s^m = 0$. Thus $a \in \overline{O}_P$. Hence $P = \overline{O}_P$. The converse follows from Lemma 2.1(iii).

(ii) By condition (CZ2), there exists m such that $a^m R s^m = 0$. Thus $a \in \overline{O}(P)$. Hence $P = \overline{O}(P)$. The converse follows from Lemma 2.1(iii) and Proposition 1.2(i), since $P = \overline{O}(P) \subseteq \overline{O}_P \subseteq P$.

From Proposition 2.2 in **[4]**, subrings of 2-primal rings are 2-primal. Since the conditions (CZ1) and (CZ2) are both inherited by subrings, we see that if a ring R satisfies the conditions of Theorem 2.3 then so do its subrings. We note that Corollary 1.10 of **[17]** could be used to show $as \in \mathrm{N}(R)$ in the above proof. However this result relies on Theorem 1.8 of **[17]** and Corollary 1.9 of **[17]**. We have included our proof which is direct and somewhat different than Shin's. The following corollary, which characterizes the 2-primal condition, is a direct consequence of Theorem 2.3 and Lemma 2.1(ii).

Corollary 2.4. *Let R be a ring which satisfies condition* (CZ1). *Then R is 2-primal if and only if $P = \overline{O}_P$ for every minimal prime ideal P of R.*

Note that Example 1.8(i) illustrates Theorem 2.3(i) and Corollary 2.4, but it is not pseudo symmetric.

In Theorem 2.3, the condition "R is 2-primal" is not superfluous. The following example shows that we cannot replace the condition "R is 2-primal" with the condition "$\mathrm{N}_r(R) = \mathrm{N}(R)$" (i.e., the nil radical equals the set of nilpotent elements). This condition was investigated in **[6]**.

Example 2.5. Let G be an abelian group which is the direct sum of a countably infinite number of infinite cyclic groups; and denote by $\{b(0), b(1), b(-1), \cdots, b(i), b(-i), \cdots\}$ a basis of G. Then there exists one and only one homomorphism $u(i)$, for $i = 1, 2, \cdots$ of G such that $u(i)(b(j)) = 0$ if $j \equiv 0 \pmod{2^i}$ and $u(i)(b(j)) = b(j-1)$ if $j \not\equiv 0 \pmod{2^i}$. Denote U the ring of endomorphisms of G generated by the endomorphisms $u(1), u(2), \cdots$. Let A be the ring obtained from U by adjoining the identity map of G. Then by **[2]** the ring A is a semiprime ring with $\mathrm{N}_r(A) = U$. Now let Q be the field of rational numbers, and $R = A \otimes_Z Q$, where Z is the ring of integers. Then as was shown in Example 3.3 of **[6]**, the ring R is semiprime, local and $\mathrm{N}_r(R) = \mathrm{N}(R) = U \otimes_Z Q$, which is the maximal ideal. Thus the ring R is not 2-primal.

Now let $M = U \otimes_Z Q$. Then it can be checked that (i) R satisfies condition (SII) (hence (CZ1)), (ii)$M = \overline{O}_M$, (iii) M is not a minimal prime ideal because R is semiprime, and (iv) $P \neq \overline{O}_P$ for some minimal prime ideal P by Corollary 2.4. Thus in Theorem 2.3 we cannot replace the hypothesis "R is 2-primal" with the condition "$\mathbf{N}_r(R) = \mathbf{N}(R)$". Furthermore since R is a local ring with nil Jacobson radical, then it satisfies condition (CZ2). However it does not satisfy condition (PSII), by Lemma 1.5(v).

The following example shows that conditions (CZ1) and (CZ2) are not superfluous in Theorem 2.3.

EXAMPLE 2.6. There is a 2-primal ring in which there is a minimal prime ideal P such that $\overline{O}_P \neq P$ (hence by Lemma 2.1(i), $P \neq \overline{O}(P)$). Since Corollary 1.10 in [**17**] shows that $P = N(P)$, we have $\overline{O}_P \neq N(P)$ and $\overline{O}(P) \neq N(P)$. Let A be a domain which is not right Ore. So there are two nonzero elements a and b in A such that $aA \cap bA = 0$. Consider the following ring

$$R = \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}.$$

Then since A is 2-primal, the ring R is also 2-primal by Proposition 2.5 of [**4**]. Next it can be easily checked that the following ideal

$$P = \begin{pmatrix} A & A \\ 0 & 0 \end{pmatrix}$$

of R is a minimal prime ideal. Thus

$$p = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

is in P. But we claim that $p \notin \overline{O}_P$. Assume to the contrary that $p \in \overline{O}_P$. Then there is a positive integer n and an element $q \in R \backslash P$ such that $p^n q = 0$. Say

$$q = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}$$

with $\alpha, \beta, \gamma \in A$. Then since $q \in R \backslash P$, it follows that $\gamma \neq 0$. Now from

$$p^n q = \begin{pmatrix} a^n & a^{n-1}b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} = 0,$$

it follows that $a^n\beta + a^{n-1}b\gamma = 0$ and so $a\beta + b\gamma = 0$. Thus $a\beta = b(-\gamma) \in aA \cap bA = 0$. So $b\gamma = 0$, but this is absurd because $\gamma \neq 0$. Consequently $p \notin \overline{O}_P$.

Our next result indicates that our characterization of minimal prime ideals P in terms of $\overline{O}(P)$ holds for a substantial class of noncommutative rings. For this result we need to recall the following definition: Let T be a ring and M a (T, T)-bimodule. Then the *split-null extension* (or *trivial extension*) $S(T, M)$ of M by T is the ring formed from the Cartesian product $T \times M$ with componentwise addition and with multiplication given by $(x, k)(y, m) = (xy, xm + ky)$.

COROLLARY 2.7. *If R satisfies any of the following conditions, then $P = \overline{O}(P)$ for every minimal prime ideal P of R:*

(i) *R is a local ring with $\mathbf{P}(R)$ equal to the Jacobson radical;*
(ii) *R has a right unity and satisfies condition* (PSII)*;*
(iii) *R is pseudo symmetric, in particular almost symmetric or a permutation identity ring;*
(iv) *$R = (T; Z)$, the Dorroh extension of a ring T by Z, where $\mathbf{P}(T) = T$;*
(v) *$R = \mathcal{S}(T, M)$, where T is a 2-primal ring satisfying condition* (CZ2).

Furthermore, if R is almost symmetric or satisfies a permutation identity, then $P/O(P)$ is the unique minimal prime ideal of $R/O(P)$ for every minimal prime ideal P of R.

Proof. (i) This part follows from the note after Definition 1.3 and Theorem 2.3(ii).

(ii) This part is a consequence of Lemma 1.5(i) and (v), and Theorem 2.3(ii).

(iii) Since condition (PSI) implies R is 2-primal, Lemma 1.5(i) and Theorem 2.3(ii) yield this assertion. From Proposition 1.6 in [**17**], every almost symmetric ring is pseudo symmetric. Proposition 1.6 shows that every permutation identity ring is pseudo symmetric.

(iv) This part follows from the fact that $\mathbf{P}(R) = (\mathbf{P}(T); 0) = (T; 0)$ is the unique prime ideal of R and $R/\mathbf{P}(R) \cong Z$.

(v) Since $\mathcal{S}(T, M)$ is isomorphic to a subring of the triangular matrix ring

$$\begin{pmatrix} T & M \\ 0 & T \end{pmatrix},$$

Propositions 2.2 and 2.5(ii) of [**4**] show that R is 2-primal. A straightforward calculation yields that if T has condition (CZ2) then so does $\mathcal{S}(T, M)$. Thus Theorem 2.3(ii) yields this assertion.

Observe that if R is either almost symmetric or a permutation identity ring, then $R/O(P)$ is 2-primal for every prime ideal P of R. From Lemma 2.1(v), we have that $P/O(P)$ is the unique minimal prime ideal of $R/O(P)$ for every minimal prime ideal P of R.

From Proposition 1.9 and Corollary 2.7, one might suspect that $\overline{O}_P = P$ for every minimal prime ideal P of a ring with condition (SI) or (PSI). But the following example negates this possibility.

EXAMPLE 2.8. There is a ring with the condition (SI) (it satisfies the (PSI) condition by Proposition 1.6 in [**17**] and is 2-primal by Theorem 1.5 of [**17**]) in which there is a minimal prime ideal P such that $\overline{O}_P \neq P$. Assume that $F\{X, Y\}$ is the free algebra over a field F generated by X and Y, and $< YX >$ is the ideal of $F\{X, Y\}$ generated by the element YX. Let $R = F\{X, Y\}/ < YX >$. Put $x = X+ < YX >$ and $y = Y+ < YX >$ in R. Then

$$R = \{f_0(x) + f_1(x)y + \cdots + f_n(x)y^n \mid n = 0, 1, 2, \ldots, \text{ and } f_i(x) \in F[x]\},$$

the polynomial ring such that $yx = 0$. Now let $\alpha, \beta \in R$ such that $\alpha\beta = 0$. Say $\alpha = f_0(x) + f_1(x)y + \cdots + f_n(x)y^n$ and $\beta = g_0(x) + g_1(x)y + \cdots + g_m(x)y^m$ with $f_n(x) \neq 0$ and $g_m(x) \neq 0$.

Case 1. $f_0(x) = 0$. Then $\alpha x\beta = f_0(x)x\beta = 0$. From the fact that $yg(x) = g(0)y$ for $g(x) \in F[x]$, it can be checked that $g_0(0) = g_1(0) = \cdots = g_m(0) = 0$. Thus $\alpha y\beta = \alpha(g_0(0) + g_1(0)y + \cdots + g_m(0)y^m)y = 0$. Thus $\alpha R\beta = 0$.

Case 2. $g_0(x) = 0$. Of course we may assume that $f_0(x) \neq 0$. In this case, it also can be checked that $g_1(x) = g_2(x) = \cdots = g_m(x) = 0$ and so $\beta = 0$. Thus $\alpha R\beta = 0$.

From Cases 1 and 2, R satisfies the condition (SI) by Lemma 1.2(d) of [**17**]. Next the ideal $P = < x >$ of R generated by x is a prime ideal, since $R/ < x > \cong F[y]$. Furthermore, assume that Q is a prime ideal of R such that $Q \subseteq P$. Since $yx = 0$, $yRx = 0$ by the (SI) condition and so either $y \in Q$ or $x \in Q$. But since $Q \subseteq P = < x >$, it follows that $x \in Q$ and thus $Q = P$. Therefore P is a minimal prime ideal of R. But for any positive integer n and any $s \in R \backslash P$, we have $x^n s \neq 0$ and so $x \notin \overline{O}_P$. Furthermore note $O(P) = 0$, and $< y >$ is also another minimal prime ideal of R.

Theorem 2.3(i) shows that condition (CZ1) distinguishes a class of rings in which $P = \overline{O}_P$, for every minimal prime ideal P. Our next result and corollary provide a large class of rings which do not satisfy condition (CZ1), but they have the property that $P = \overline{O}_P$, for every prime ideal P.

Recall a ring R is called *strongly π-regular* if for every $x \in R$ there exists a positive integer $n = n(x)$, depending on x, such that $x^n \in x^{n+1}R$. Strong π-regularity is left-right symmetric.

THEOREM 2.9. *Let*

$$R = \begin{pmatrix} A_1 & A_{12} & \cdots & A_{1n} \\ 0 & A_2 & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{pmatrix},$$

where each A_i is a ring with unity and A_{ij} is a left A_i-right A_j-bimodule for $i < j$. Then R is a 2-primal strongly π-regular ring if and only if each A_i is a 2-primal strongly π-regular ring.

Proof. Let M denote the strictly upper triangular submatrix of R. Assume R is a 2-primal strongly π-regular ring. Then $\mathbf{P}(R) = M + \sum \mathbf{P}(A_i) = \mathbf{N}(R) = M + \bigcup \mathbf{N}(A_i)$. Consequently, $\mathbf{P}(A_i) = \mathbf{N}(A_i)$. Hence each A_i is a 2-primal ring. Observe there exists a set of orthogonal idempotents $\{e_1, \ldots, e_n\}$ such that $1 = e_1 + \cdots + e_n$ and $A_i \cong e_iRe_i$. Define $h_i : R \to e_iRe_i$ by $h_i(x) = e_ixe_i$. Then h_i is a surjective ring homomorphism. Hence each A_i is strongly π-regular.

Conversely, assume each A_i is a 2-primal strongly π-regular ring. Observe $R/\mathbf{P}(R) \cong \oplus A_i/\mathbf{P}(A_i)$. Since each $A_i/\mathbf{P}(A_i)$ is reduced and strongly π-regular, then $R/\mathbf{P}(R)$ is reduced and strongly π-regular. In particular, R is 2-primal and every prime factor ring of R is strongly π-regular. By Theorem 2.1 of [7], R is strongly π-regular.

COROLLARY 2.10. *If R satisfies any of the following conditions, then $P = \overline{O}_P$ for every prime ideal P of R:*

(i) *R is a 2-primal ring with unity which satisfies condition* (CZ1), *and every prime ideal of R is a maximal ideal of R.*

(ii) *R is the ring as in Theorem 2.9, where each A_i is a 2-primal strongly π-regular ring (e.g., R is the $n \times n$ upper triangular matrix ring over a 2-primal strongly π-regular ring with unity).*

Proof. (i) This part follows directly from Theorem 2.8 in [**6**].

(ii) Since a strongly π-regular ring is weakly π-regular, Lemma 2.2(2) in [**6**], Theorem 2.8 in [**6**], and Theorem 2.9 yield the assertion.

Since, in general, idempotents are not central in the rings indicated in Corollary 2.10(ii), these rings do not satisfy conditions (CZ1). (See Lemma 1.5(iii).)

3. Properties of $R/O(P)$. In this section we use our previous results to investigate $R/O(P)$. We focus on conditions which guarantee that $P/O(P)$ is the *unique* minimal prime ideal of $R/O(P)$. Our first proposition shows that a minimal prime ideal P in a 2-primal ring has an essential decomposition in terms of $O(P)$ and $\mathbf{P}(R)$.

PROPOSITION 3.1. *Let R be a 2-primal ring and P a prime ideal of R. Then:*
(i) $O(P) + \mathbf{P}(R) \subseteq \overline{O}(P) \subseteq \overline{O}_P \subseteq P$.
(ii) *If P is a minimal prime ideal, then $O(P) + \mathbf{P}(R)$ is right essential in P.*

Proof. (i) This follows from Proposition 1.2.
(ii) Let $0 \neq x \in P$. Then there exists $s \in R \backslash P$ such that $xs \in \mathbf{N}(R) = \mathbf{P}(R)$ (see proof of Theorem 2.3). If $xRs = 0$, then $0 \neq x \in O(P) + \mathbf{P}(R)$. If $xRs \neq 0$, then there exists $r \in R$ such that $xrs \neq 0$. Since $\mathbf{P}(R)$ is completely semiprime, Lemma 7 of [5] yields $xrs \in \mathbf{P}(R)$. Therefore $O(P) + \mathbf{P}(R)$ is right essential in P.

LEMMA 3.2. *Let P be a prime ideal. Then $O(P/O(P)) \subseteq O_P/O(P)$.*

Proof. Let $x + O(P) \in O(P/O(P))$. Then $xRb \subseteq O(P)$ for some $b \in R\backslash P$. Thus $bRb \nsubseteq P$. So there exists $r \in R$ such that $brb \notin P$. Hence $xbrb \in xRb \subseteq O(P)$. Then there exists $s \in R\backslash P$ such that $xbrbRs = 0$. Now since $brb \notin P$ and $s \notin P$, $brbRs \nsubseteq P$. Take $t \in brbRs$ such that $t \notin P$. Then $xt = 0$ with $t \in R\backslash P$. So $x \in O_P$. Therefore $x + O(P) \in O_P/O(P)$.

We say a right ideal X of R is *properly right essential* in a right ideal Y of R if X is right essential in Y and $X \neq Y$.

THEOREM 3.3. *Let P be a minimal prime ideal of R with $O(P) = O_P$ and $R/O(P)$ a 2-primal ring. Then exactly one of the following conditions holds:*
(i) *$P = \overline{O}(P)$ (i.e., $P/O(P)$ is the unique minimal prime ideal of $R/O(P)$); or*
(ii) *$0 \neq \mathbf{P}(R/O(P))$ is properly right essential in $P/O(P)$ and $[P/O(P)]^n$ is left essential in $R/O(P)$ for all positive integers n.*

Proof. Since $\mathbf{P}(R/O(P)) \neq P/O(P)$ in condition (ii), then at most only one of the conditions (i) or (ii) can hold. So assume $P \neq \overline{O}(P)$. Since $R/O(P)$ is 2-primal, $\mathbf{P}(R/O(P)) = \overline{O}(P)/O(P)$. Observe that $P/O(P)$ is a minimal prime ideal in $R/O(P)$. By Proposition 1.11 of [17], $P/O(P)$ is completely prime. Hence P is completely prime in R. By Proposition 3.1(i) and Lemma 3.2, we have that $O(P/O(P)) + \mathbf{P}(R/O(P)) \subseteq \overline{O}_P/O(P) = \overline{O}(P)/O(P) = \mathbf{P}(R/O(P))$. Now Proposition 3.1(ii) shows that $\mathbf{P}(R/O(P))$ is right essential in $P/O(P)$. From Lemma 3.2, $\overline{O}(P/O(P)) \subseteq \overline{O}_P/O(P) = \overline{O}(P)/O(P)$. Assume $[P/O(P)]^n$ is not left essential in R for some positive integer n. From Proposition 2.2(ii), $P/O(P) \subseteq \overline{O}(P/O(P))$. Hence $P/O(P) \subseteq \overline{O}(P)/O(P)$. Since P is completely prime, $\overline{O}(P) \subseteq P$, and so $P/O(P) = \overline{O}(P)/O(P)$. Thus $P = \overline{O}(P)$, a contradiction.

We leave open the following question: does there exist a right duo ring (i.e., every right ideal is an ideal) which satisfies Theorem 3.3(ii) for some minimal prime ideal P? Observe that if R is a right duo ring, then R satisfies condition (SI) (hence $O_P = O(P)$) and $R/O(P)$ is right duo (hence 2-primal). Thus if R is right duo, then every minimal prime ideal satisfies the hypothesis of Theorem 3.3.

LEMMA 3.4. *Let P be a nonzero minimal prime ideal such that* $\mathbf{P}(R)$ *is right essential in P. If Q is any prime ideal such that $P \neq Q$, then Q is right essential in R.*

Proof. Let $x \in Q$ such that $x \in R \backslash P$. Let Y a right ideal of R which satisfies $[\mathbf{P}(R)+ < x >_R] \bigcap Y = 0$. Then $Y < x >_R = 0 \in P$ yields $Y \subseteq P$. But $\mathbf{P}(R) \bigcap Y = 0$ implies $Y = 0$. Hence $\mathbf{P}(R) + < x >_R$ is right essential in R. Thus Q is right essential in R.

COROLLARY 3.5. *Let P be a minimal prime ideal of R with $O(P) = O_P$ and $R/O(P)$ a 2-primal ring. If $Q/O(P)$ is a nonzero prime ideal of $R/O(P)$ such that $P \neq Q$, then $Q/O(P)$ is right essential in $R/O(P)$.*

Proof. If $P = O(P)$, then $R/O(P)$ is a domain and we are finished. So assume $P \neq O(P)$. Then from Theorem 3.3, $0 \neq \mathbf{P}(R/O(P))$ is right essential in $P/O(P)$. The result now follows from Lemma 3.4.

PROPOSITION 3.6. *Let R be a 2-primal ring with unity satisfying condition* (CZ2). *Then the following conditions are equivalent.*

(i) *Every prime ideal is maximal;*
(ii) *Every prime ideal is a minimal prime ideal;*
(iii) $P = \overline{O}(P)$, *for each prime ideal P;*
(iv) $P/O(P) = \mathbf{P}(R/O(P))$, *for each prime ideal P;*
(v) $R/\mathbf{P}(R)$ *is biregular;*
(vi) *R is weakly π-regular.*

Furthermore, if $R/O(P)$ is 2-primal for every prime ideal P of R, then these are equivalent to

(vii) *$P/O(P)$ is the unique prime ideal of $R/O(P)$, for each prime ideal P,*
(viii) *$P/O(P)$ is the unique maximal ideal of $R/O(P)$, for each prime ideal P,*
(ix) *$P/N(P)$ is the unique maximal ideal of $R/N(P)$, for each prime ideal P.*

Proof. (i)⇒(ii). Since every prime ideal is maximal, then obviously every prime ideal is a minimal prime ideal.

(ii)⇒(i). Obviously, if every prime ideal of R is a minimal prime ideal, then every prime ideal is maximal.

(ii)⇒(iii). By Theorem 2.3(ii), $P = \overline{O}(P)$, for each prime ideal P.

(iii)⇒(i). Since R is 2-primal, $\mathbf{P}(R) = \mathbf{N}(R) = \mathbf{N}_r(R)$, where $\mathbf{N}_r(R)$ is the nil radical of R. By Lemma 1.5(iii), every idempotent of R is central. By Lemma 2.1(i), $P = \overline{O}_P$ for each prime ideal P of R. Now Proposition 2.11(1) of [**6**] and Theorem 2.8 of [**6**] yield that every prime ideal is maximal.

(iii)⇔(iv). This equivalence is a consequence of Lemma 2.1(iv).

(i)⇔(v)⇔(vi). This equivalence is also a consequence of Theorem 2.8 in [**6**].

(iii)⇒(vii). By Lemma 2.1(v), $P/O(P)$ is the unique minimal prime ideal of $R/O(P)$. Since (iii) is equivalent to (i), then $P/O(P)$ is the unique maximal ideal of $R/O(P)$.

(vii)⇒(viii). This implication follows from the fact that every maximal ideal of $R/O(P)$ is a prime ideal of $R/O(P)$.

(viii)⇒(i). This implication is obvious.

(vii)⇒(ix). By Lemma 2.1(v) and Proposition 1.2(ii), $P = \overline{O}(P) = N(P)$. So it follows that $R/N(P) \cong (R/O(P))/(P/O(P))$ which is a simple domain, since $R/O(P)$ is 2-primal. Hence $P/N(P)$ is the unique maximal ideal of $R/N(P)$.

(ix)⇒(i). This implication is obvious.

Observe that a ring R, which is either right duo with the (CZ2) condition or almost symmetric, is 2-primal, satisfies condition (CZ2), and has $R/O(P)$ a 2-primal ring for each prime ideal P of R. Thus Proposition 3.6 is an extension of Shin's Theorem 4.2 in [**15**].

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