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ON THE MASS FORMULA OF SUPERSINGULAR ABELIAN VARIETIES WITH REAL MULTIPLICATIONS

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Abstract

A geometric mass concerning supersingular abelian varieties with real multiplications is formulated and related to an arithmetic mass. We determine the exact geometric mass formula for superspecial abelian varieties of Hilbert-Blumenthal type. As an application, we compute the number of the irreducible components of the supersingular locus of some Hilbert-Blumenthal varieties in terms of special values of the zeta function.

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1. Introduction

For a reductive group G over $\mathbb{Q}$ which is $\mathbb{R}$-anisotropic and an open compact subgroup K of $G(\mathbb{A}_f)$, one can define the mass of (G, K). The attached mass is a weighted class number. It measures the size of K as well as the class number; but more importantly it behaves well when the group G or the level K varies. The importance of the mass formula is its uniform way to organize arithmetic objects and the measurement of these arithmetic objects. The computation of the mass formula has a long history, dating from the pioneers Smith, Minkowski, and Siegel. There are many important contributions to the Tamagawa numbers due to the works of Weil, Ono, Langlands, Harder, Lai and Kottwitz, and to the exact formulas due to those of Shimura, Hashimoto-Ibukiyama, Prasad, Gross, Gan-Yu and many others.

For a small class of groups G, the mass can also arise from special abelian varieties in characteristic p. The celebrated Deuring mass formula says

$$\sum_E \frac{1}{\# \operatorname{Aut} E} = \frac{p-1}{24},$$

where E runs over the isomorphism classes of supersingular elliptic curves. The goal of this paper is to establish the connection between the geometrically defined mass and the arithmetically defined mass. Then one can either access the existing mass formulas, or verify a (arithmetic or geometric) mass formula by a different (geometric or arithmetic) method.

In this paper we show that the mass for certain special polarized abelian varieties with real multiplications can be an arithmetic mass for some (G, K); see Corollary 2.5. The features are: these are supersingular points and need not to be superspecial; the polarizations can be inseparable; and the formulation does not require the existence or definition of the moduli space. The latter indicates that there is no new difficulty to establish such connection even when the moduli space either has bad reduction or is not well-behaved.

We use Shimura's exact mass formula [9] to express the geometric mass in term of special values of the zeta function up to precise local terms; see equation (3) and Theorem 2.7. The special case when the totally real number field F is $\mathbb{Q}$, the abelian varieties are superspecial, and the polarizations are principal, the geometric mass formula reduces to a result of Ekedahl and van der Geer [3] obtained by the geometric method. Our formulation can be generalized to basic points in arbitrary PEL-Shimura variety modulo p (with modification due to the Hasse principle). Doing that will require much more work and we plan to carry it out in a subsequent paper.

In a part of this paper we determine the remaining local terms in the case of superspecial abelian varieties of Hilbert-Blumenthal type; see Theorem 3.7. Using this geometric mass formula, we can determine the number of the irreducible components of the supersingular locus, following the methods in [6] and [11]. In the last section we determine this number in some restricted cases, particularly when p is totally ramified in F; see Theorem 4.4. We note that our approach gives an explanation of the interesting result of Bachmat and Goren [1] on the supersingular locus of Hilbert modular surfaces. Finally we note that there is an interesting connection of supersingular points with the theory of modular forms modulo p; see [4] and [8].

2. Supersingular points and the mass formula

2.1. Shimura's exact mass formula Let B be a quaternion division algebra over a totally real number field F, and let σ denote the standard involution of B. Let V be a left B-module of rank m, and let φ be a quaternion Hermitian form on V, that is,

$$\varphi : V \times V \to B$$

such that $\varphi(x, y) = \varphi(y, x)^\sigma$ and $\varphi(ax, by) = a\varphi(x, y)b^\sigma$ for all $x, y \in V$ and $a, b \in B$. Let G^φ denote the unitary group attached to φ. It is a reductive group over F and we will regard it as a reductive algebraic group over $\mathbb{Q}$ via the Weil restriction of scalars from F to $\mathbb{Q}$. Assume that $G^\varphi(\mathbb{R})$ is compact. For any open compact subgroup K of $G^\varphi(\mathbb{A}_f)$, the mass of K is defined as follows. Let $\{c_1, c_2, \ldots, c_h\}$ be a (complete) set of representatives of the double coset space $G^\varphi(\mathbb{Q})\backslash G^\varphi(\mathbb{A}_f)/K$, and let $\Gamma_i := G^\varphi(\mathbb{Q}) \cap c_i K c_i^{-1}$. Note that each Γ_i is finite by the assumption of $G^\varphi(\mathbb{R})$. The mass of K is defined to be

$$\mathrm{mass}(K) := \sum_{i=1}^{h} \frac{1}{\#\Gamma_i}.$$

For the general definition of the mass of K (without the compactness assumption), we refer to [9, Introduction, page 67]. It follows easily from the definition that for two open compact subgroups $K_1 \subset K_2$, one has $\mathrm{mass}(K_1) = [K_2 : K_1]\,\mathrm{mass}(K_2)$.

Choose a maximal order O_B of B. Let L be an O_B-lattice in V which is maximal along the lattices on which φ takes its value in O_B. Let K_0 be the maximal compact subgroup of $G^\varphi(\mathbb{A}_f)$ which stabilizes the adelic lattice $L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$. In [9, Introduction, page 68] Shimura gives an explicit formula

$$\mathrm{mass}(K_0) = |D_F|^{m^2} \prod_{k=1}^{m} \left\{ D_F^{1/2} \left[(2k-1)!(2\pi)^{-2k}\right]^d \zeta_F(2k) \right\} \times \prod_{i=1}^{s} \prod_{k=1}^{m} \left\{ (N(\mathfrak{p}_i)^k + (-1)^k \right\}, \tag{1}$$

where D_F is the discriminant of F over $\mathbb{Q}$, d is the degree of F, and $\{\mathfrak{p}_i\}_{1 \le i \le s}$ are the primes of O_F at which B is ramified.

Note that the maximality of L is a local property. That is, L is maximal if and only if each $L_\mathfrak{p}$ is maximal for all primes $\mathfrak{p}$ of F.

2.2. One can express the exact formula in terms of special values of zeta functions at negative integers. Let F be a totally real field of degree d. Set

$$\Lambda_F(s) := A^s \Gamma(s/2)^d \zeta_F(s), \quad A := D_F^{1/2} \pi^{-d/2}.$$

Then one has the functional equation for $\zeta_F(s)$

$$\Lambda_F(1-s) = \Lambda_F(s).$$

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It gives

$$\zeta_F(s) = D_F^{1/2} D_F^{-s} \pi^{-d/2+ds} \frac{\Gamma((1-s)/2)^d}{\Gamma(s/2)^d} \zeta_F(1-s).$$

Let $s = 2k$, one gets

$$\zeta_F(2k) = D_F^{1/2} D_F^{-2k} \pi^{-d/2+2dk} \frac{\Gamma((1-2k)/2)^d}{\Gamma(k)^d} \zeta_F(1-2k).$$

On the other hand, one has

$$\frac{\Gamma((1-2k)/2)}{\Gamma(k)} = \frac{(-1)^k \frac{2^{2k-1}(k-1)!}{(2k-1)!} \Gamma(1/2)}{(k-1)!} = \frac{(-1)^k 2^{2k-1} \sqrt{\pi}}{(2k-1)!}.$$

The factor in the middle of (1) can be rewritten as follows.

(2) $$D_F^{1/2} \left[(2k-1)!(2\pi)^{-2k}\right]^d \zeta_F(2k)$$
$$= D_F^{1-2k} \left[(2k-1)!(2\pi)^{-2k}\right]^d \left[\frac{(-1)^k 2^{2k-1}\sqrt{\pi}}{(2k-1)!}\right]^d \pi^{-\frac{d}{2}+2dk} \zeta_F(1-2k)$$
$$= D_F^{1-2k} \left[\frac{(-1)^k}{2}\right]^d \zeta_F(1-2k).$$

Hence we get

(3) $$\operatorname{mass}(K_0) = \prod_{k=1}^{m} \left\{ \left[\frac{(-1)^k}{2}\right]^d \zeta_F(1-2k) \right\} \prod_{i=1}^{s} \prod_{k=1}^{m} \{(N(\mathfrak{p}_i)^k + (-1)^k\}.$$

2.3. If $F = \mathbb{Q}$ and B is the quaternion algebra which is ramified at $\{\infty, p\}$, then we have

(4) $$\operatorname{mass}(K_0) = \frac{(-1)^{m(m+1)/2}}{2^m} \left\{ \prod_{k=1}^{m} \zeta(1-2k) \right\} \prod_{k=1}^{m} \{(p^k + (-1)^k\}.$$

If $m = 1$ and B is ramified only at the primes over p and infinite places, then we have

(5) $$\operatorname{mass}(K_0) = \left[\frac{-1}{2}\right]^d \zeta_F(-1) \prod_{i=1}^{s} \{(N(\mathfrak{p}_i) + (-1)\}.$$

2.4. In the following, the ground field k is algebraically closed of characteristic p. Let $x = (A_0, \lambda_0, \iota_0)$ be a supersingular polarized abelian O_F-variety over k of dimension $g = md$. Let G_x denote the group scheme over Spec $\mathbb{Z}$ whose group of R-points, for each commutative ring R, is $G_x(R) = \{\phi \in (\operatorname{End}_{O_F}(A_0) \otimes R)^\times; \phi'\phi = 1\}$, where the map $\phi \mapsto \phi'$ is the Rosati involution induced by λ_0.

Let Λ_x denote the set of the isomorphism classes of polarized abelian O_F-varieties (A, λ, ι) of dimension g over k such that

(i) the Dieudonné module $M(A)$ is isomorphic to $M(A_0)$ as quasi-polarized Dieudonné $O_F \otimes \mathbb{Z}_p$-modules, and

(ii) the Tate module $T_l(A)$ is isomorphic to $T_l(A_0)$ as non-degenerate alternating $O_F \otimes \mathbb{Z}_l$-modules for all $l \neq p$.

The condition (i) implies that A is supersingular. Let Λ'_x be the subset of Λ_x that consists of objects (A, λ, ι) such that

(iii) there exists an element $\phi \in \mathrm{Hom}_{O_F}(A_0, A) \otimes \mathbb{Q}$ such that $\phi^*\lambda = \lambda_0$.

THEOREM 2.1 ([11, Theorem 10.5]). *There is a natural bijection of pointed sets between* Λ'_x *and* $G_x(\mathbb{Q})\backslash G_x(\mathbb{A}_f)/G_x(\hat{\mathbb{Z}})$.

PROPOSITION 2.2. *Let* $(A_i, \lambda_i, \iota_i)$, $i = 1, 2$ *be two supersingular polarized abelian* O_F*-varieties over* k. *Then there exists* $\varphi \in \mathrm{Hom}_{O_F}(A_1, A_2) \otimes \mathbb{Q}$ *such that* $\varphi^*\lambda_2 = \lambda_1$. *That is, the condition* (iii) *of Subsection* 2.4 *is automatic.*

PROOF. Choose an isogeny $\varphi : A_1 \to A_2$. By Noether-Skolem's theorem, φ can be chosen to be O_F-linear. Let $\lambda'_1 := \varphi^*\lambda_2$ and $*_1$ and $*'_1$ be the Rosati involutions induced by λ_1 and λ'_1 on $E_1 := \mathrm{End}(A_1) \otimes \mathbb{Q}$, respectively. By [12, Satz 1.1], there exists a positive element $c \in E_1^\times$ with $c = c^{*_1}$ such that $x^{*'_1} = c^{-1}x^{*_1}c$ for all $x \in E_1$. As $x \in F$, c lies in $C := \mathrm{End}_{O_F}(A_1) \otimes \mathbb{Q}$.

Let P be the the algebraic variety over $\mathbb{Q}$ defined by $\{X \in C; X^{*_1}X = c\}$. It is a torsor under the algebraic group $G_1 := \{g \in C^\times; g^{*_1}g = 1\}$ over $\mathbb{Q}$ by left action. Hence it forms an element ξ in $H^1(\mathbb{Q}, G_1)$. By [5, Lemma 2.11], $P(\mathbb{R}) \neq \emptyset$. Since G_1 is semi-simple and simply-connected, $H^1(\mathbb{Q}_p, G_1) = 0$. Therefore ξ is locally trivial everywhere. By the Hasse principle for G_1, ξ is trivial. Then there exists an element $g \in C(\mathbb{Q})$ such that $g^{*_1}g = c$. Replacing φ by φg, we have $*_1 = *'_1$, hence that $\lambda'_1 = q\lambda_1$ for some $q \in \mathbb{Q}^\times$. Note that q is positive. This follows from the fact that the polarizations lie in the positive cone of the Néron-Severi group $\mathrm{NS}(A_1) \otimes \mathbb{R}$. Therefore q is a norm of C and we can find a $\varphi \in \mathrm{Hom}_{O_F}(A_1, A_2) \otimes \mathbb{Q}$ such that $\varphi^*\lambda_2 = \lambda_1$. This completes the proof. □

COROLLARY 2.3. $\Lambda'_x = \Lambda_x$.

LEMMA 2.4. *Let* $(A, \lambda, \iota) \in \Lambda_x$ *and* $[c]$ *be the corresponding double coset. Then* $\mathrm{Aut}(A, \lambda, \iota) \simeq \Gamma_c$, *where* $\Gamma_c := G_x(\mathbb{Q}) \cap cG_x(\hat{\mathbb{Z}})c^{-1}$.

PROOF. Let G' be the group scheme over $\mathrm{Spec}\,\mathbb{Z}$ attached to (A, λ, ι) defined as in Subsection 2.4. That is, for any commutative ring R,

$$G'(R) = \{\alpha \in (\mathrm{End}_{O_F}(A) \otimes R); \alpha'\alpha = 1\},$$

where the map $\alpha \mapsto \alpha'$ is the Rosati involution induced by λ.

Choose a map $\phi \in \mathrm{Hom}_{O_F}(A_0, A) \otimes \mathbb{Q}$ such that $\phi^*\lambda = \lambda_0$. Then the element $c \in G_x(\mathbb{A}_f)$ has the property

$$(*) \qquad \phi c_l \in \mathrm{Isom}((A_0(l), \lambda_0, \iota_0), (A(l), \lambda, \iota)), \quad \forall l.$$

Note that $\alpha \in \mathrm{Aut}(A, \lambda, \iota)$ if and only if $\alpha \in G'(\mathbb{Q})$ and $\alpha_l \in \mathrm{Aut}(A(l), \lambda, \iota)$ for all l.

The map ϕ gives an isomorphism $G_x(\mathbb{Q}) \to G'(\mathbb{Q})$ which sends β to $\phi\beta\phi^{-1} =: \alpha$. From $(*)$, we have $\alpha \in G'(\hat{\mathbb{Z}})$ if and only if $(\phi c)^{-1}\alpha(\phi c) \in G_x(\hat{\mathbb{Z}})$. Hence, $\alpha \in G'(\hat{\mathbb{Z}})$ if and only if $c^{-1}\beta c \in G_x(\hat{\mathbb{Z}})$. This completes the proof. □

COROLLARY 2.5. $\mathrm{mass}(\Lambda_x) := \sum_{(A,\lambda,\iota)\in\Lambda_x} \dfrac{1}{\#\mathrm{Aut}(A, \lambda, \iota)} = \mathrm{mass}(G_x(\hat{\mathbb{Z}}))$.

2.5. The semi-simple algebra $C_x := \mathrm{End}_{O_F}(A_0) \otimes \mathbb{Q}$ is the centralizer of F in the simple algebra $M_g(\mathrm{End}^0(E))$, where E is a supersingular elliptic curve. We have $C_x \simeq M_m(B)$, where B is a quaternion algebra over the totally real field F ramified at all real places and unramified at all primes not dividing p. Since the local invariants of the commutant C_x coincides with those of $F \otimes_{\mathbb{Q}} \mathrm{End}^0(E)$, we have $B \simeq F \otimes_{\mathbb{Q}} \mathrm{End}^0(E)$. Therefore,

$$C_x \otimes \mathbb{R} \simeq M_m(\mathbb{H}), \quad C_x \otimes \mathbb{Q}_l \simeq M_{2m}(F \otimes \mathbb{Q}_l), \quad l \neq p$$
$$C_x \otimes \mathbb{Q}_p \simeq \prod_{\mathfrak{p}|p,\, g_\mathfrak{p}:\mathrm{odd}} M_m(B_\mathfrak{p}) \times \prod_{\mathfrak{p}|p,\, g_\mathfrak{p}:\mathrm{even}} M_{2m}(F_\mathfrak{p}),$$

where $g_\mathfrak{p} := [F_\mathfrak{p} : \mathbb{Q}_p]$. The Rosati involution induces the standard involution on $M_m(\mathbb{H})$ and $M_m(B_\mathfrak{p})$, which we will denote by $*$. It also induces the symplectic involution on $M_{2m}(F \otimes \mathbb{Q}_l)$ and $M_{2m}(F_\mathfrak{p})$. Therefore, we have

$$G_x(\mathbb{R}) \simeq \{g \in M_m(\mathbb{H}); g^*g = 1\}, \quad G_x(\mathbb{Q}_l) \simeq \mathrm{Sp}_{2m}(F \otimes \mathbb{Q}_l), \quad l \neq p$$
$$G_x(\mathbb{Q}_p) \simeq \prod_{\mathfrak{p}|p,\, g_\mathfrak{p}:\mathrm{odd}} G_\mathfrak{p} \times \prod_{\mathfrak{p}|p,\, g_\mathfrak{p}:\mathrm{even}} \mathrm{Sp}_{2m}(F_\mathfrak{p}),$$

where $G_\mathfrak{p} = \{g \in M_m(B_\mathfrak{p}); g^*g = 1\}$. We will choose such isomorphisms and replace $\simeq$ by $=$.

LEMMA 2.6. *There exist a left B-module V, a quaternion Hermitian from φ on V, and a maximal lattice L in the sense of Shimura (see Subsection 2.1) such that $G^\varphi \simeq G_x$ over $\mathbb{Q}$ and K_0 defined in Subsection 2.1 is*

$$K_0 = \prod_{l\neq p} \mathrm{Sp}_{2m}(O_F \otimes \mathbb{Q}_l) \times \prod_{\mathfrak{p}|p,\, g_\mathfrak{p}:odd} K_{0,\mathfrak{p}} \times \prod_{\mathfrak{p}|p,\, g_\mathfrak{p}:even} \mathrm{Sp}_{2m}(O_\mathfrak{p}),$$

where $O_\mathfrak{p}$ is the ring of integers in $F_\mathfrak{p}$ and $K_{0,\mathfrak{p}} = \{g \in M_m(O_{B_\mathfrak{p}}); g^*g = 1\}$.

PROOF. Let $V = B^{\oplus g}$ with $\varphi(\underline{x}, \underline{y}) = \sum x_i \bar{y}_i^\sigma$ and let $L = O_B^{\oplus g}$. □

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2.6. The computation of $\mathrm{mass}(G_x(\hat{\mathbb{Z}}))$ is to relate with the mass of the standard one, K_0, and then to apply Shimura's mass formula.

For any two open compact subgroups K_1, K_2 of $G_x(\mathbb{A}_f)$, of $G_x(\mathbb{Q}_p)$, or of $G_x(\mathbb{Q}_l)$, we define a rational number $\mu_\bullet(K_2/K_1)$ with $\bullet = f$, p or l in three respective cases, by

$$\mu_\bullet(K_2/K_1) := [K_1 : K_1 \cap K_2]^{-1}[K_2 : K_1 \cap K_2].$$

We have the following properties

(i) $\mu_\bullet(K_2/K_1) = \begin{cases} [K_2 : K_1] & \text{if } K_1 \subset K_2; \\ [K_1 : K_2]^{-1} & \text{if } K_1 \supset K_2. \end{cases}$

(ii) If $K_1 = \prod_l K_{1,l}$ and $K_2 = \prod_l K_{2,l}$, where l runs over all primes of $\mathbb{Q}$, then $\mu_f(K_2/K_1) = \prod_l \mu_l(K_{2,l}/K_{1,l})$.

(iii) $\mathrm{mass}(K_1) = \mu_f(K_2/K_1)\,\mathrm{mass}(K_2)$.

From Corollary 2.5, we have

$$\mathrm{mass}(\Lambda_x) = \mu_f(K_0/G_x(\hat{\mathbb{Z}}))\,\mathrm{mass}(K_0). \tag{6}$$

If l is prime to the degree of the polarization λ_0, then $G_x(\mathbb{Z}_l)$ is isomorphic to a product of $\mathrm{Sp}_{2m}(O_v)$ and $\mu_l(K_{0,l}/G_x(\mathbb{Z}_l)) = 1$. Therefore, we have

$$\mathrm{mass}(\Lambda_x) = \left\{ \prod_{l=p,\ \text{or } l \mid \deg \lambda_0} \mu_l(K_{0,l}/G_x(\mathbb{Z}_l)) \right\} \mathrm{mass}(K_0). \tag{7}$$

THEOREM 2.7. *Let notation be as above. If $m = 1$ or $\deg \lambda_0$ is a power of p, then we have the simplified mass formula* $\mathrm{mass}(\Lambda_x) = \mu_p(K_{0,p}/K_{M_0})\,\mathrm{mass}(K_0)$, *where K_{M_0} is the group of automorphisms of the quasi-polarized Dieudonné $O_F \otimes \mathbb{Z}_p$-module M_0 attached to x.*

As a consequence, the following corollary is a generalization of the Deuring mass formula. Recently, Ekedahl and van der Geer [3] compute the intersections of cycle classes on the moduli space $\mathscr{A}_g$. They obtained this as a consequence of the Hirzebruch-Mumford proportionality principle.

COROLLARY 2.8. *Let Λ be the set of the isomorphism classes of principally polarized superspecial abelian varieties over k of dimension g. Then*

$$\sum_{(A,\lambda)\in\Lambda} \frac{1}{\#\mathrm{Aut}(A,\lambda)} = \frac{(-1)^{g(g+1)/2}}{2^g} \left\{ \prod_{k=1}^{g} \zeta(1-2k) \right\} \prod_{k=1}^{g} \left\{ (p^k + (-1)^k \right\}.$$

PROOF. Take $x = (A_0, \lambda_0)$ to be a principally polarized superspecial abelian variety over k of dimension g. Note that the Dieudonné module M_0 of x is isomorphic to the product of g copies of separably quasi-polarized rank two supersingular Dieudonné modules. It follows that $\Lambda = \Lambda_x$ and the group K_{M_0} of the automorphisms of M_0 is $K_{M_0} = \{\alpha \in M_g(O_{B_p}); \alpha^*\alpha = 1\}$, where B_p is the quaternion division algebra over $\mathbb{Q}_p$. Therefore, we obtain $\mathrm{mass}(\Lambda) = \mathrm{mass}(K_0)$ and finish the proof by (4). □

3. Superspecial points of HB type and the mass formula

We keep the notations in the previous section, except that we denote the totally real field by $\mathbf{F}$ and reserve the symbol F for the Frobenius operator on a Dieudonné module. Let $\mathfrak{p}$ be a prime of $O_{\mathbf{F}}$ over p, and let $e_{\mathfrak{p}}$ and $f_{\mathfrak{p}}$ denote the ramification degree and residue degree of $\mathfrak{p}$, respectively. In the rest of this paper, we will only treat the Hilbert-Blumenthal cases, namely $m = 1$ and $d = g$. Let $\mathcal{O} := O_{\mathbf{F}} \otimes \mathbb{Z}_p = \oplus_{\mathfrak{p}|p} \mathcal{O}_{\mathfrak{p}}$.

3.1. We first recall the classification of superspecial quasi-polarized Dieudonné $\mathcal{O}$-modules [10]. Let $x = (A_0, \lambda_0, \iota_0)$ be a superspecial polarized abelian $O_{\mathbf{F}}$-variety over k and let M_0 be its covariant Dieudonné module. We have $M_0 = \oplus_{\mathfrak{p}|p} M_{\mathfrak{p}}$ and will describe $M_{\mathfrak{p}}$ for each $\mathfrak{p}$.

Let $\underline{e}$ be the Lie type of $M_{\mathfrak{p}}$ and $\underline{a}$ be the a-type of $M_{\mathfrak{p}}$. We recall their definitions in [10, Section 1] below:

$$\underline{e}(M_{\mathfrak{p}}) = (\{e_1^i, e_2^i\})_{i\in\mathbb{Z}/f_{\mathfrak{p}}\mathbb{Z}} \iff \mathrm{Lie}(M_{\mathfrak{p}}) \simeq \bigoplus_{i\in\mathbb{Z}/f_{\mathfrak{p}}\mathbb{Z}} k[\pi]/(\pi^{e_1^i}) \oplus k[\pi]/(\pi^{e_2^i})$$

$$\underline{a}(M_{\mathfrak{p}}) = (\{a_1^i, a_2^i\})_{i\in\mathbb{Z}/f_{\mathfrak{p}}\mathbb{Z}} \iff M_{\mathfrak{p}}/(F, V)M_{\mathfrak{p}} \simeq \bigoplus_{i\in\mathbb{Z}/f_{\mathfrak{p}}\mathbb{Z}} k[\pi]/(\pi^{a_1^i}) \oplus k[\pi]/(\pi^{a_2^i}),$$

where π is a uniformizer of $O_{\mathfrak{p}}$ and the right-hand sides are isomorphisms of $O_{\mathfrak{p}} \otimes_{\mathbb{Z}_p} k \simeq \oplus_{i\in\mathbb{Z}/f_{\mathfrak{p}}\mathbb{Z}} k[\pi]/(\pi^{e_{\mathfrak{p}}})$-modules.

As $M_{\mathfrak{p}}$ is superspecial, $\underline{e}$ is equal to $\underline{a}$ and it has the form

$$(\{e_1, e_2\}, \{e_{\mathfrak{p}} - e_1, e_{\mathfrak{p}} - e_2\}, \{e_1, e_2\}, \dots)$$

for some integers e_1, e_2 with $0 \le e_1 \le e_2 \le e_{\mathfrak{p}}$; see [10, Section 2]. When $f_{\mathfrak{p}}$ is odd, it satisfies an additional condition $e_1 + e_2 = e_{\mathfrak{p}}$. We say $M_{\mathfrak{p}}$ is of type (e_1, e_2) for convenience.

Let $\mathcal{D}_{\mathfrak{p}}^{-1} = (\pi^{-d})$ be the inverse of the different of $\mathcal{O}_{\mathfrak{p}}$ over $\mathbb{Z}_p$. There is a unique $W \otimes \mathcal{O}_{\mathfrak{p}}$-bilinear pairing $(\cdot, \cdot) : M_{\mathfrak{p}} \times M_{\mathfrak{p}} \to W \otimes \mathcal{O}_{\mathfrak{p}}$ such that $\langle x, y\rangle = \mathrm{Tr}_{W\otimes\mathcal{O}_{\mathfrak{p}}/W}(\pi^{-d}(x, y))$. We have $O_{\mathfrak{p}} \otimes W = \oplus_{i\in\mathbb{Z}/f_{\mathfrak{p}}\mathbb{Z}} W^i$ by the action of $\mathcal{O}_{\mathfrak{p}}^{\mathrm{ur}}$ and this also gives the decomposition $M_{\mathfrak{p}} = \oplus_{i\in\mathbb{Z}/f_{\mathfrak{p}}\mathbb{Z}} M_{\mathfrak{p}}^i$.

LEMMA 3.1. (1) *If $f_{\mathfrak{p}}$ is even, then there is a W^i-basis X_i, Y_i for $M_{\mathfrak{p}}^i$ for each $i \in \mathbb{Z}/f_{\mathfrak{p}}\mathbb{Z}$ such that*

(i) $$(X_i, Y_i) = \begin{cases} \pi^n & \text{if } i \text{ is odd;} \\ \pi^{n+e_{\mathfrak{p}}-e_1-e_2} & \text{if } i \text{ is even,} \end{cases}$$

for all $i \in \mathbb{Z}/f_{\mathfrak{p}}\mathbb{Z}$ and some $n \in \mathbb{Z}$.

(ii) $$FX_i = \begin{cases} -\pi^{e_1} Y_{i+1} & \text{if } i \text{ is odd;} \\ -\pi^{e_{\mathfrak{p}}-e_2} Y_{i+1} & \text{if } i \text{ is even,} \end{cases} \qquad FY_i = \begin{cases} v\pi^{e_2} X_{i+1} & \text{if } i \text{ is odd;} \\ v\pi^{e_{\mathfrak{p}}-e_1} X_{i+1} & \text{if } i \text{ is even,} \end{cases}$$

where $v\pi^{e_{\mathfrak{p}}} = p$.

(2) *If $f_{\mathfrak{p}}$ is odd, then there is a W^i-basis X_i, Y_i for $M_{\mathfrak{p}}^i$ for each $i \in \mathbb{Z}/f_{\mathfrak{p}}\mathbb{Z}$ such that*

(i) $(X_i, Y_i) = \pi^n$ *for some $n \in \mathbb{Z}$.*

(ii) $FX_i = -\pi^{e_1} Y_{i+1}$, $FY_i = v\pi^{e_2} Y_{i+1}$ *for $i \in \mathbb{Z}/f_{\mathfrak{p}}\mathbb{Z}$, where $v\pi^{e_{\mathfrak{p}}} = p$.*

(3) *In particular, if the pairing $(\cdot, \cdot)$ is perfect and $M_{\mathfrak{p}}$ satisfies the Rapoport condition (in this case, $M_{\mathfrak{p}}$ is of type $(0, e_{\mathfrak{p}})$), then there exists a W^i-basis $\{X_i, Y_i\}$ of $M_{\mathfrak{p}}^i$ for each $i \in \mathbb{Z}/f_{\mathfrak{p}}\mathbb{Z}$ such that*

(i) $Y_i \in (VM)^i$ *and* $(X_i, Y_i) = 1$,

(ii) $FX_i = -Y_{i+1}$, $FY_i = pX_{i+1}$,

for all $i \in \mathbb{Z}/f_{\mathfrak{p}}\mathbb{Z}$.

PROOF. See [10, Lemmas 4.5–4.6]. □

Let $\mathbf{F}'_{\mathfrak{p}}$ be the unique quadratic unramified extension of $\mathbf{F}_{\mathfrak{p}}$ and τ be the generator of $\mathrm{Gal}(\mathbf{F}'_{\mathfrak{p}}/\mathbf{F}_{\mathfrak{p}})$.

LEMMA 3.2. *Let $K_{M_{\mathfrak{p}}}$ be the group of automorphisms of the quasi-polarized Dieudonné $\mathcal{O}_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$.*

(1) *If the residue degree $f_{\mathfrak{p}}$ is even, then*

$$K_{M_{\mathfrak{p}}} \simeq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(O_{\mathbf{F}_{\mathfrak{p}}}); c \equiv 0 \mod \pi^{e_2-e_1} \right\}.$$

(2) *If the residue degree $f_{\mathfrak{p}}$ is odd, then*

$$K_{M_{\mathfrak{p}}} \simeq \left\{ \begin{pmatrix} a & b \\ \pi^{e_2-e_1} b^{\tau} & a^{\tau} \end{pmatrix} \in \mathrm{SL}_2(O_{\mathbf{F}'_{\mathfrak{p}}}) \right\}.$$

PROOF. Let $\phi \in K_{M_{\mathfrak{p}}}$. Choose a W^i-basis $\{X_i, Y_i\}$ for $M_{\mathfrak{p}}^i$ as in Lemma 3.1. Write

$$\begin{pmatrix} \phi(X_i) \\ \phi(Y_i) \end{pmatrix} = A_i \begin{pmatrix} X_i \\ Y_i \end{pmatrix}, \quad A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \mathrm{GL}_2(W^i).$$

It follows from $(\phi(X_i), \phi(Y_i)) = (X_i, Y_i)$ that $A_i \in \mathrm{SL}_2(W^i)$. Note that we have $F^2(X_i) = -pX_{i+2}$ and $F^2(Y_i) = -p\,Y_{i+2}$ for all i. The condition $\phi F^2 = F^2\phi$ gives $A_{i+2} = A_i^{(2)}$ for all $i \in \mathbb{Z}/f_{\mathfrak{p}}\mathbb{Z}$, where we write $A^{(n)}$ for A^{σ^n}.

(1) If $f_{\mathfrak{p}}$ is even, then $\{A_i\}$ are determined by A_0 and A_1, and $A_0, A_1 \in \mathrm{SL}_2(O_{\mathbf{F}_{\mathfrak{p}}})$. We have

$$\begin{pmatrix} F(X_0) \\ F(Y_0) \end{pmatrix} = J_1 \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}, \quad J_1 = \begin{pmatrix} 0 & -\pi^{e_{\mathfrak{p}}-e_2} \\ \upsilon\pi^{e_{\mathfrak{p}}-e_1} & 0 \end{pmatrix},$$
$$\begin{pmatrix} F(X_1) \\ F(Y_1) \end{pmatrix} = J_0 \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}, \quad J_0 = \begin{pmatrix} 0 & -\pi^{e_1} \\ \upsilon\pi^{e_2} & 0 \end{pmatrix}.$$

The condition $F\phi = \phi F$ gives $J_1 A_1 = A_0^{(1)} J_1$ and $J_0 A_0^{(1)} = A_1 J_0$. These give the relations

$$a_1 = d_0^{(1)}, \quad d_1 = a_0^{(1)}, \quad c_0^{(1)} = -\upsilon\pi^{e_2-e_1} b_1, \quad c_1 = -\upsilon\pi^{e_2-e_1} b_0^{(1)}.$$

We see that A_1 is determined by A_0 and $c_0 \equiv 0 \mod \pi^{e_2-e_1}$.

(2) If $f_{\mathfrak{p}}$ is odd, then $\{A_i\}$ are determined by A_0 and $A_0 \in \mathrm{SL}_2(O_{\mathbf{F}'_{\mathfrak{p}}})$. We have

$$\begin{pmatrix} F(X_0) \\ F(Y_0) \end{pmatrix} = J \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -\pi^{e_1} \\ \upsilon\pi^{e_2} & 0 \end{pmatrix}.$$

Applying $F\phi = \phi F$, we have $JA_1 = A_0^{(1)} J$. As $A_1 = A_0^{(f_{\mathfrak{p}}+1)}$, this equation gives

$$d_0 = a_0^{\tau}, \quad c_0 = -\upsilon\pi^{e_2-e_1} b_0^{\tau}.$$

As $-\upsilon$ is a norm of some element in $O_{\mathbf{F}'_{\mathfrak{p}}}^{\times}$ (over $O_{\mathbf{F}_{\mathfrak{p}}}^{\times}$), we choose an isomorphism

$$K_{M_{\mathfrak{p}}} \simeq \left\{ \begin{pmatrix} a & b \\ \pi^{e_2-e_1} b^{\tau} & a^{\tau} \end{pmatrix} \in \mathrm{SL}_2(O_{\mathbf{F}'_{\mathfrak{p}}}) \right\}.$$

This completes the proof. □

3.2. It remains to compute the local term $\mu_{\mathfrak{p}}(K_{0,\mathfrak{p}}/K_{M_{\mathfrak{p}}})$. We write q for $N(\mathfrak{p})$, the cardinality of the residue field $k(\mathfrak{p})$.

When $f_{\mathfrak{p}}$ is even, $K_{M_{\mathfrak{p}}} = \Gamma_0(\pi^{e_2-e_1})$ is an open compact subgroup of $\mathrm{SL}_2(\mathbf{F}_{\mathfrak{p}})$. If $e_1 = e_2$, then $\mu_{\mathfrak{p}}(K_{0,\mathfrak{p}}/K_{M_{\mathfrak{p}}}) = 1$; if $e_2 > e_1$, then

$$\mu_{\mathfrak{p}}(K_{0,\mathfrak{p}}/K_{M_{\mathfrak{p}}}) = \#\mathbf{P}^1(O_{\mathbf{F}_{\mathfrak{p}}}/\pi^{e_2-e_1}) = q^{e_2-e_1-1}(q+1).$$

Consider now the case where $f_{\mathfrak{p}}$ is odd. Put

$$K(n) := \left\{ \begin{pmatrix} a & b \\ \pi^n b^{\tau} & a^{\tau} \end{pmatrix} \in \mathrm{SL}_2(O_{\mathbf{F}'_{\mathfrak{p}}}) \right\}.$$

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If n is odd, then $K(n)$ is an open compact subgroup of the group $B^\times_{\mathfrak{p},1}$ of reduced norm one and $K(1) = K_{0,\mathfrak{p}}$. If n is even, then $K(n)$ is an open compact subgroup of $SL_2(\mathbf{F}_\mathfrak{p})$.

For $\epsilon = 0$ or 1, we define a ring A_ϵ as follows. Let B_ϵ be the $\mathbf{F}'_\mathfrak{p}$-algebra generated by $1, \xi$ with relations $\xi^2 = \pi^\epsilon$ and $\xi a = a^\tau \xi$ for all $a \in \mathbf{F}'_\mathfrak{p}$, and let A_ϵ be the $O_{\mathbf{F}'_\mathfrak{p}}$-lattice generated by 1 and ξ. Note that A_ϵ is endowed with the usual reduced norm and trace. For $d \geq 1$, let $A_{\epsilon,d} := \{a + \pi^d b\xi\} \subset A_\epsilon$ be a subring. We have $(A_\epsilon)^\times_1 = K(\epsilon)$ and $(A_{\epsilon,d})^\times_1 \simeq K(2d+\epsilon)$, where $(\ )_1$ denotes the set of elements of reduced norm one. It is clear that

$$(1+\pi^k A_\epsilon)^\times_1/(1+\pi^{k+1}A_\epsilon)^\times_1 \simeq \{a \in A_\epsilon/\pi A_\epsilon; \operatorname{Tr} d(a) = 0\}, \quad k \geq 1$$

hence that

$$\#(1+\pi^k A_\epsilon)^\times_1/(1+\pi^{k+1}A_\epsilon)^\times_1 = q^3. \tag{8}$$

If $\epsilon = 1$, then $(A_\epsilon/\pi)^\times_1 \simeq (O_{\mathbf{F}'_\mathfrak{p}}/\pi)^\times_1 \ltimes (O_{\mathbf{F}'_\mathfrak{p}}/\pi)$ and its has $q^2(q+1)$ elements. When $\epsilon = 0$, $(A_\epsilon/\pi)^\times_1 \simeq \{a, b \in O_{\mathbf{F}'_\mathfrak{p}}/\pi; N(a) - N(b) = 1\}$. Write $\alpha = N(a)$ and $\beta = N(b)$. There are $q+1$ solutions for $(\alpha, \beta) = (1, 0)$ or $(0, -1)$ and $(q+1)^2$ solutions for others. This group has $(q+1)2 + (q-2)(q+1)^2 = q(q^2-1)$ elements. Namely, we have

$$\#(A_\epsilon/\pi)^\times_1 = \begin{cases} q^2(q+1) & \epsilon = 1; \\ q(q^2-1) & \epsilon = 0, \end{cases} \tag{9}$$

and hence that

$$\#(A_\epsilon/\pi^d)^\times_1 = \begin{cases} q^{3d-1}(q+1) & \epsilon = 1; \\ q^{3d-2}(q^2-1) & \epsilon = 0. \end{cases} \tag{10}$$

On the other hand, $(A_{\epsilon,d}/\pi^d A_\epsilon)^\times_1 \simeq (O_{\mathbf{F}'_\mathfrak{p}}/\pi^d)^\times_1$ and it has $q^{d-1}(q+1)$ elements. We conclude

LEMMA 3.3. *Notation being as above, then*

$$\mu_\mathfrak{p}(K(\epsilon)/K(2d+\epsilon)) = \begin{cases} q^{2d} & \epsilon = 1; \\ q^{2d-1}(q-1) & \epsilon = 0, \end{cases}$$

where q is $N(\mathfrak{p})$, the cardinality of the residue field $k(\mathfrak{p})$.

Finally we need to compute $\mu_\mathfrak{p}(K_{0,\mathfrak{p}}/K(0))$ when $e_\mathfrak{p}$ is even.

LEMMA 3.4. *Notation being as above, let $O_{\mathbf{F}'_\mathfrak{p}} = O_{\mathbf{F}_\mathfrak{p}}[\sqrt{c}]$ for some non-square residue c in $O^\times_{\mathbf{F}_\mathfrak{p}}$. Then for $\epsilon = 0$,*

$$A_\epsilon \simeq \left\{ \begin{pmatrix} \alpha & \beta \\ c\gamma & \delta \end{pmatrix} \in M_2(O_{\mathbf{F}_\mathfrak{p}});\ \alpha \equiv \delta \mod 2,\ \beta \equiv \gamma \mod 2 \right\}.$$

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In particular, if the residue characteristic p is not 2, then $K(0) \simeq \mathrm{SL}_2(O_{\mathbf{F}_\mathfrak{p}})$ and $\mu_\mathfrak{p}(K_{0,\mathfrak{p}}/K(0)) = 1$.

PROOF. The $O_{\mathbf{F}_\mathfrak{p}}$-order A_ϵ is generated by x, ξ with relations $x^2 = c, \xi^2 = 1$, and $\xi x = -x\xi$. Choose an isomorphism $A \otimes \mathbf{F}_\mathfrak{p} \simeq M_2(\mathbf{F}_\mathfrak{p})$ with $x \mapsto \left(\begin{smallmatrix} 0 & 1 \\ c & 0 \end{smallmatrix}\right), \xi \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Then the image of A_ϵ in $M_2(\mathbf{F}_\mathfrak{p})$ is

$$\left\{ \begin{pmatrix} a_0 + b_0 & a_1 - b_1 \\ c(a_1 + b_1) & a_0 - b_0 \end{pmatrix} ; \; a_0, a_1, b_0, b_1 \in O_{\mathbf{F}_\mathfrak{p}} \right\},$$

and the assertion follows. □

LEMMA 3.5. *If $f_\mathfrak{p}$ is odd, $e_\mathfrak{p}$ is even, and $p = 2$, then*

$$\mu_\mathfrak{p}(K_{0,\mathfrak{p}}/K(0)) = q^{2e_\mathfrak{p}-1}(q-1).$$

PROOF. It is clear that $(A_\epsilon/2)_1^\times \simeq (O_{\mathbf{F}'_\mathfrak{p}}/2)_1^\times$, hence that this group has $q^{e_\mathfrak{p}-1}(q+1)$ elements. It follows from the equality $\#\mathrm{SL}_2(O_{\mathbf{F}_\mathfrak{p}}/2) = q^{3(e_\mathfrak{p}-1)}(q+1)(q^2-q)$ that $\mu_\mathfrak{p}(K_{0,\mathfrak{p}}/K(0)) = q^{2e_\mathfrak{p}-1}(q-1)$. □

PROPOSITION 3.6. *Notation is as above.*

(1) *If $f_\mathfrak{p}$ is even, then*

$$\mu_\mathfrak{p}(K_{0,\mathfrak{p}}/K_{M_\mathfrak{p}}) = \begin{cases} 1 & e_1 = e_2; \\ q^{e_2-e_1-1}(q+1) & e_1 < e_2. \end{cases}$$

(2) *If $f_\mathfrak{p}$ is odd, then*

$$\mu_\mathfrak{p}(K_{0,\mathfrak{p}}/K_{M_\mathfrak{p}}) = \begin{cases} q^{e_2-e_1-1} & e_\mathfrak{p} \text{ is odd;} \\ o_\mathfrak{p} & e_\mathfrak{p} \text{ is even and } e_1 = e_2; \\ q^{e_2-e_1-1}(q-1)o_\mathfrak{p} & e_\mathfrak{p} \text{ is even and } e_1 < e_2, \end{cases}$$

where

$$o_\mathfrak{p} = \begin{cases} 1 & p \neq 2; \\ q^{2e_\mathfrak{p}-1}(q-1) & p = 2. \end{cases}$$

THEOREM 3.7. *Let $x = (A_0, \lambda_0, \iota_0)$ be a superspecial polarized abelian $O_\mathbf{F}$-variety and let Λ_x be the set defined in Subsection 2.4. The attached Dieudonné module M_0 decomposes as $\oplus_{\mathfrak{p}|p} M_\mathfrak{p}$ and suppose that each $M_\mathfrak{p}$ has type $(e_{1,\mathfrak{p}}, e_{2,\mathfrak{p}})$ (see Subsection 3.1). Then*

$$\sum_{(A,\lambda,\iota)\in\Lambda_x} \frac{1}{\#\mathrm{Aut}(A,\lambda,\iota)} = \left[\frac{-1}{2}\right]^g \zeta_\mathbf{F}(-1) \prod_{\mathfrak{p}|p} c_\mathfrak{p} o_\mathfrak{p}, \tag{11}$$

where

$$c_{\mathfrak{p}} = \begin{cases} 1 & f_{\mathfrak{p}} \text{ is even, } e_{1,\mathfrak{p}} = e_{2,\mathfrak{p}}; \\ q_{\mathfrak{p}}^{e_{2,\mathfrak{p}}-e_{1,\mathfrak{p}}-1}(q_{\mathfrak{p}}+1) & f_{\mathfrak{p}} \text{ is even, } e_{1,\mathfrak{p}} < e_{2,\mathfrak{p}}; \\ q_{\mathfrak{p}}^{e_{2,\mathfrak{p}}-e_{1,\mathfrak{p}}-1}(q_{\mathfrak{p}}-1) & f_{\mathfrak{p}} \text{ is odd, } e_{\mathfrak{p}} \text{ is odd}; \\ 1 & f_{\mathfrak{p}} \text{ is odd, } e_{\mathfrak{p}} \text{ is even, } e_{1,\mathfrak{p}} = e_{2,\mathfrak{p}}; \\ q_{\mathfrak{p}}^{e_{2,\mathfrak{p}}-e_{1,\mathfrak{p}}-1}(q_{\mathfrak{p}}-1) & f_{\mathfrak{p}} \text{ is odd, } e_{\mathfrak{p}} \text{ is even, } e_{1,\mathfrak{p}} < e_{2,\mathfrak{p}}. \end{cases}$$

and

$$o_{\mathfrak{p}} = \begin{cases} q_{\mathfrak{p}}^{2e_{\mathfrak{p}}-1}(q_{\mathfrak{p}}-1) & f_{\mathfrak{p}} \text{ is odd, } e_{\mathfrak{p}} \text{ is even, and } p = 2; \\ 1 & \text{otherwise.} \end{cases}$$

3.3. We say $(A_0, \lambda_0, \iota_0)$ is *minimal* if it reaches the minimal mass among all the types of superspecial points. In this case the mass formula above is simplified as

$$\text{(12)} \qquad \sum_{(A,\lambda,\iota)\in\Lambda_x} \frac{1}{\#\operatorname{Aut}(A,\lambda,\iota)} = \left[\frac{-1}{2}\right]^g \zeta_{\mathbf{F}}(-1) \prod_{\mathfrak{p}|p,\ g_{\mathfrak{p}}:\text{odd}} (q_{\mathfrak{p}}-1) \prod_{\mathfrak{p}|p} o_{\mathfrak{p}},$$

where $o_{\mathfrak{p}}$ is as above.

4. Supersingular locus

4.1. Let p be a fixed prime number. Let $\mathcal{M}^{(p)}$ denote the moduli stack over $\operatorname{Spec}\hat{\mathbb{Z}}_{(p)}$ of polarized abelian $O_{\mathbf{F}}$-varieties (A, λ, ι) of dimension $g = [\mathbf{F} : \mathbb{Q}]$ with the polarization λ of prime-to-p degree. It is a separated Deligne-Mumford algebraic stack over $\operatorname{Spec}\hat{\mathbb{Z}}_{(p)}$ locally of finite type. In [2], Deligne and Pappas showed that the algebraic stack $\mathcal{M}^{(p)}$ is flat and a locally complete intersection over $\operatorname{Spec}\hat{\mathbb{Z}}_{(p)}$ of relative dimension g, and that the closed fiber $\mathcal{M}^{(p)} \otimes \mathbb{F}_p$ is geometrically normal and has singularities of codimension at least two. It follows from Deligne-Pappas's results and the compactification of Rapoport [7] that the irreducible components of geometric special fiber $\mathcal{M}^{(p)} \otimes \overline{\mathbb{F}_p}$ are in bijection correspondence with those of geometric generic fiber $\mathcal{M}^{(p)} \otimes \overline{\mathbb{Q}}$. Those are parameterized by the isomorphism classes of non-degenerate skew-symmetric $O_{\mathbf{F}}$-modules $H_1(A(\mathbb{C}), \mathbb{Z})$ for all $(A, \lambda, \iota) \in \mathcal{M}^{(p)}(\mathbb{C})$.

4.2. Let n be an integer such that $n \geq 3$ and $(n, p) = 1$ and we choose a primitive n-th root of unity ζ_n in $\overline{\mathbb{Q}} \subset \mathbb{C}$. For any geometric point $\operatorname{Spec} k \to \operatorname{Spec}\hat{\mathbb{Z}}_{(p)}[\zeta_n]$, the choice of ζ_n determines a $(1 + n\hat{\mathbb{Z}}^{(p)})^{\times}$-orbit of isomorphisms $\alpha(k) : \hat{\mathbb{Z}}^{(p)} \xrightarrow{\sim} \prod_{l\neq p} \mu_{l^\infty}(k)$. Let S be a connected $\mathbb{Z}_{(p)}[\zeta_n]$-scheme and (A, λ, ι) be polarized abelian $O_{\mathbf{F}}$-scheme over S. A (full) symplectic level-n structure on (A, λ, ι) w.r.t. ζ_n we understand is a $\pi_1(S, s)$-invariant K_n-orbit of $O_{\mathbf{F}} \otimes \hat{\mathbb{Z}}^{(p)}$-linear isomorphisms

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$\eta : V_{\mathbb{Z}} \otimes \hat{\mathbb{Z}}^{(p)} \xrightarrow{\sim} \prod_{l\neq p} T_l(A_s)$ for some non-degenerate skew-symmetric $O_{\mathbf{F}}$-module $(V_{\mathbb{Z}}, \psi, i)$ such that the pull-back of the Weil pairing is $\alpha(k(s)) \circ \psi$, where K_n is the kernel of $G(\hat{\mathbb{Z}}^{(p)}) \to G(\mathbb{Z}/n\mathbb{Z})$, G is the automorphism group scheme $\mathrm{Aut}_{O_{\mathbf{F}}}(V_{\mathbb{Z}}, \psi)$ over $\mathrm{Spec}\,\mathbb{Z}$ and s is a geometric point of S. Note that $(V_{\mathbb{Z}} \otimes \mathbb{Z}[1/p], \psi, i)$ is uniquely determined by (A, λ, ι) by the strong approximation. When $\deg \lambda$ is prime to n, it is the same to an $O_{\mathbf{F}}/n\,O_{\mathbf{F}}$-isomorphism $V_{\mathbb{Z}}/n\,V_{\mathbb{Z}} \xrightarrow{\sim} A[n](S)$ such that the pull-back of the Weil pairing is $\alpha \circ \psi$.

4.3. Let $\mathscr{M}_n^{(p)}$ be the moduli stack over $\mathrm{Spec}\,\mathbb{Z}_{(p)}[\zeta_n]$ of objects in $\mathscr{M}^{(p)}$ together with a symplectic level-n structure w.r.t. ζ_n, and let $\mathscr{M}_n$ be an irreducible component of $\mathscr{M}_n^{(p)}$. If $\mathscr{M}_n$ is the one that classifies the principally polarized objects in $\mathscr{M}_n^{(p)}$, then it is the connected (and irreducible) component of the moduli space $\mathscr{M}_n^{O_{\mathbf{F}}}$ denoted in [2, Section 2] by the choice of the element ζ_n in $\mathrm{Isom}(\mu_n, \mathbb{Z}/n)$.

Let $(V_{\mathbb{Z}}, \psi, i)$ be the non-degenerate skew-symmetric $O_{\mathbf{F}}$-module corresponding to $\mathscr{M}_n$. Let G denote the automorphism group scheme $\mathrm{Aut}_{O_{\mathbf{F}}}(V_{\mathbb{Z}}, \psi)$ over $\mathbb{Z}$. We have $\mathscr{M}_n(\mathbb{C}) \simeq \Gamma(n)\backslash G(\mathbb{R})/SO_2(\mathbb{R})^g$, where $\Gamma(n)$ is the kernel of the map $G(\mathbb{Z}) \to G(\mathbb{Z}/n)$.

PROPOSITION 4.1. *Any $\mathscr{M}_n$ is isomorphic to $\mathscr{M}[L, L^+]_n$ for some (L, L^+), where (L, L^+) is a projective rank one $O_{\mathbf{F}}$-module together with a notion of positivity, and $\mathscr{M}[L, L^+]$ is the Deligne-Pappas space corresponding to the class (L, L^+). Conversely, any Deligne-Pappas space $\mathscr{M}[L, L^+]_n$ is isomorphic to some $\mathscr{M}_n$.*

PROOF. Recall that $\mathscr{M}[L, L^+]$ classifies the objects (A, i, ι), where (A, ι) is an abelian $O_{\mathbf{F}}$-variety, and $i : (L, L^+) \to (\mathscr{P}(A), \mathscr{P}(A)^+)$ such that the $L \otimes A \cong A^t$ (the DP condition). By the weak approximation, there is $\lambda_0 \in L^+$ such that $(\#L/O_{\mathbf{F}}\lambda_0, p) = 1$. The map $(A, i, \iota, \eta) \mapsto (A, i(\lambda_0), \iota, \eta)$ gives a morphism $\mathscr{M}[L, L^+]_n \to \mathscr{M}_n^{(p)}$, which factors through an irreducible component $\mathscr{M}_n$ by the irreducibility of $\mathscr{M}[L, L^+]_n$. It follows from $\mathrm{Aut}(A, i, \iota, \eta) = \mathrm{Aut}(A, i(\lambda_0), \iota, \eta)$ that it is isomorphic.

Conversely, let $\widetilde{A}$ be the universal family over $\mathscr{M}_n$. By [10, Theorem 2.12], the polarization sheaf $\mathscr{P}(\widetilde{A})$ is constant and $\widetilde{A}$ satisfies the DP condition. Take $(L, L^+) = (\mathscr{P}(\widetilde{A}), \mathscr{P}(\widetilde{A})^+)$ and this finishes the proof. □

4.4. In the remaining part of this paper we treat the supersingular locus $\mathscr{S}$ of $\mathscr{M}_n \otimes \overline{\mathbb{F}}_p =: \mathscr{M}_{n,p}$ in the restricted case that *all residue degrees $f_{\mathfrak{p}}$ are one*. The method of computing the number of irreducible components and describing each component has been documented in [6]. Based on *loc. cit.* and earlier works, the work [11] indicates that one needs to find out all possible types of isogenies such that models constructed have the right dimension. In this restricted case, on one hand, we know the right dimension as the Grothendieck conjecture has been proved [10, Theorem 7.3]. On the other hand, the isogeny type of the generic supersingular point is simple, just one step. Hence the recipe in [11] gives the direct connection to a class number. Combining the Shimura mass formula and the computation of local factors in the previous section, we express the number $\mathrm{irrd}(\mathscr{S})$ of irreducible components of $\mathscr{S}$ in terms of special values of the zeta function.

We assume $f_{\mathfrak{p}} = 1$ for all $\mathfrak{p}|p$ from Proposition 4.2 to Theorem 4.4.

PROPOSITION 4.2. *The points that lie in the Rapoport locus are dense in each Newton stratum of* $\mathscr{M}_{n,p}$.

PROOF. We may assume that there is one prime over p as the problem is local. The Lie stratum $\mathscr{N}_i$ of type $\{i, g-i\}$ has dimension $g - 2i$ and each generic point has slope sequence $s(i)$ [10, Section 6], where

$$s(i) = \{i/g, \ldots, i/g, (g-i)/g, \ldots, (g-i)/g\}$$

(with multiplicity g). Let $\mathscr{N}_i^{(j)}$ be the Newton stratum with slope sequence $s(j)$ in $\mathscr{N}_i$. We know that each generic point of $\mathscr{N}_i^{(j)}$ has a-type $(i, \lceil j \rceil)$. By [10, Lemma 6.19], the codimension of $\mathscr{N}_i^{(j)}$ in $\mathscr{N}_i$ is not less than $\lceil j \rceil - i$, hence that $\dim \mathscr{N}_i^{(j)} \leq g - i - \lceil j \rceil$. □

4.5. Let $(A, \lambda, \iota, \eta)$ be a point in $\mathscr{S}(k)$ which satisfies the Rapoport condition (see [10, Section 1]). Let M be the Dieudonné module of A and write $M = \oplus_{\mathfrak{p}|p} M_{\mathfrak{p}}$. By [10, Proposition 4.4], we can choose a basis X, Y of $M_{\mathfrak{p}}$ such that

$$FX = \alpha X + Y, \quad FY = \pi^{e_{\mathfrak{p}}} X.$$

As A is supersingular, we have $\mathrm{ord}_\pi(\alpha) \geq e_{\mathfrak{p}}/2$ [10, Section 6]. We compute the Dieudonné modules appearing in the canonical isogenies attached to M (see [11, Section 8]):

Case 1: $e_{\mathfrak{p}}$ is even, write $e_{\mathfrak{p}} = 2c$.

$$\begin{aligned} M_0 &:= M_{\mathfrak{p}} = W[\pi]\langle X, Y\rangle, \\ M_1 &:= (F, V)M_0 = W[\pi]\langle Y, \pi^c X\rangle, \\ M_2 &:= (F, V)M_1 = W[\pi]\langle \pi^c Y, \pi^{2c} X\rangle. \end{aligned} \tag{13}$$

One sees that $M_1/M_2 \simeq k[\pi]/\pi^c \oplus k[\pi]/\pi^c$, so M_1 is superspecial of type (c, c).

Case 2: $e_{\mathfrak{p}}$ is odd, write $e_{\mathfrak{p}} = 2c + 1$.

$$\begin{aligned} M_0 &:= M_{\mathfrak{p}} = W[\pi]\langle X, Y\rangle, \\ M_1 &:= (F, V)M_0 = W[\pi]\langle Y, \pi^{c+1} X\rangle, \\ M_2 &:= (F, V)M_1 = W[\pi]\langle \pi^{c+1} Y, \pi^{2c+1} X\rangle. \end{aligned} \tag{14}$$

One sees that $M_1/M_2 \simeq k[\pi]/\pi^c \oplus k[\pi]/\pi^{c+1}$, so M_1 is superspecial of type $(c, c+1)$.

4.6. Let $x = (A_0, \lambda_0, \iota_0, \eta_0)$ be a *polarized* abelian $O_{\mathbf{F}}$-variety over k with a symplectic level-n structure for $(V_{\mathbb{Z}}, \psi, i)$ such that each factor $M_{\mathfrak{p}}$ of the Dieudonné module M_0 of A_0 is isomorphic to M_1 above (that is, as described in (13) if $e_{\mathfrak{p}}$ is even and in (14) if $e_{\mathfrak{p}}$ is odd). Let G_x be the automorphism group scheme of x (see Subsection 2.4) and Λ_x be the set of objects defined as in Subsection 2.4 together with a symplectic level-n structure.

It is clear that an element $g \in G_x(\hat{\mathbb{Z}})$ preserves η_0 if and only if $g \equiv 1 \mod n$. By Theorem 2.1, we have

$$\#\Lambda_x = \#G_x(\mathbb{Q})\backslash G_x(\mathbb{A}_f)/K_n = [G_x(\hat{\mathbb{Z}}) : K_n]\,\mathrm{mass}(G_x(\hat{\mathbb{Z}})),$$

where K_n is the kernel of the map $G_x(\hat{\mathbb{Z}}) \to G_x(\mathbb{Z}/n\mathbb{Z})$. We can choose an isomorphism $T_l(A_0) \simeq V_{\mathbb{Z}} \otimes \mathbb{Z}_l$ compatible with the additional structure for each $l \neq p$, and obtain an isomorphism $G_x(\hat{\mathbb{Z}}^{(p)}) \simeq G(\hat{\mathbb{Z}}^{(p)})$. It follows that

$$[G_x(\hat{\mathbb{Z}}) : K_n] = [G_x(\hat{\mathbb{Z}}^{(p)}) : K_n^p] = [G(\hat{\mathbb{Z}}^{(p)}) : \widehat{\Gamma(n)}^p] = [G(\mathbb{Z}) : \Gamma(n)],$$

where $\widehat{\Gamma(n)}$ is the closure of $\Gamma(n)$ in $G(\hat{\mathbb{Z}})$. As the corresponding Dieudonné module is minimal, we conclude that

$$\#\Lambda_x = [G(\mathbb{Z}) : \Gamma(n)]\,\mathrm{mass}(G_x(\hat{\mathbb{Z}})) \tag{15}$$

where the formula $\mathrm{mass}(G_x(\hat{\mathbb{Z}}))$ is given in (12).

4.7. Let $\xi \in \Lambda_x$, we consider the functor $\mathbf{X}_\xi$ which classifies the isomorphism classes of polarized $O_{\mathbf{F}}$-linear isogenies $\varphi : (A_1, \lambda_1, \iota_1, \eta_1) \to (A_0, \lambda_0, \iota_0, \eta_0)$ over k-schemes S such that

(i) $(A_1, \lambda_1, \iota_1, \eta_1) \simeq \xi \times S$.

(ii) $\ker\varphi = \oplus_{\mathfrak{p}}(\ker\varphi)_{\mathfrak{p}}$ is an α-group and the α-sheaf $\mathrm{Lie}((\ker\varphi)_{\mathfrak{p}}^t)$ is a rank one locally free $O_S \otimes k[\pi]/\pi^{\lceil e_{\mathfrak{p}}/2\rceil}$-module.

Let $\alpha(\xi)$ denote the α-group of ξ. The isogeny φ gives a finite flat subgroup scheme $\ker\varphi$ of $\alpha(\xi)$ satisfying (ii). Conversely, given such finite flat subgroup scheme H, one has an $O_{\mathbf{F}}$-linear isogeny $\varphi : A_1 \to A_1/H =: A_0$. The condition (ii) implies that H is isotropic for the Weil pairing of λ_1, hence the polarization descends to A_0. Since the isogeny has a p-power degree, the symplectic level-n structure of A_1 identifies with that of A_0.

Let V_0 is the α-sheaf of the $\mathfrak{p}$-component of $\alpha(\xi)$. One has that

$$V_0 \simeq \begin{cases} k[\pi]/\pi^c \oplus k[\pi]/\pi^{c+1} & e_{\mathfrak{p}} = 2c+1; \\ k[\pi]/\pi^c \oplus k[\pi]/\pi^c & e_{\mathfrak{p}} = 2c. \end{cases}$$

Then $\mathbf{X}_\xi = \prod_{\mathfrak{p}|p} \mathbf{X}_\mathfrak{p}$, where $\mathbf{X}_\mathfrak{p}$ classifies the quotient bundles $\mathscr{F}$ of $V_0 \otimes S$ such that $\mathscr{F}$ is a rank one locally free $O_S \otimes k[\pi]/\pi^{\lceil e_\mathfrak{p}/2 \rceil}$-module. The freeness is an open condition, therefore $\mathbf{X}_\mathfrak{p}$ is representable by an quasi-projective scheme over k.

LEMMA 4.3. *$\mathbf{X}_\mathfrak{p}$ is a smooth irreducible quasi-projective scheme of dimension* $[e_\mathfrak{p}/2]$.

PROOF. Using duality, we may identify $\mathbf{X}_\mathfrak{p}$ with the space of subspaces W of V_0 such that $W \simeq k[\pi]/\pi^{\lceil e_\mathfrak{p}/2 \rceil}$. The algebraic group $H := \mathrm{Aut}_{k[\pi]}(V_0)$ acts naturally on the space $\mathbf{X}_\mathfrak{p}$. If $e_\mathfrak{p} = 2c$, then $H = \mathrm{GL}_2(k[\pi]/\pi^c)$, viewed as an algebraic group over k. If $e_\mathfrak{p} = 2c+1$, then

$$H = \left\{ \begin{pmatrix} a & b \\ \pi c & d \end{pmatrix}; a, b \in (k[\pi]/\pi^c),\ c, d \in k[\pi]/\pi^{c+1},\ a, d : \text{invertible} \right\}$$

with the natural multiplication. Namely, we lift g_1, g_2 with entries in $k[\pi]/\pi^{e_\mathfrak{p}}$, multiply them and project to quotient rings. It is clear that the group H is connected and the action is transitive. Then the space $\mathbf{X}_\mathfrak{p}$ is a homogeneous space of H, hence it is a smooth quasi-projective variety.

As $\mathbf{X}_\mathfrak{p}$ is smooth, the dimension can be computed by tangent spaces. Let $W_0 \in \mathbf{X}_\mathfrak{p}(k)$, then the tangent space at W_0 is given by $\mathrm{Hom}_{k[\pi]}(W_0, V_0/W_0)$, which has dimension c. □

Let pr denote the projection that sends the objects in $\mathbf{X}_\xi$ to their targets. The morphism $\mathrm{pr} : \mathbf{X}_\xi \to \mathscr{M}_{n,p}$ factors through the supersingular locus and we have

$$\mathrm{pr} : \coprod_{\xi \in \Lambda_x} \mathbf{X}_\xi \to \mathscr{S}.$$

Let $\mathscr{S}^R$ be the intersection of $\mathscr{S}$ with the Rapoport locus. For any $x \in \mathscr{S}^R(k)$, there is a unique $\xi \in \Lambda_x$ and $y \in \mathbf{X}_\xi(k)$ such that $\mathrm{pr}(y) = x$. By Proposition 4.2, the morphism pr is dominant. It follows from the irreducibility of $\mathbf{X}_\xi$ that $\#\Lambda_x = \mathrm{irrd}(\mathscr{S})$. By (12) and (15), we have

THEOREM 4.4. *Assume that $f_\mathfrak{p} = 1$ for all primes $\mathfrak{p}$ of $O_\mathbf{F}$ over p. Then the number of the irreducible components of the supersingular locus $\mathscr{S}$ of $\mathscr{M}_{n,p}$ is*

$$(16) \qquad [G(\mathbb{Z}) : \Gamma(n)] \left[\frac{-1}{2}\right]^g \zeta_\mathbf{F}(-1) \prod_{\mathfrak{p}|p,\ g_\mathfrak{p}:odd} (q_\mathfrak{p} - 1) \prod_{\mathfrak{p}|p} o_\mathfrak{p},$$

where $q_\mathfrak{p} = N(\mathfrak{p})$ and

$$(17) \qquad o_\mathfrak{p} = \begin{cases} q_\mathfrak{p}^{2e_\mathfrak{p}-1}(q_\mathfrak{p} - 1) & e_\mathfrak{p} \text{ is even and } p = 2; \\ 1 & \text{otherwise.} \end{cases}$$

For $e_\mathfrak{p} = 1$ and $f_\mathfrak{p}$ small, the description of the supersingular locus is given in [11]. The following theorem is a reformulation of the results in *loc. cit.* by the geometric mass formula in Section 3.

THEOREM 4.5. *Assume that $e_\mathfrak{p} = 1$ and $f_\mathfrak{p} \le 4$ for all primes $\mathfrak{p}$ of $O_\mathbf{F}$ over p.*

(1) *The number* $\operatorname{irrd}(\mathcal{S})$ *of the irreducible components of the supersingular locus $\mathcal{S}$ is*

$$\prod_{\mathfrak{p}|p} c(\mathfrak{p})[G(\mathbb{Z}) : \Gamma(n)] \left[\frac{-1}{2}\right]^g \zeta_\mathbf{F}(-1) \prod_{\mathfrak{p}|p,\ g_\mathfrak{p}:odd} (q_\mathfrak{p} - 1) \prod_{\mathfrak{p}|p} o_\mathfrak{p}, \tag{18}$$

where $q_\mathfrak{p} = N(\mathfrak{p})$, $o_\mathfrak{p}$ is given in (17), *and*

$$c(\mathfrak{p}) = \begin{cases} 1 & f_\mathfrak{p} = 1; \\ 2 & f_\mathfrak{p} = 2; \\ 3 & f_\mathfrak{p} = 3; \\ 6 & f_\mathfrak{p} = 4. \end{cases}$$

(2) *Each irreducible component of $\mathcal{S}$ is isomorphic to $\prod_{\mathfrak{p}|p} X_\mathfrak{p}$, where*

$$X_\mathfrak{p} \simeq \begin{cases} \{\text{point}\} & f_\mathfrak{p} = 1; \\ \mathbf{P}^1 & f_\mathfrak{p} = 2, 3; \\ \mathbf{P}^2 \text{ or } \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(2)) & f_\mathfrak{p} = 4. \end{cases}$$

With Theorems 4.4–4.5, it is natural to expect the following.

CONJECTURE 4.6. *In the general case* $\operatorname{irrd}(\mathcal{S})$ *has the form* (18), *where $c(\mathfrak{p})$ only depends on $e_\mathfrak{p}$ and $f_\mathfrak{p}$.*

REMARK. (1) Specializing Theorems 4.4 and 4.5 to the case when $g = 2$ and $p \neq 2$, one recovers the main results of Bachmat and Goren in [1]. All ingredients of the method in *loc. cit.* are tied to the assumption that $g = 2$: Zagier's explicit formula for quadratic zeta value $\zeta_\mathbf{F}(-1)$, the explicit description of $O_\mathbf{F}$ as $\mathbb{Z}[(d + \sqrt{d})/2]$, the fact that supersingular locus is codimension one, and the work of Katsura-Oort on moduli space of abelian surfaces. For details, see loc. cit. and references therein. Therefore, a different approach for the generalization of their theorem is required.
(2) It is not hard to see from the methods of the work [11] that $\operatorname{irrd}(\mathcal{S})$ is a sum of some class numbers. Indeed, we consider the type of canonical isogenies for generic points. The number of generic points with same type is the number of superspecial points appearing in the canonical isogenies, which is a class number [11]. Therefore,

$\mathrm{irrd}(\mathscr{S})$ is a sum of some class numbers. The point of Conjecture 4.6 says that the superspecial points appearing in such canonical isogenies are minimal (12). If it is true, then the number $c(\mathfrak{p})$ should have an interesting group-theoretic meaning. A further computation (for $f_{\mathfrak{p}} \leq 8$) suggests that

$$c(\mathfrak{p}) = \binom{f_{\mathfrak{p}}}{[f_{\mathfrak{p}}/2]}$$

when $e_{\mathfrak{p}} = 1$. This result and the analysis of the canonical isogenies will be published elsewhere.

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