5

Long-time asymptotics

For any dynamical system one of the first qualitative issues is to understand whether there are general patterns governing the long-time behavior. In this spirit we plan to study the long-time asymptotics of the Abraham model with prescribed external potentials. The basic mechanism at work is the loss of energy radiated to infinity, which is proportional to $\dot{\boldsymbol{v}}(t)^2$ according to Larmor's formula. Since the energy is bounded from below, we expect

$$\lim_{t\to\infty} \dot{\boldsymbol{v}}(t) = 0 \tag{5.1}$$

under rather general conditions. In fact, one would also expect that the velocity tends to a definite limit,

$$\lim_{t\to\infty} \boldsymbol{v}(t) = \boldsymbol{v}_\infty \in \mathbb{V}, \tag{5.2}$$

which leaves us with two qualitatively rather different cases.

(i) $\boldsymbol{v}_\infty = 0$. The charged particle comes to rest confined by the external potentials.
(ii) $\boldsymbol{v}_\infty \neq 0$. The charge escapes into a region with zero external potentials and travels there with constant velocity.

If we take also the asymptotics for $t \to -\infty$ into account, then four familiar cases arise: excitation by incident radiation and subsequent relaxation, (i) → (i); ionization, (i) → (ii); capture through radiation losses, (ii) → (i); and scattering of light from a freely moving charged particle, (ii) → (ii).

There must be a corresponding long-time asymptotic for the radiation field. It consists of a part attached to the motion of the particle and a part scattered to infinity. Thus a more complete description of the long-time solution is

$$Y(t) \cong S_{\boldsymbol{q}(t),\boldsymbol{v}(t)} + (\boldsymbol{E}_{\text{out}}(t), \boldsymbol{B}_{\text{out}}(t), 0, 0) \tag{5.3}$$

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for large t. Here $S_{\boldsymbol{q}(t),\boldsymbol{v}(t)}$ is the charge soliton at the current position and momentum and $\boldsymbol{E}_{\mathrm{out}}(t)$, $\boldsymbol{B}_{\mathrm{out}}(t)$ are the solution of the homogeneous Maxwell equations with appropriately adjusted initial conditions, the scattering data which depend on $Y(0)$.

At present two techniques are at hand for establishing a limit like (5.3). The first one exploits the fact that energy cannot be radiated to infinity forever. This route requires that all field modes are coupled to the particle as expressed by the

Wiener condition (W):

$$\widehat{\varphi}(\boldsymbol{k}) > 0\,. \tag{5.4}$$

The second route is based on a contraction method. It needs no extra condition and gives explicit convergence rates. However, it requires $|e|$ to be sufficiently small, i.e. $|e| < \bar{e}$ with a suitable $\bar{e}$ depending only on the initial energy. Presumably (W) and $\bar{e}$ are artifacts of our mathematical technique.

5.1 Radiation damping and the relaxation of the acceleration

We will establish the limit (5.1) under the Wiener condition, but otherwise in complete generality. The proof follows rather closely physical intuition and leads to an equation of convolution type which has a definite long-time limit.

Let us consider a ball of radius R centered at the origin. At time t the sum of the field energy in this ball and of the mechanical energy of the particle is given by

$$\mathcal{E}_R(t) = \mathcal{E}(t) - \frac{1}{2}\int_{\{|\boldsymbol{x}|\geq R\}} \mathrm{d}^3x\big(\boldsymbol{E}(\boldsymbol{x},t)^2 + \boldsymbol{B}(\boldsymbol{x},t)^2\big) \tag{5.5}$$

provided R is sufficiently large. Using the conservation of total energy, $\mathcal{E}(t) = \mathcal{E}(0)$, $\mathcal{E}_R$ changes in time as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_R(t) = -R^2\int \mathrm{d}^2\omega\, \widehat{\boldsymbol{\omega}}\cdot[\boldsymbol{E}(R\widehat{\boldsymbol{\omega}},t)\times\boldsymbol{B}(R\widehat{\boldsymbol{\omega}},t)]\,, \tag{5.6}$$

where $\widehat{\boldsymbol{\omega}}$ is a vector on the unit sphere, $\mathrm{d}^2\omega$ the surface measure normalized to 4π, and $\boldsymbol{E}\times\boldsymbol{B}$ the Poynting vector for the flux in energy at the surface of the ball under consideration. Since the total energy is bounded from below, we conclude that

$$\mathcal{E}_R(R) - \mathcal{E}_R(R+t) = -\int_R^{R+t} \mathrm{d}s\,\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}_R(s) \leq C \tag{5.7}$$

with the constant $C = \mathcal{E}(0) - \bar{\phi}$ independent of R and t.

In (5.7) we first take the limit $R \to \infty$, which yields the energy radiated to infinity during the time interval $[0, t]$ through a large sphere centered at the origin. Subsequently we take the limit $t \to \infty$ to obtain the total radiated energy. To state the result let us define

$$\begin{aligned} \boldsymbol{E}_\infty(\widehat{\boldsymbol{\omega}}, t) = -\frac{e}{4\pi} \int \mathrm{d}^3 y \, \varphi(\boldsymbol{y} - \boldsymbol{q}(t + \widehat{\boldsymbol{\omega}} \cdot \boldsymbol{y})) \\ \times \big[(1 - \widehat{\boldsymbol{\omega}} \cdot \boldsymbol{v})^{-1} \dot{\boldsymbol{v}} + (1 - \widehat{\boldsymbol{\omega}} \cdot \boldsymbol{v})^{-2} (\widehat{\boldsymbol{\omega}} \cdot \dot{\boldsymbol{v}})(\boldsymbol{v} - \widehat{\boldsymbol{\omega}})\big]\big|_{t + \widehat{\boldsymbol{\omega}} \cdot \boldsymbol{y}} \end{aligned} \tag{5.8}$$

which is a functional of the actual trajectory of the particle. Whatever its motion we must have

$$\int_0^\infty \mathrm{d}t \int \mathrm{d}^2 \omega \, |\boldsymbol{E}_\infty(\widehat{\boldsymbol{\omega}}, t)|^2 \le C < \infty \, . \tag{5.9}$$

Note that the integrand in (5.9) is proportional to $\dot{\boldsymbol{v}}(t)^2$, which therefore is expected to decay to zero for large t.

To establish (5.9) is somewhat tedious with pieces of the argument explained in the section below and in section 8.5. One imagines that the trajectory $t \mapsto \boldsymbol{q}(t)$ is given and solves the inhomogeneous Maxwell–Lorentz equations according to (2.16), (2.17). If the time-zero fields are in $\mathcal{M}^\sigma$, $0 < \sigma \le 1$, see the definition (2.49), then $\boldsymbol{E}_{\mathrm{ini}}(t)$ and $\boldsymbol{B}_{\mathrm{ini}}(t)$ decay as stated in (5.28). Therefore $|\frac{\mathrm{d}}{\mathrm{d}s} \mathcal{E}_R(s)| < C R^2 (1 + s)^{-2 - 2\sigma}$ and the contribution to (5.7) from the initial fields vanishes in the limit $R \to \infty$. Next one has to study the asymptotics of the retarded fields, which is carried out in section 8.5. There ε is fixed, and for our purpose we may set $\varepsilon = 1$. In addition in (8.48) the sphere of radius R is centered at $\boldsymbol{q}^\varepsilon(t)$, rather than at the origin. This means, in the present context one can use the asymptotics (8.51), (8.52) as $R \to \infty$ with $\boldsymbol{q}^\varepsilon(t)$ replaced by 0. Combining both arguments proves that (5.9) follows from (5.7) in the limit $R \to \infty$.

The real task is to extract from (5.9) that the acceleration vanishes for long times.

Theorem 5.1 (Long-time limit of the acceleration). *For the Abraham model satisfying* (C), (P), *and the Wiener condition* (W) *let the initial data be* $Y(0) = (\boldsymbol{E}^0, \boldsymbol{B}^0, \boldsymbol{q}^0, \boldsymbol{v}^0) \in \mathcal{M}^\sigma$ *with* $0 < \sigma \le 1$. *Then*

$$\lim_{t \to \infty} \dot{\boldsymbol{v}}(t) = 0 \, . \tag{5.10}$$

Proof: By energy conservation $|\boldsymbol{v}(t)| \le \bar{v} < 1$. Inserting in (2.41) and using (P) we conclude that $|\dot{\boldsymbol{v}}(t)| \le C$. Differentiating (2.41) and using again (P) also $|\ddot{\boldsymbol{v}}(t)| \le C$ uniformly in t. Therefore $\boldsymbol{E}_\infty(\widehat{\boldsymbol{\omega}}, t)$ is Lipschitz continuous jointly in

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$\widehat{\boldsymbol{\omega}}$, t. Since the energy dissipation (5.9) is bounded, this implies

$$\lim_{t\to\infty} \boldsymbol{E}_\infty(\widehat{\boldsymbol{\omega}}, t) = 0 \tag{5.11}$$

uniformly in $\widehat{\boldsymbol{\omega}}$.

We analyze the structure of the integrand in (5.8). The retarded argument depends only on $y_\parallel = \widehat{\boldsymbol{\omega}} \cdot \boldsymbol{y}$. Therefore the integration over $\boldsymbol{y}_\perp = \boldsymbol{y} - y_\parallel \widehat{\boldsymbol{\omega}}$ can be carried out and we are left with a one-dimensional integral of convolution type. We set $\varphi_{\mathrm{a}}(x_3) = \int \mathrm{d}x_1 \mathrm{d}x_2\, \varphi(\boldsymbol{x})$. Then

$$\begin{aligned}
\boldsymbol{E}_\infty(\widehat{\boldsymbol{\omega}}, t) &= \frac{e}{4\pi} \int \mathrm{d}y_\parallel\, \varphi_{\mathrm{a}}(y_\parallel - q_\parallel(t + y_\parallel)) \\
&\quad \times \left[(1 - \widehat{\boldsymbol{\omega}} \cdot \boldsymbol{v})^{-2} \widehat{\boldsymbol{\omega}} \times ((\widehat{\boldsymbol{\omega}} - \boldsymbol{v}) \times \dot{\boldsymbol{v}})\right]\Big|_{t+y_\parallel} \\
&= \frac{e}{4\pi} \int \mathrm{d}s \varphi_{\mathrm{a}}(t - (s - q_\parallel(s))) \\
&\quad \times \left[(1 - \widehat{\boldsymbol{\omega}} \cdot \boldsymbol{v})^{-2} \widehat{\boldsymbol{\omega}} \times ((\widehat{\boldsymbol{\omega}} - \boldsymbol{v}) \times \dot{\boldsymbol{v}})\right]\Big|_{s} .
\end{aligned} \tag{5.12}$$

Since $|\dot{q}_\parallel(s)| < 1$, we can substitute $\theta = s - q_\parallel(s)$ and obtain the convolution representation

$$\boldsymbol{E}_\infty(\widehat{\boldsymbol{\omega}}, t) = \int \mathrm{d}\theta\, \varphi_{\mathrm{a}}(t - \theta) \boldsymbol{g}_{\widehat{\omega}}(\theta) = \varphi_{\mathrm{a}} * \boldsymbol{g}_{\widehat{\omega}}(t) , \tag{5.13}$$

where

$$\boldsymbol{g}_{\widehat{\omega}}(\theta) = \frac{e}{4\pi} \left[(1 - \widehat{\boldsymbol{\omega}} \cdot \boldsymbol{v})^{-2} \widehat{\boldsymbol{\omega}} \times ((\widehat{\boldsymbol{\omega}} - \boldsymbol{v}) \times \dot{\boldsymbol{v}})\right]\Big|_{s(\theta)} . \tag{5.14}$$

From (5.11) we know that $\lim_{t\to\infty} \varphi_{\mathrm{a}} * \boldsymbol{g}_{\widehat{\omega}}(t) = 0$. If $\widehat{\varphi}(\boldsymbol{k}_0) = 0$ for some $\boldsymbol{k}_0$, hence $\widehat{\varphi}$ violating the Wiener condition, then we could choose $\boldsymbol{g}_{\widehat{\omega}}(\theta)$ periodic with frequency $|\boldsymbol{k}_0|$ and still have $\varphi_{\mathrm{a}} * \boldsymbol{g}_{\widehat{\omega}}(t) = 0$. At this point no further progress seems to be possible. However under the Wiener condition (W) and with the smoothness of $\boldsymbol{g}_{\widehat{\omega}}(\theta)$ already established, Pitt's extension to the Tauberian theorem of Wiener assures us that

$$\lim_{\theta\to\infty} \boldsymbol{g}_{\widehat{\omega}}(\theta) = 0 , \tag{5.15}$$

which, since $\theta(t) \to \infty$ as $t \to \infty$, implies

$$\lim_{t\to\infty} \widehat{\boldsymbol{\omega}} \times ((\widehat{\boldsymbol{\omega}} - \boldsymbol{v}(t)) \times \dot{\boldsymbol{v}}(t)) = 0 \tag{5.16}$$

for every $\widehat{\boldsymbol{\omega}}$ in the unit sphere. Replacing $\widehat{\boldsymbol{\omega}}$ by $-\widehat{\boldsymbol{\omega}}$ and summing both expressions yields $\widehat{\boldsymbol{\omega}} \times (\widehat{\boldsymbol{\omega}} \times \dot{\boldsymbol{v}}(t)) \to 0$ as $t \to \infty$. Since this is true for every $\widehat{\boldsymbol{\omega}}$, the claim follows. □

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Note that by fiat Theorem 5.1 avoids any claims as regards the convergence of $(\boldsymbol{q}(t), \boldsymbol{v}(t))$ as $t \to \infty$.

Since the acceleration vanishes for large times, the comoving electromagnetic fields will adjust locally to the appropriate charge soliton. We established already that $\boldsymbol{E}_{\mathrm{ini}}(t)$ and $\boldsymbol{B}_{\mathrm{ini}}(t)$ decay. Thus one only has to consider the retarded fields $\boldsymbol{E}_{\mathrm{ret}}(\boldsymbol{x}+\boldsymbol{q}(t), t)$, $\boldsymbol{B}_{\mathrm{ret}}(\boldsymbol{x}+\boldsymbol{q}(t), t)$ relative to the position of the particle and compare them with the soliton fields $\boldsymbol{E}_{v(t)}(\boldsymbol{x})$, $\boldsymbol{B}_{v(t)}(\boldsymbol{x})$ at the current velocity. For this purpose one uses the representations (4.31), (4.32) for the charge soliton and (2.16), (2.17) for the retarded fields. We insert the explicit form (2.15) of the propagator. This yields

$$\begin{aligned}\boldsymbol{E}_v(\boldsymbol{x}) = e\int \mathrm{d}^3y\,(4\pi|\boldsymbol{x}-\boldsymbol{y}|)^{-1}\big(|\boldsymbol{x}-\boldsymbol{y}|^{-1}\varphi(\boldsymbol{y}-\boldsymbol{v}|\boldsymbol{x}-\boldsymbol{y}|)\widehat{\boldsymbol{n}}\\ +\boldsymbol{v}\cdot\nabla\varphi(\boldsymbol{y}-\boldsymbol{v}|\boldsymbol{x}-\boldsymbol{y}|)(\boldsymbol{v}-\widehat{\boldsymbol{n}})\big)\,, \qquad (5.17)\end{aligned}$$

$$\begin{aligned}\boldsymbol{B}_v(\boldsymbol{x}) = e\int \mathrm{d}^3y(4\pi|\boldsymbol{x}-\boldsymbol{y}|)^{-1}\widehat{\boldsymbol{n}}\times\big(-|\boldsymbol{x}-\boldsymbol{y}|^{-1}\varphi(\boldsymbol{y}-|\boldsymbol{x}-\boldsymbol{y}|\boldsymbol{v})\boldsymbol{v}\\ +\boldsymbol{v}\cdot\nabla\varphi(\boldsymbol{y}-|\boldsymbol{x}-\boldsymbol{y}|\boldsymbol{v})\boldsymbol{v}\big)\,, \qquad (5.18)\end{aligned}$$

where $\widehat{\boldsymbol{n}} = (\boldsymbol{x}-\boldsymbol{y})/|\boldsymbol{x}-\boldsymbol{y}|$. Similarly for the retarded fields

$$\begin{aligned}\boldsymbol{E}_{\mathrm{ret}}(\boldsymbol{x}+\boldsymbol{q}(t), t) = \int \mathrm{d}^3y\,(4\pi|\boldsymbol{x}-\boldsymbol{y}|)^{-1}\big(|\boldsymbol{x}-\boldsymbol{y}|^{-1}\varphi(\boldsymbol{y}+\boldsymbol{q}(t)-\boldsymbol{q}(\tau))\widehat{\boldsymbol{n}}\\ +\boldsymbol{v}(\tau)\cdot\nabla\varphi(\boldsymbol{y}+\boldsymbol{q}(t)-\boldsymbol{q}(\tau))(\boldsymbol{v}(\tau)-\widehat{\boldsymbol{n}})\\ -\varphi(\boldsymbol{y}+\boldsymbol{q}(t)-\boldsymbol{q}(\tau))\dot{\boldsymbol{v}}(\tau)\big)\,, \qquad (5.19)\end{aligned}$$

$$\begin{aligned}\boldsymbol{B}_{\mathrm{ret}}(\boldsymbol{x}+\boldsymbol{q}(t), t) = \int \mathrm{d}^3y(4\pi|\boldsymbol{x}-\boldsymbol{y}|)^{-1}\widehat{\boldsymbol{n}}\times\big(-|\boldsymbol{x}-\boldsymbol{y}|^{-1}\varphi(\boldsymbol{y}+\boldsymbol{q}(t)\\ -\boldsymbol{q}(\tau))\boldsymbol{v}(\tau)+\boldsymbol{v}(\tau)\cdot\nabla\varphi(\boldsymbol{y}+\boldsymbol{q}(t)-\boldsymbol{q}(\tau))\boldsymbol{v}(\tau)\\ -\varphi(\boldsymbol{y}+\boldsymbol{q}(t)-\boldsymbol{q}(\tau))\dot{\boldsymbol{v}}(\tau)\big)\,, \qquad (5.20)\end{aligned}$$

where $\tau = t - |\boldsymbol{x}-\boldsymbol{y}|$ and $t \geq t_\varphi = 2R_\varphi/(1-\bar{v})$.

We compare the fields locally and use the result that $\lim_{t\to\infty}\dot{\boldsymbol{v}}(t) = 0$. Then, for any fixed $R > 0$,

$$\begin{aligned}\lim_{t\to\infty}\int_{\{|\boldsymbol{x}|\leq R\}} \mathrm{d}^3x\Big(\big(\boldsymbol{E}(\boldsymbol{x}+\boldsymbol{q}(t), t)-\boldsymbol{E}_{v(t)}(\boldsymbol{x})\big)^2\\ +\big(\boldsymbol{B}(\boldsymbol{x}+\boldsymbol{q}(t), t)-\boldsymbol{B}_{v(t)}(\boldsymbol{x})\big)^2\Big) = 0\,. \qquad (5.21)\end{aligned}$$

The scattered fields are not covered by (5.21) and will be studied in section 5.3.

5.2 Convergence to the soliton manifold

In the case of *zero* external potentials, in essence any solution $Y(t)$ rapidly converges to the soliton manifold $\mathcal{S}$ as $t \to \infty$, in particular $\boldsymbol{v}(t) \to \boldsymbol{v}_\infty$. Such behavior will be of importance in the discussion of the adiabatic limit, see chapter 6, where it will be explained that in the matching to a comparison dynamics one cannot use the naive $\boldsymbol{v}(0)$ but instead must take $\boldsymbol{v}_\infty$. For hydrodynamic boundary value problems such a property is known as the slip condition, since the extrapolation from the bulk does not coincide with the boundary conditions imposed externally.

To prove the envisaged behavior we need a little preparation. Firstly we must have some decay and smoothness of the initial fields at infinity. We already introduced such a set of "good" initial data, $\mathcal{M}^\sigma$, compare with (2.49), and therefore require here $Y(0) \in \mathcal{M}^\sigma$, $0 < \sigma \le 1$. Secondly, we need a notion for two field configurations being close to each other. At a given time and far away from the particle the fields are determined by their initial data. Only close to the particle are they Coulombic. Therefore it is natural to measure closeness in the *local energy norm* defined by

$$\|(\boldsymbol{E},\boldsymbol{B})\|_R^2 = \frac{1}{2} \int_{\{|\boldsymbol{x}|\le R\}} \mathrm{d}^3x \big(\boldsymbol{E}(\boldsymbol{x})^2 + \boldsymbol{B}(\boldsymbol{x})^2\big) \tag{5.22}$$

for given radius R.

The true solution is $Y(t) = (\boldsymbol{E}(\boldsymbol{x},t), \boldsymbol{B}(\boldsymbol{x},t), \boldsymbol{q}(t), \boldsymbol{v}(t))$ which is to be compared with the charge soliton approximation $\big(\boldsymbol{E}_{\boldsymbol{v}(t)}(\boldsymbol{x}-\boldsymbol{q}(t)), \boldsymbol{B}_{\boldsymbol{v}(t)}(\boldsymbol{x}-\boldsymbol{q}(t)), \boldsymbol{q}(t), \boldsymbol{v}(t)\big)$. We set $\boldsymbol{Z}_1(\boldsymbol{x},t) = \boldsymbol{E}(\boldsymbol{x},t) - \boldsymbol{E}_{\boldsymbol{v}(t)}(\boldsymbol{x}-\boldsymbol{q}(t))$, $\boldsymbol{Z}_2(\boldsymbol{x},t) = \boldsymbol{B}(\boldsymbol{x},t) - \boldsymbol{B}_{\boldsymbol{v}(t)}(\boldsymbol{x}-\boldsymbol{q}(t))$, $Z = (\boldsymbol{Z}_1, \boldsymbol{Z}_2)$ and want to establish that $\|Z(\cdot + \boldsymbol{q}(t), t)\|_R \to 0$ for large times at fixed R.

Proposition 5.2 (Long-time limit for the velocity). *For the Abraham model with zero external potentials and satisfying* (*C*) *let* $|e| \le \bar{e}$ *with sufficiently small* $\bar{e}$ *and let the initial data be* $Y(0) \in \mathcal{M}^\sigma$ *for some* $\sigma \in (0,1]$. *Then for every* $R > 0$ *we have*

$$\|Z(\cdot + \boldsymbol{q}(t), t)\|_R \le C_R(1+|t|)^{-1-\sigma} . \tag{5.23}$$

In addition, the acceleration is bounded as

$$|\dot{\boldsymbol{v}}(t)| \le C(1+|t|)^{-1-\sigma} \tag{5.24}$$

and there exists a $\boldsymbol{v}_\infty \in \mathbb{V}$ *such that*

$$\lim_{t\to\infty} \boldsymbol{v}(t) = \boldsymbol{v}_\infty . \tag{5.25}$$

Proof: Using the Maxwell equations together with the identities $(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{E}_v = -\nabla\times\boldsymbol{B}_v + e\varphi\boldsymbol{v}$, $(\boldsymbol{v}\cdot\nabla)\boldsymbol{B}_v = \nabla\times\boldsymbol{E}_v$ one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}Z(t) = AZ(t) - g(t)\,, \tag{5.26}$$

where A is defined in (2.18) and $g(t)$ has the components $(\dot{\boldsymbol{v}}(t)\cdot\nabla_v)\boldsymbol{E}_v(\boldsymbol{x}-\boldsymbol{q}(t))$, $(\dot{\boldsymbol{v}}(t)\cdot\nabla_v)\boldsymbol{B}_v(\boldsymbol{x}-\boldsymbol{q}(t))$, and therefore

$$Z(t) = U(t)Z(0) - \int_0^t \mathrm{d}s\; U(t-s)g(s) \tag{5.27}$$

with $U(t) = \mathrm{e}^{At}$.

For the first term we note that $\boldsymbol{Z}_1(\boldsymbol{x},0) = \boldsymbol{E}^0(\boldsymbol{x}) - \boldsymbol{E}_{v^0}(\boldsymbol{x}-\boldsymbol{q}^0)$, $\boldsymbol{Z}_2(\boldsymbol{x},0) = \boldsymbol{B}^0(\boldsymbol{x}) - \boldsymbol{B}_{v^0}(\boldsymbol{x}-\boldsymbol{q}^0) \in \mathcal{M}^\sigma$ by assumption. Using the solution of the inhomogeneous Maxwell–Lorentz equations in position space and the bound (2.49) one has

$$\begin{aligned}
|\boldsymbol{Z}_1(\boldsymbol{x},t)| + |\boldsymbol{Z}_2(\boldsymbol{x},t)| &\le C\,t^{-2}\int \mathrm{d}^3y\,\delta(|\boldsymbol{x}-\boldsymbol{y}|-t)\big(|\boldsymbol{Z}_1(\boldsymbol{y},0)| + |\boldsymbol{Z}_2(\boldsymbol{y},0)|\big)\\
&\quad + C\,t^{-1}\int \mathrm{d}^3y\,\delta(|\boldsymbol{x}-\boldsymbol{y}|-t)\big(|\nabla\boldsymbol{Z}_1(\boldsymbol{y},0)|\\
&\quad + |\nabla\boldsymbol{Z}_2(\boldsymbol{y},0)|\big)\\
&\le C\,t^{-2}\int \mathrm{d}^3y\,\delta(|\boldsymbol{x}-\boldsymbol{y}|-t)(1+|\boldsymbol{y}|)^{-1-\sigma}\\
&\quad + C\,t^{-1}\int \mathrm{d}^3y\,\delta(|\boldsymbol{x}-\boldsymbol{y}|-t)(1+|\boldsymbol{y}|)^{-2-\sigma}\\
&\le C\,(1+t)^{-1-\sigma}\,.
\end{aligned} \tag{5.28}$$

The integrand in the second term of (5.27) will be estimated in section 7.3 with the bound

$$\|U(t-s)g(s)\|_{R_\varphi} \le C(\bar{v})e^2(1+(t-s)^2)^{-1}\|Z(\cdot+\boldsymbol{q}(s),s)\|_{R_\varphi}\,; \tag{5.29}$$

compare with (7.36).

We choose $R \ge R_\varphi$. From (5.29) and (5.28)

$$\begin{aligned}
&\|Z(\cdot+\boldsymbol{q}(t),t)\|_R \le C(1+t)^{-1-\sigma}\\
&\quad + C(\bar{v})e^2\int_0^t \mathrm{d}s\,(1+(t-s)^2)^{-1}\|Z(\cdot+\boldsymbol{q}(s),s)\|_R\,.
\end{aligned} \tag{5.30}$$

Let $\kappa = \sup_{t \geq 0} (1+t)^{1+\sigma} \|Z(\cdot + \boldsymbol{q}(t), t)\|_R$. Then

$$\kappa \leq C + C(\bar{v}) e^2 \left(\int_0^t \mathrm{d}s \, (1 + (t-s)^2)^{-1} (1+s)^{-1-\sigma} \right) \kappa \,, \tag{5.31}$$

which implies $\kappa < \infty$ provided $C(\bar{v})\, e^2$ is sufficiently small.

To estimate the decay rate for the acceleration we start from Newton's equations of motion in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(m_{\mathrm{b}} \gamma \boldsymbol{v}(t)\big) = e\big(\boldsymbol{E}_\varphi(\boldsymbol{q}(t)) - \boldsymbol{E}_{\boldsymbol{v}(t)\varphi}(0) + \boldsymbol{v} \times (\boldsymbol{B}_\varphi(\boldsymbol{q}(t)) - \boldsymbol{B}_{\boldsymbol{v}(t)\varphi}(0))\big) \,, \tag{5.32}$$

which uses the fact that the force from the soliton field vanishes. By energy conservation $|\boldsymbol{v}(t)| \leq \bar{v} < 1$. Therefore (5.32) implies

$$|\dot{\boldsymbol{v}}(t)| \leq C\, e \, \|Z(\cdot + \boldsymbol{q}(t), t)\|_{R_\varphi} \tag{5.33}$$

and (5.24) follows from (5.23). Since $\boldsymbol{v}(t) = \boldsymbol{v}(0) + \int_0^t \mathrm{d}s \, \dot{\boldsymbol{v}}(s)$, one has $|\boldsymbol{v}(t) - \boldsymbol{v}_\infty| \leq C\,(1+|t|)^{-\sigma}$. □

5.3 Scattering theory

We still have to provide an analysis of the scattered wave. Our results are somewhat fragmentary and we start with an easy and sufficient integrability condition.

Theorem 5.3 (Existence of scattering solutions). *For the Abraham model satisfying* (*C*) *and* (*P*) *let* $Y(t) \in \mathcal{M}$ *be a solution. If*

$$\int_0^\infty \mathrm{d}t \, |\dot{\boldsymbol{v}}(t)| < \infty \,, \tag{5.34}$$

then there exist scattering data $(\boldsymbol{E}_{\mathrm{sc}}, \boldsymbol{B}_{\mathrm{sc}})$ *such that*

$$\lim_{t\to\infty} \big(\|\boldsymbol{E}(t) - \boldsymbol{E}_{\boldsymbol{v}(t)}(\cdot - \boldsymbol{q}(t)) - \boldsymbol{E}_{\mathrm{sc}}(t)\| + \|\boldsymbol{B}(t) - \boldsymbol{B}_{\boldsymbol{v}(t)}(\cdot - \boldsymbol{q}(t)) - \boldsymbol{B}_{\mathrm{sc}}(t)\| \big) = 0 \,, \tag{5.35}$$

where $(\boldsymbol{E}_{\mathrm{sc}}(t), \boldsymbol{B}_{\mathrm{sc}}(t)) = U(t)(\boldsymbol{E}_{\mathrm{sc}}, \boldsymbol{B}_{\mathrm{sc}})$ *propagate according to the homogeneous Maxwell–Lorentz equations.*

Note that in (5.35) the difference is in the global energy norm and therefore carries the information on the scattered wave.

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Proof: The difference in (5.35) is $Z(t)$ by definition. (5.26) remains valid in the presence of external forces, which means that

$$Z(t) = U(t)\big(Z(0) - \int_0^t \mathrm{d}s\, U(-s)g(s)\big)\,. \tag{5.36}$$

We set

$$\begin{aligned} \boldsymbol{E}_{\mathrm{sc}}(\boldsymbol{x}) &= \boldsymbol{E}^0(\boldsymbol{x}) - \boldsymbol{E}_{v^0}(\boldsymbol{x}-\boldsymbol{q}^0) - \int_0^\infty \mathrm{d}t\,(\dot{\boldsymbol{v}}(t)\cdot\nabla_v)\boldsymbol{E}_v(\boldsymbol{x}-\boldsymbol{q}(t))\,,\\ \boldsymbol{B}_{\mathrm{sc}}(\boldsymbol{x}) &= \boldsymbol{B}^0(\boldsymbol{x}) - \boldsymbol{B}_{v^0}(\boldsymbol{x}-\boldsymbol{q}^0) - \int_0^\infty \mathrm{d}t\,(\dot{\boldsymbol{v}}(t)\cdot\nabla_v)\boldsymbol{B}_v(\boldsymbol{x}-\boldsymbol{q}(t))\,. \end{aligned} \tag{5.37}$$

Since $|\boldsymbol{v}(t)| \le \bar{v} < 1$, the integrands have uniformly bounded energy norm. Thus by assumption (5.34) the integrals converge in $\mathcal{M}$ and define $(\boldsymbol{E}_{\mathrm{sc}}, \boldsymbol{B}_{\mathrm{sc}}) \in \mathcal{M}$. Hence (5.35) follows. □

There are two cases of interest for which the integrability condition (5.34) can be checked.

(i) *Compton scattering (zero external potential).* If $|e| \le \bar{e}$, then by (5.24) $|\dot{\boldsymbol{v}}(t)| \le C(1+|t|)^{-1-\sigma}$ which implies (5.34). For a freely moving charge the asymptotic motion is rectilinear and the scattered waves propagate according to the free Maxwell equations. Such a result also applies to a charge reaching an essentially potential-free region. The standard example is a charge scattered by an infinitely heavy nucleus. For sufficiently long times the incident fields have decayed already and we assume that the charge has reached, with its velocity pointing outwards, a large sphere centered at the nucleus. Then the external force decays as $1/t^2$ which combined with Theorem 5.3 yields the desired asymptotics.

(ii) *Rayleigh scattering (bounded motion).* Under the Wiener condition (W) we already know that $\lim_{t\to\infty} \dot{\boldsymbol{v}}(t) = 0$. If in addition the motion is bounded,

$$|\boldsymbol{q}(t)| \le \bar{q} \tag{5.38}$$

for all t, then necessarily

$$\lim_{t\to\infty} \boldsymbol{v}(t) = 0\,, \tag{5.39}$$

i.e. the particle comes to rest. Inserting in Newton's equations of motion (2.34) and using the fact that the fields become locally soliton-like, we infer that

$$\lim_{t\to\infty} \nabla\phi_{\mathrm{ex}}(\boldsymbol{q}(t)) = 0\,. \tag{5.40}$$

Let us define the set $\mathcal{A}$ of critical points for the potential ϕ_{ex}, $\mathcal{A} = \{\boldsymbol{q} \mid \nabla\phi_{\mathrm{ex}}(\boldsymbol{q}) = 0\}$. By (5.40), $\boldsymbol{q}(t)$ approaches $\mathcal{A}$ as a set. If $\mathcal{A}$ happens to be a discrete set, then, by the continuity of solutions in t, $\boldsymbol{q}(t)$ has to converge to some definite $\boldsymbol{q}^* \in \mathcal{A}$.

Such reasoning yields no rate of convergence. The situation improves in the case where $\boldsymbol{q}^*$ is a stable local minimum of ϕ_{ex}. We linearize the Maxwell equations at $Y^* = S_{\boldsymbol{q}^*,\mathbf{0}}$. The solution to the linearized equations converges exponentially fast to zero. Therefore, once $\boldsymbol{q}(t)$ is in the vicinity of $\boldsymbol{q}^*$, the velocity decays exponentially ensuring (5.34). In particular, if ϕ_{ex} is strictly convex and if (W) holds, then the asymptotics (5.35) of Theorem 5.3 hold for every $Y(0) \in \mathcal{M}$.

A standard situation not covered by (i) and (ii) is the motion in a uniform magnetic field. Even if one assumes that the motion is bounded, one can only conclude that $\boldsymbol{v}(t) \to 0$. The attractor $\mathcal{A}$ equals $\mathbb{R}^3$. Physically one would expect the charge to spiral inwards and to come to rest at its center of gyration. Another instructive example is the motion in a confining ϕ_{ex} with a flat bottom, say $\{\boldsymbol{x} \mid |\boldsymbol{x}| \le 1\}$ and $\boldsymbol{A}_{\text{ex}} = 0$. Each time the particle is reflected by the confining potential, it loses energy. Thus $\boldsymbol{v}(t) \to 0$ as $t \to \infty$, but $\boldsymbol{q}(t)$ has no limit.

Notes and references

Section 5.1

The long-time asymptotics are studied in Komech and Spohn (2000), where the details of the proof can be found. See also Komech, Spohn and Kunze (1997). Pitt's version of the Wiener theorem is proved in Rudin (1977), Theorem 9.7(b). We remark that Theorem 5.1 provides no rate of convergence. Thus to investigate the asymptotics of the velocity and position requires extra considerations.

Theorem 5.1 can also be read that under the Wiener condition the Abraham model admits no periodic solution. In the literature, Bohm and Weinstein (1948), Eliezer (1950), and in particular the review by Pearle (1982), periodic solutions of the Abraham model have been reported repeatedly for the case of a charged sphere, i.e. $\varphi(\boldsymbol{x}) = (4\pi a^2)^{-1}\delta(|\boldsymbol{x}| - a)$, which is not covered by Theorem 5.1 since (W) is violated. These computations invoke certain approximations and it is not clear whether the full model, as defined by (2.39)–(2.41), has periodic solutions. Pearle (1977) argues that in the Nodvik model there are no periodic solutions. Kunze (1998) proves that if there is a periodic solution, its frequency is determined by the zeros of the radial part of the form factor $\widehat{\varphi}$, which under (C) form a discrete set. If $\widehat{\varphi}$ has a zero, then the linearized system admits a periodic solution. However, the full nonlinear equations have no periodic solution, at least in a small neighborhood of the linearized periodic solution.

As will be explained in chapter 11, the Abraham model extends in the obvious way to the dynamics of many charges. The argument of Theorem 5.1 applied to this case yields that the acceleration of the center of mass relaxes to zero. One would expect particles to form neutral lumps, each of which is traveling at constant velocity for large t. No argument towards a proof is in sight.

Section 5.2

The contraction method was first developed in Komech, Kunze and Spohn (1999). Komech and Spohn (1998) prove the convergence to the soliton manifold in the case of a scalar wave field requiring only (W) and not the restriction $|e| < \bar{e}$. No convergence rates are obtained. Their result is extended to the Abraham model by Imaikin, Komech and Mauser (2003). Orbital stability was established before by Bambusi and Galgani (1993). Bambusi (1994) investigates the long-time stability in the case of an attractive central potential.

Section 5.3

Our results are based on Imaikin, Komech and Spohn (2002). The linearization argument is fully carried out in Komech, Spohn and Kunze (1997).