Numerous formulae have been given which exhibit the asymptotic behaviour as \( t \to \infty \) of solutions of
\[
x'' + F(t)x = 0,
\]
where \( F(t) \) is essentially positive and \( \int_0^\infty tF(t)\,dt = \infty \). Several of these results have been unified by a theorem of F. V. Atkinson [1]. It is the purpose of this paper to establish results, analogous to the theorem of Atkinson, for the third order equation
\[
x''' + F(t)x = 0,
\]
and for the fourth order equation
\[
x^{(iv)} + F(t)x = 0.
\]
However, rather than assume that \( F(t) \) is essentially positive, we shall instead assume that \( F(t) \) is essentially of one sign. We assume that \( F \) has a decomposition for either \( l = 1 \) or \( l = 2 \),
\[
(-1)^{l+1}F(t) = f(t) + m(t),
\]
where \( f(t) \) is a "smooth" part of \( F(t) \) and \( m(t) \) is a "small" part. It is assumed throughout that \( f(t) \) and \( m(t) \) are continuous on a ray \([a, \infty)\) with \( f(t) > 0 \) and continuously differentiable. The analysis is similar to that used in [4] for a two term \( n \)th order equation with sufficiently smooth coefficients.

It is convenient to express (1) and (2) in the vector forms
\[
S' = MS
\]
and
\[
T' = NT
\]
where \( S, M, T, \) and \( N \) are, respectively
\[
\begin{bmatrix}
x \\ x' \\ x'' \\
\end{bmatrix}, \quad \begin{bmatrix}
0 & 1 & 0 \\ 0 & 0 & 1 \\ -F & 0 & 0
\end{bmatrix}, \quad \begin{bmatrix}
x \\ x' \\ x'' \\
\end{bmatrix}, \quad \begin{bmatrix}
0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -F & 0 & 0 & 0
\end{bmatrix}.
\]

Theorem 1. Let
\[
f'f^{-4/3} = h(t) + h_1(t)
\]
and suppose that the following conditions hold:

\[ \int_0^\infty f^{-2/3} |m| \, dt < \infty, \quad (7) \]

\[ \int_0^\infty f^{1/3} |h_1| \, dt < \infty, \quad (8) \]

and

\[ \int_0^\infty |h'| \, dt < \infty \quad \text{with} \quad h(\infty)^2 \neq 27.4^{-1/3}. \quad (9) \]

Then there is a fundamental matrix \( S(t) \) of (4) and a \( t_0 \) such that, as \( t \to \infty \),

\[
\begin{bmatrix}
  f^{1/3}(t) & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & f^{-1/3}(t)
\end{bmatrix}
\begin{bmatrix}
  \exp(-\Lambda_1(t)) & 0 & 0 \\
  0 & \exp(-\Lambda_2(t)) & 0 \\
  0 & 0 & \exp(-\Lambda_3(t))
\end{bmatrix} \to L, \quad (10)
\]

where, for \( i = 1, 2, 3 \),

\[ \Lambda_i(t) = \int_{t_0}^t f^{1/3}(\tau) \lambda_i(\tau) \, d\tau, \]

\[ \lambda_i(\tau) = \frac{w_i(\tau) + h(\tau)^2}{27w_i(\tau)}, \]

\[ w_i(\tau) = \mu_i [(1 + \{1 - 4h(\tau)^3/3^9\}^{1/2})/2]^{1/3}, \]

\( \mu_1, \mu_2, \mu_3 \) are the cube roots of \( (-1)^i \), and \( L = \{l_{ij}\} \) is the \( 3 \times 3 \) matrix given by \( l_{1j} = 1 \), \( l_{2j} = \lambda_j(\infty) - h(\infty)/3 \), and \( l_{3j} = \lambda_j(\infty)l_{2j} \) for \( j = 1, 2, 3 \).

Proof. We first transform (4) by defining \( Z = QS \), where \( Q \) is the diagonal matrix \( Q = \text{diag}[f^{1/3}, 1, f^{-1/3}] \). Then

\[
Z' = [QM^{-1} + Q'Q^{-1}]Z = f^{1/3} (A + B + C)Z, \quad (11)
\]

where

\[
A = \begin{bmatrix}
  h(\infty)/3 & 1 & 0 \\
  0 & 0 & 1 \\
  (-1)^t & 0 & -h(\infty)/3
\end{bmatrix}, \quad B = \begin{bmatrix}
  1/3 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & -1/3
\end{bmatrix},
\]

and

\[
C = \begin{bmatrix}
  h_1(t)/3 & 0 & 0 \\
  0 & 0 & 0 \\
  (-1)^t m(t)/f(t) & 0 & -h_1(t)/3
\end{bmatrix}.
\]
If \( \int_0^\infty f^{1/3} \, dt < \infty \), it follows from (6), (8), and (9) that \( \int_0^\infty \frac{f'}{f} \, dt < \infty \); hence \( \log f \) has a limit at \( \infty \). Thus \( f \) has a positive lower bound, contrary to \( \int_0^\infty f^{1/3} \, dt < \infty \). Let \( k(t) = \int_t^\infty f^{1/3} \, d\tau \), and denote the inverse of \( k \) by \( g \). The change of variable \( W(s) = Z(g(s)) \) in (11) yields for \( W \),

\[
W'(s) = [A + \bar{B}(s) + \bar{C}(s)]W(s),
\]

where \( \bar{B}(s) = B(g(s)) \) and \( \bar{C}(s) = C(g(s)) \). By conditions (7) and (8),

\[
\int_0^\infty |\bar{C}(s)| \, ds = \int_0^\infty f^{1/3}(t) |C(t)| \, dt < \infty.
\]

By condition (9), \( \bar{B}(s) \to 0 \) as \( s \to \infty \) and \( \int_0^\infty |\bar{B}'(s)| \, ds = \int_0^\infty |B'(t)| \, dt < \infty \). Hence, if the characteristic roots of \( A \) are distinct and the real parts of the roots of \( A + \bar{B}(s) \) are well-behaved, we may apply the asymptotic theorem due to Levinson [3, Chap. 3, Theorem 8.1]. A calculation shows that the roots \( \lambda(s) \) of \( A + \bar{B}(s) \) satisfy the equation

\[
\lambda^3 - \lambda h(g(s))^2/9 - (-1)^I = 0.
\]

By recalling that, if \( w \neq 0 \) satisfies the equation

\[
w^3 = \frac{-q/2 \pm \sqrt{q^2/4 + p^3/27}}{3},
\]

then \( z = w - p/3w \) satisfies the equation \( z^3 + pz + q = 0 \) (cf. [2, p. 112]), the roots of (13) may be written (for \( I = 1 \) use the \(-\) sign in (14) and for \( I = 2 \) use the \(+\) sign), for \( I = 1, 2, 3 \),

\[
\lambda_I(s) = \tilde{w}_I(s) + h(g(s))^2/27 \tilde{w}_I(s),
\]

where

\[
\tilde{w}_I(s) = \mu_I((1 + \{1 - 4h(g(s))^6/3^9\})^{1/3})/2)^{1/3},
\]

and \( \mu_1, \mu_2, \) and \( \mu_3 \) are the cube roots of \((-1)^I\). For \( 4h(g(s))^6 > 3^9 \) in (16), the exponent 1/2 denotes the root in the upper half plane and the exponent 1/3 denotes the first quadrant root. Thus the characteristic roots of \( A \) are distinct and a short calculation shows that the columns of \( L \) are characteristic vectors of \( A \). From (13) and \( h(\infty)^2 \neq 27.4^{-1/3} \), it follows that the roots \( \lambda_I(s) \) \((s \text{ sufficiently large and } s \to \infty)\) must occur in one of the following combinations: (i) one negative root and a pair of complex conjugate roots with positive real part, (ii) one positive root and a pair of complex conjugate roots with negative real part, or (iii) three distinct real roots. In either case, we have, for each \( i, j \), that either \( \text{Re} [\tilde{\lambda}_I(s) - \tilde{\lambda}_J(s)] \equiv 0 \) or \( \text{Re} [\tilde{\lambda}_I(s) - \tilde{\lambda}_J(s)] \to 0 \) as \( s \to \infty \); thus the theorem of Levinson applies. There is then a number \( s_0 \) and a fundamental matrix \( W \) of (12) such that, as \( s \to \infty \),

\[
W(s) = \begin{bmatrix}
\exp \left(-\int_{s_0}^s \tilde{\lambda}_1(u) \, du\right) & 0 & 0 \\
0 & \exp \left(-\int_{s_0}^s \tilde{\lambda}_2(u) \, du\right) & 0 \\
0 & 0 & \exp \left(-\int_{s_0}^s \tilde{\lambda}_3(u) \, du\right)
\end{bmatrix} \to L.
\]
Since \( W(k(t)) = Q(t)S(t) \) and (for \( t_0 = g(s_0) \))
\[
\int_{s_0}^{s(t)} \lambda(u) du = \int_{t_0}^{t} f^{1/3}(\tau)\lambda(\tau) d\tau,
\]
the above asymptotic behaviour for \( W \) yields (10) for \( S \).

For the perturbed Euler equation
\[
x'''' + (K/\xi + \xi)x = 0,
\]
Theorem 1 is applicable if \( \int_{0}^{\infty} t^2 |\dot{x}| dt < \infty \) and \( K^2 \neq 4/27 \). Also applicable to perturbations of the Euler equation is the result that, if \( m(t) = 0 \) and \( \int_{0}^{\infty} \left|\left((f^{1/3})'\right)\right| dt < \infty \), with \( f^{1/3} \) tending as \( t \to \infty \) to a positive limit \( L \), \( L^6 \neq 4/27 \), then the hypothesis of Theorem 1 is satisfied with \( h(t) = -3/(f^{1/3}) \) and \( h_1(t) = 3/(f^{1/3})'/(f^{2/3}) \).

**THEOREM 2.** Let
\[
f f^{-5/4} = h(t) + h_1(t)
\]
and suppose that the following conditions hold:
\[
\int_{a}^{\infty} f^{-3/4} |m| dt < \infty,
\]
\[
\int_{a}^{\infty} f^{1/4} |h_1| dt < \infty,
\]
and
\[
\int_{a}^{\infty} |h'| dt < \infty,
\]
with \( h(\infty)^2 \neq 16 \) for \( l = 1 \), \( h(\infty)^2 \neq 64/3 \) for \( l = 2 \). Then there is a number \( t_0 \) and a fundamental matrix \( T \) of (5) such that, as \( t \to \infty \),
\[
Q(t)T(t)E(t) \to K,
\]
where \( Q(t) \) and \( E(t) \) are the diagonal matrices
\[
Q(t) = \text{diag}[f(t)^{3/8}, f(t)^{1/8}, f(t)^{-1/8}, f(t)^{-3/8}]
\]
and
\[
E(t) = \text{diag}[\exp(-\Lambda_1(t)), \ldots, \exp(-\Lambda_4(t))],
\]
in which, for \( i = 1, 2, 3, 4 \),
\[
\Lambda_i(t) = \int_{t_0}^{t} f^{1/4}(\tau)\lambda_i(\tau) d\tau,
\]
and the \( \lambda_i(\tau) \) are the 4 roots of the equation
\[
\lambda^2 = 5h(\tau)^2/64 \pm \left((-1)^i + h(\tau)^4/4^4\right)^{1/2}.
\]
Also $K = \{k_{ij}\}$ is given by $k_{11} = 1$, $k_{2j} = \lambda_j(\infty) - 3h(\infty)/8$, $k_{3j} = \lambda_j(\infty)k_{2j} - h(\infty)/8$, and $k_{4j} = \lambda_j(\infty)k_{3j} + h(\infty)/8$ for $j = 1, 2, 3, 4$.

**Proof.** The proof is similar to that of Theorem 1. The transformation $Z = Q^T$ with $Q$ as in (18) yields

$$Z' = f^{-1/4}(A + B + C)Z$$

where

$$A = \begin{bmatrix}
3h(\infty)/8 & 1 & 0 & 0 \\
0 & h(\infty)/8 & 1 & 0 \\
0 & 0 & -h(\infty)/8 & 1 \\
(-1)^l & 0 & 0 & -3h(\infty)/8
\end{bmatrix},$$

$$B = [h(t) - h(\infty)] \text{diag} [3/8, 1/8, -1/8, -3/8],$$

and

$$C = \begin{bmatrix}
3h_1(t)/8 & 0 & 0 & 0 \\
0 & h_1(t)/8 & 0 & 0 \\
0 & 0 & -h_1(t)/8 & 0 \\
(-1)^l m(t)/f(t) & 0 & 0 & -3h_1(t)/8
\end{bmatrix}.$$  

Define $k(t) = \int_0^t f^{-1/4} \, dt$, and let $g$ be the inverse of $k$. As in Theorem 1, $k(t) \to \infty$ as $t \to \infty$ and, if $W(s) = Z(g(s))$, then

$$W'(s) = [A + B(s) + C(s)]W(s),$$

where $B(s) = B(g(s))$ and $C(s) = C(g(s))$. The characteristic roots $\lambda_i(s)$ of $A + B(s)$ satisfy the equation

$$\lambda^4 - 5\lambda^2 h(g(s))^2/32 + 9h(g(s))^4/8^4 = (-1)^l.$$  \hfill (19)

From (19) it follows that

$$\lambda^2 = 5h(g(s))^2/64 \pm \{(-1)^l + h(g(s))^4/4^4\}^{1/2}. \hfill (20)$$

For $l = 1$, the condition $h(\infty)^2 \neq 16$ implies that (20) has, for $s$ sufficiently large and $\leq \infty$ and $h(\infty)^2 > 16$, four distinct real roots; for $h(\infty)^2 < 16$, roots of the form $\pm a \pm ib$ and $-\alpha \pm ib$ with $\alpha > 0$, $\beta > 0$. For $l = 2$, the condition $h(\infty)^2 \neq 64/3$ implies that (20) has, for $s$ sufficiently large and $\leq \infty$ and $h(\infty)^2 > 64/3$, four distinct real roots; for $h(\infty)^2 < 64/3$, roots of the form $\pm a \pm ib$ with $\alpha > 0$, $\beta > 0$. Thus, as in Theorem 1, we have, for each $l, j$, that either $\Re{[\lambda_i(s) - \lambda_j(s)]} \equiv 0$ or $\Re{[\lambda_i(s) - \lambda_j(s)]}$ tends to a nonzero constant as $s \to \infty$. Application of the theorem of Levinson as in Theorem 1 yields (17).
REFERENCES


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