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NEARNESS CONVERGENCE

BY

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ABSTRACT. In this paper, a concept of nearness convergence is introduced which contains the proximal convergence of Leader as a special case. It is proved that uniform convergence and this nearness convergence are equivalent on totally bounded uniform nearness spaces. One of the features of this convergence is that it lies between uniform convergence and pointwise convergence, and implies uniform convergence on compacta. Some other weaker notions of nearness convergence which are sufficient to preserve nearness maps are also discussed.

1. **Introduction.** It is well known that if a net $\{f_n : n \in D\}$ of continuous functions from a topological space X to a topological space Y converges pointwise (P.C.) to a function f, then f is not necessarily continuous. This prompted Weierstrass in 1841 to introduce uniform convergence (U.C.) which preserves continuity as well as uniform continuity. However, U.C. is rather strong because it is possible for f to be continuous without the convergence being uniform. This led a number of mathematicians to explore necessary and sufficient conditions for f to be continuous. Arzelà discovered quasi uniform convergence (Q.U.C.) and Dini discovered simple uniform convergence (S.U.C.). Two results stand out in the case X is compact: the well known Dini's theorem (f is continuous iff P.C. = U.C. for monotone nets) and the little known Arzelà's theorem (f is continuous iff P.C. = Q.U.C.). In 1937 Weil discovered uniform spaces in which one can introduce uniform continuity and U.C. Consequently, the earlier results were generalized in this setting and the above results of Arzelà and Dini proved to be of great value in Functional Analysis (Bartle [1], Dunford and Schwartz [3] page 268). A topological space is uniformizable if and only if it is completely regular. So a question naturally arises: is it possible to describe in a general topological space a convergence that preserves continuity? It is the purpose of this paper to study some notions of convergence in the setting of nearness spaces. A nearness space is a generalization of a uniform space and unlike uniformity, one can introduce nearness in an R_0-space (see Herrlich [4], Naimpally [8]). However, to make the exposition a bit simpler, we assume that all topological spaces in this paper are T_1. In the sequel $\overline{A}$ denotes the closure of a set A in a space X. If $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(X)$

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(the power set of X), then $\mathcal{A} \vee \mathcal{B} = \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\}$. $C(X,Y)$ denotes the space of continuous functions from X to Y.

DEFINITION (1.1). *Let X be a nonempty set and $\eta \subset \mathcal{P}(\mathcal{P}(X))$. Then η is called a nearness on X iff (a) $\mathcal{A} \in \eta$ implies $\emptyset \notin \mathcal{A}$, (b) $\cap \mathcal{A} \neq \emptyset$ implies $\mathcal{A} \in \eta$, (c) $(\mathcal{A} \vee \mathcal{B}) \in \eta$ if and only if $\mathcal{A} \in \eta$ or $\mathcal{B} \in \eta$, and (d) if $\mathcal{A} \in \eta$ and for each $B \in \mathcal{B}$ there is an $A \in \mathcal{A}$ such that $A \subseteq B$, then $\mathcal{B} \in \eta$.*

If X is a topological space then a nearness η on X is called *compatible* with the topology on X provided for each $A \subset X$, $p \in X$, $p \in \overline{A}$ if and only if $\{\{p\}, A\} \in \eta$. Every T_1-space X has compatible nearnesses η_0 and η_k defined by

(1.2) $\mathcal{A} \in \eta_0$ if and only if $\cap \{\overline{A} : A \in \mathcal{A}\} \neq \emptyset$,

(1.3) $\mathcal{A} \in \eta_k$ if and only if $\mathcal{A} \in \eta_0$ or all the members of $\mathcal{A}$ are infinite.

If $(Y, \mathcal{V})$ is a uniform space, it induces a compatible nearness $\eta = \eta(\mathcal{V})$ defined by

(1.4) $\mathcal{A} \in \eta$ if and only if for each $V \in \mathcal{V}$, $\cap \{V[A] : A \in \mathcal{A}\} \neq \emptyset$.

This nearness $\eta(\mathcal{V})$ not only satisfies (1.1) (a)–(d), but also

(1.5) $\mathcal{A} \notin \eta$ implies there is a $\mathcal{B} \notin \eta$ such that for each $B \in \mathcal{B}$, there is an $A \in \mathcal{A}$ with $A \subseteq \cap \{C \in \mathcal{B} : B \cup C \neq X\}$ (Herrlich [4]).

A nearness space (Y, η) satisfying (1.5) is called a *uniform nearness space*. A nearness space (Y, η) is called *totally bounded* if $\mathcal{A} \notin \eta$ implies that there is a finite subset $\mathcal{B} \subset \mathcal{A}$ such that $\cap \mathcal{B} = \emptyset$.

In this paper (X, ξ) and (Y, η) denote T_1-nearness spaces, $\{f_n : n \in D\}$ is a net of maps on X to Y converging to a function $f : X \longrightarrow Y$, the mode of convergence to be specified. A function $f : (X, \xi) \longrightarrow (Y, \eta)$ is called a *near map* (or nearness map) iff $\mathcal{A} \in \xi$ implies $f(\mathcal{A}) \in \eta$. If ξ and η are induced by uniformities, then 'near map' is equivalent to 'uniform continuity' and thus the concept of a near map is a generalization of uniform continuity.

If our aim is merely the preservation of nearness (continuity, proximal continuity, contiguity) then the following definition would have sufficed.

DEFINITION (1.6). *$f_n \xrightarrow{\text{n.c.}} f$ iff for each $\mathcal{A} \subset \mathcal{P}(X)$, $f(\mathcal{A}) \notin \eta$ implies eventually $f_n(\mathcal{A}) \notin \eta$.*

THEOREM (1.7). *If each f_n is a near map and $f_n \xrightarrow{\text{n.c.}} f$, then f is a near map.*

PROOF. Suppose $f(\mathcal{A}) \notin \eta$. Then, eventually $f_n(\mathcal{A}) \notin \eta$, and, since each f_n is a near map, $\mathcal{A} \notin \xi$. So, f is a near map.

We can similarly show that n.c. preserves continuity, proximal-continuity or contigual-continuity. However, n.c. is not comparable to P.C.

EXAMPLE (1.8). Let $X = Y = \mathbf{R}$ with usual metric nearnesses. Define $f_n(x) = n + x, x \in X$ and $f(x) = x, x \in X$. Then $f_n \xrightarrow{\text{n.c.}} f$ but $f_n \overset{\text{P.C.}}{\not\longrightarrow} f$. Since P.C. does not preserve continuity, it follows that n.c. and P.C. are independent.

DEFINITION (1.9). $\text{n}^*.\text{c.} = \text{n.c.} + \text{P.C.}$

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THEOREM (1.10). U.C. *is stronger than* n*.c.

PROOF. It suffices to show that U.C. implies n.c. Suppose $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ are uniform spaces and $\xi = \xi(\mathcal{U})$, $\eta = \eta(\mathcal{V})$, $f_n \xrightarrow{\text{U.C.}} f$. Suppose $f(\mathcal{A}) \notin \eta$. Then there is a $V \in \mathcal{V}$ such that $\cap\{V[f(A)] : A \in \mathcal{A}\} = \emptyset$. Since $f_n \xrightarrow{\text{U.C.}} f$, there is a $W \in \mathcal{V}$ such that $W^2 \subseteq V$ and $f_n(A) \subseteq W[f(A)]$ eventually. So, eventually, $\cap\{W[f_n(A)] : A \in \mathcal{A}\} = \emptyset$, whence $f_n(\mathcal{A}) \notin \eta$ eventually. Thus, $f_n \xrightarrow{\text{n.c.}} f$. □

We now show that n*.c. does not imply U.C.

EXAMPLE (1.11). Let $X = Y = \mathbf{R}$ with the usual metric nearness. $f_n(x) = \big(1 + (1/n)\big)x$, $x \in X$ and $f(x) = x$, $x \in X$. Then $f_n \xrightarrow{\text{n}^*\text{.c.}} f$, but $f_n \overset{\text{U.C.}}{\not\longrightarrow} f$.

One can define other weaker notions of nearness convergence which are sufficient to preserve near maps. For example, analogous to Dini Convergence and Arzelà convergence (see [2] for definitions) we may, in the nearness setting, define the following notions respectively.

DEFINITION (1.12). *(a)* $f_n \xrightarrow{\text{d.c.}} f$ *iff* $f_n \xrightarrow{\text{P.C.}} f$ *and for all* $\mathcal{A} \subset \mathcal{P}(X)$, $f(\mathcal{A}) \notin \eta$ *implies* $f_n(\mathcal{A}) \notin \eta$ *frequently. (b)* $f_n \xrightarrow{\text{a.c.}} f$ *iff* $f_n \xrightarrow{\text{P.C.}} f$ *and for all* $\mathcal{A} \subset \mathcal{P}(X)$, $|\mathcal{A}| > 1$ *and* $f(\mathcal{A}) \notin \eta \Rightarrow$ *for each* $m \in D$ *there exist* $n_1, n_2, \ldots, n_p \in D$, $n_i > m$, *and* $\mathcal{A}_i \subset \mathcal{P}(X)$, $i = 1, 2, \ldots, p$, *such that* $\mathcal{A} = \bigcup_{i=1}^{p} \mathcal{A}_i$ *and* $f_i(\mathcal{A}_i) \notin \eta$.

The proof of the following theorem is straightforward and P.C. is not needed in the hypothesis.

THEOREM (1.13). *If* $f_n \xrightarrow{\text{d.c.}} f$ *(resp.* $f_n \xrightarrow{\text{a.c.}} f$*) and each* f_n *is a near map, then* f *is a near map.*

Obviously n*.c. implies d.c. and n*.c. implies a.c. but the converse is not true.

EXAMPLE (1.14). Let $X = Y = \mathbf{R}$ with the usual metric nearness. Let

$$u_{2n-1}(x) = \frac{x}{nx + (1 - nx)^2}, \quad u_{2n}(x) = \frac{-x}{[(n+1)x^2 + (1 - (n+1)x)^2]}.$$

Take

$$f_m(x) = \sum_{n=1}^{m} u_n(x), \quad f(x) = \frac{x}{x^2 + (1 - x)^2}.$$

Then $f_m \xrightarrow{\text{d.c.}} f$ but $f_m \overset{\text{n}^*\text{.c.}}{\not\longrightarrow} f$.

Our definition of near convergence should be stronger than P.C. and we ensure this in the next section using the analogy of Leader convergence.

2. **Near Convergence.** Leader [6] first defined convergence in proximity spaces which we call Leader convergence (L.C.).

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DEFINITION (2.1). *Let* (X,δ_1), (Y,δ_2) *be proximity spaces,* $f \in Y^X$ *and* $(f_n : n \in D)$ *a net in* Y^X. $f_n \xrightarrow{\text{L.C.}} f$ *iff for each* $A \subset X$, $E \subset Y$, $f(A)\not\delta_2 E$ *implies eventually* $f_n(A)\not\delta_2 E$.

It is known that if δ_2 is induced by a uniformity $\mathcal{V}$ then U.C. implies L.C. and the two are equivalent if (a) Y is totally bounded, or (b) (f_n) is a sequence (or D is linearly ordered), or (c) X is compact (Leader [6], Njåstad [10], DiConcilio and Naimpally [2]). Nachman [7] has shown that L.C. does not imply U.C. However, L.C. implies P.C. and on $C(X,Y)$ L.C. implies U.C.C. (uniform convergence on compacta). This provides a motivation for our definition of *Near Convergence* (N.C.).

DEFINITION (2.2). *Let* X *be a set* (Y,η) *a nearness space* $f \in Y^X$, $(f_n : n \in D)$ *a net in* $Y^X \cdot f_n \xrightarrow{\text{N.C.}} f$ *iff for each* $\mathcal{A} \subset \mathcal{P}(X)$, $\mathcal{B} \subset \mathcal{P}(Y)$, $f(\mathcal{A}) \cup \mathcal{B} \notin \eta$ *implies eventually* $f_n(\mathcal{A}) \cup \mathcal{B} \notin \eta$.

REMARK (2.3). Certainly, N.C. implies n.c., and (as will be shown in the sequel) N.C. is stronger than P.C. If (Y,δ) is a proximity space, then (1.1) gives L.C. by taking $\mathcal{A}$ and $\mathcal{B}$ as singleton families in $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ respectively. Also, by taking $\mathcal{A}$ such that $|\mathcal{A}| = 2$ and $\mathcal{B} = \emptyset$ (the empty set), we get the notion of the proximal convergence defined in [2]. The proof of the following theorem is omitted.

THEOREM (2.4). *If* (X,ξ) *and* (Y,η) *are nearness spaces and* $\{f_n : n \in D\}$ *a net of nearness maps from* X *into* Y *such that* $f_n \xrightarrow{\text{N.C.}} f \in Y^X$, *then* f *is also a nearness map.*

THEOREM (2.5). *Let* (Y,η) *be a uniform nearness space and* $\{f_n : n \in D\}$ *a net of maps from a set* X *into* Y *converging uniformly to* $f \in Y^X$, *then* $f_n \xrightarrow{\text{N.C.}} f$.

PROOF. Suppose $f_n \xrightarrow{\text{U.C.}} f$. Let $f(\mathcal{A}) \cup \mathcal{B} \notin \eta$, $\mathcal{A} \subset \mathcal{P}(X)$ and $\mathcal{B} \subset \mathcal{P}(Y)$. Then there exists an entourage U such that $\cap\{U[C] : C \in f(\mathcal{A}) \cup \mathcal{B}\} = \emptyset$. Because of uniform convergence there exists an entourage V such that $V^2 \subseteq U$ and $f_n(A) \subseteq V[f(A)]$ for each $A \in \mathcal{A}$ eventually. Then $V[f_n(A)] \subseteq V^2[f_n(A)] \subseteq U[f(A)]$ eventually. Also $V^2 \subseteq U$ implies $V[B] \subseteq U[B]$ for each $B \in \mathcal{B}$. Thus, eventually,

$$\cap\{V[E] : E \in f_n(\mathcal{A}) \cup \mathcal{B}\} \subseteq \cap\{U[C] : C \in f(\mathcal{A}) \cup \mathcal{B}\} = \emptyset.$$

Hence $f_n(\mathcal{A}) \cup \mathcal{B} \notin \eta$ eventually, proving $f_n \xrightarrow{\text{N.C.}} f$. □

Now U. C. = N. C. when X is compact.

The next theorem shows that N.C. and U.C. are equivalent on totally bounded uniform nearness spaces.

THEOREM (2.6). *If* $\{f_n : n \in D\}$ *is a net of maps from a set* X *to a totally bounded uniform nearness space* (Y,η) *such that* $f_n \xrightarrow{\text{N.C.}} f \in Y^x$, *then* $f_n \xrightarrow{\text{U.C.}} f$.

PROOF. Suppose $f_n \xrightarrow{\text{N.C.}} f$. If possible assume that $f_n \overset{\text{U.C.}}{\not\longrightarrow} f$. Then there exists a symmetric entourage U, a cofinal subset D_0 of D and for each $n \in D_0$ an element

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$x_n \in X$ such that $f_n(x_n) \notin U(f(x_n))$. Since, Y is totally bounded, we can choose a symmetric entourage V such that $V^2 \subseteq U$ and a family $\{E_i\}_{i=1}^k$ of subsets of Y where $Y = \bigcup_{i=1}^k E_i$ and $E_i \times E_i \subseteq V$ for each $i = 1, 2, \ldots, k$. Consequently, there exists a cofinal subset D_1 of D_0 such that for all $n \in D_1$, $f_n(x_n) \in E_{i_0}$ for some $i_0 = 1, 2, \ldots, k$. Moreover $(f_n(x_n), f_m(x_m)) \in E_{i_0} \times E_{i_0} \subseteq V$ for all $n, m \in D_1$. Take $A = \{x_n : n \in D_1\}$, $B = \{f_n(x_n) : n \in D_1\}$. Then for all $p, q \in D_1$, $(f(x_p), f_q(x_q)) \notin V$. For, otherwise it would follow that $(f(x_p), f_p(x_p)) \in V^2 \subseteq U$ contradicting the definition of U. Thus, $f(A) \cap V[B] = \emptyset$ and hence $f(\mathcal{A}) \cup \mathcal{B} \notin \eta$, (where $\mathcal{A} = \{A\}$, $\mathcal{B} = \{B\}$). On the other hand $f_n(A) \cap B \neq \emptyset$ for all $n \in D_1$, contradicting the fact that $f_n \xrightarrow{\text{N.C.}} f$. □

COROLLARY (2.7). *(Leader [6]) If $\{f_n : n \in D\}$ is a net of maps from a set X to a proximity space (Y, δ) induced by a totally bounded uniformity such that $f_n \xrightarrow{\text{L.C.}} f \in Y^x$ then $f_n \xrightarrow{\text{U.C.}} f$.*

The following theorem shows that N.C. lies between pointwise convergence and uniform convergence.

THEOREM (2.8). N.C. *implies* P.C.

PROOF. Suppose $f_n \xrightarrow{\text{N.C.}} f$. Let O be an open set in Y containing $f(x)$ where $x \in X$. Take $E = Y \backslash O$. Then $f(\{x\}) \cup \{E\} \notin \eta$. Hence eventually $f_n(\{x\}) \cup \{E\} \notin \eta$. In other words $f_n(x) \in O$ eventually. □

REMARK (2.9). If X is a topological space and (Y, η) is an EF-nearness space then N.C. implies uniform convergence on compacta.

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