Compositio Mathematica **108:** 123–149, 1997.

Addition formula for q-disk polynomials

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Received 27 December 1995; accepted in final form 4 January 1996

Abstract. Explicit models are constructed for irreducible $*$-representations of the quantised universal enveloping algebra $U_q(\mathfrak{gl}(n))$. The irreducible decomposition of these modules with respect to the subalgebra $U_q(\mathfrak{gl}(n-1))$ is given, and the corresponding spherical and associated spherical elements are determined in terms of little q-Jacobi polynomials. This leads to a proof of an addition theorem for the spherical elements, the so-called q-disk polynomials.

Mathematics Subject Classifications (1991): 16W30, 17B37, 33D45, 33D80, 81R50.

Key words: CQG algebras, quantum unitary group, quantized universal enveloping algebra, spherical elements, q-disk polynomials, addition formula

1. Introduction

Over the past few years quantum groups have shown to be powerful tools in the study of q-hypergeometric functions. They enabled proofs of identities which would have been hard to guess without quantum group theoretic motivation. We mention for instance the papers [10], [16], [18] and [22]. See also [11], [12], [15], [20] and [29], and references therein, for surveys on the connection between quantum groups and basic hypergeometric functions.

The purpose of this paper is to present an addition theorem for so-called q-disk polynomials, using quantum group theory. This result is a q-analogue of a result which was proved around 1970 by Šapiro [25] and Koornwinder [13], [14] independently. They considered the homogeneous space $U(n)/U(n-1)$, were $U(n)$ denotes the group of unitary transformations of the vector space $\mathbb{C}^n$, and identified the corresponding zonal spherical functions as polynomials in two variables z and $\bar{z}$, whose orthogonality measure is supported by the closed unit disk D in the complex plane. These so-called disk polynomials are denoted $R^{(\alpha)}_{l,m}(z)$ ($l, m \in \mathbb{Z}_+; \alpha > -1$) and they can be expressed in terms of the normalised Jacobi polynomials $R^{(\alpha,\beta)}_n(x) = P^{(\alpha,\beta)}_n(x)/P^{(\alpha,\beta)}_n(1)$, cf. [5, Sect. 10.8]. Explicitly;

$$R^{(\alpha)}_{l,m}(z) = \begin{cases} z^{l-m} R^{(\alpha,l-m)}_m(2|z|^2-1) & (l \geqslant m) \\ \bar{z}^{m-l} R^{(\alpha,m-l)}_l(2|z|^2-1) & (l \leqslant m). \end{cases} \tag{1.1}$$

https://doi.org/10.1023/A:1000179602766 Published online by Cambridge University Press

The orthogonality reads

$$\iint_D R^{(\alpha)}_{l,m}(r\,\mathrm{e}^{i\phi})\overline{R^{(\alpha)}_{l',m'}(r\,\mathrm{e}^{i\phi})}r(1-r^2)^{\alpha}\mathrm{d}r\,\mathrm{d}\phi = 0$$

$$\text{if } (l,m) \neq (l',m'). \tag{1.2}$$

The zonal spherical functions on $U(n)/U(n-1)$ are then the constant multiples of the polynomials $R^{(n-2)}_{l,m}(z)$. The associated spherical functions were also shown to be expressible in terms of disk polynomials, and from this an addition formula was derived for disk polynomials with positive integer values of α. By an easy argument this identity is then extended to all complex values of α. With this result Koornwinder was able to prove an addition formula for general Jacobi polynomials; see [14]. In fact the line of arguing and the results in this paper are very similar to (part of) the ones in [14].

q-Disk polynomials (quantum disk polynomials in the terminology of [17]) are polynomials in two non-commuting variables which are expressed by means of little q-Jacobi polynomials, and which can be understood as a q-analogue of disk polynomials. They appeared in [23] where the authors studied a quantum analogue of $U(n)/U(n-1)$, or rather the coordinate ring of this quantum homogeneous space. They investigated its $\mathcal{U}_q(\mathfrak{gl}(n))$-module structure, where $\mathcal{U}_q(\mathfrak{gl}(n))$ denotes the quantised universal enveloping algebra corresponding to the Lie algebra $\mathfrak{gl}(n)$, using the theory of highest weight representations, and ended with identifying the zonal spherical functions as q-disk polynomials. These polynomials are defined as follows (cf. [17]): let $\mathcal{Z}$ be the complex unital $*$-algebra generated by the elements z and z^*, subject to the relation $z^*z = qzz^* + 1 - q$ and with involution $(z)^* = z^*$. Then the *q-disk polynomials* $R^{(\alpha)}_{l,m}(z, z^*; q)$, with $\alpha > -1$ and $l, m \in \mathbb{Z}_+$, are defined as

$$R^{(\alpha)}_{l,m}(z,z^*;q) = \begin{cases} z^{l-m}p_m^{(\alpha,l-m)}(1-zz^*;q) & (l \geqslant m) \\ p_l^{(\alpha,m-l)}(1-zz^*;q)(z^*)^{m-l} & (l \leqslant m). \end{cases} \tag{1.3}$$

Here $p_l(x;a,b;q) = {}_2\phi_1(q^{-l}, abq^{l+1}; aq; q, qx)$ is the little q-Jacobi polynomial [8], and $p_l^{(\alpha,\beta)}(x;q) = p_l(x;q^{\alpha},q^{\beta};q)$.

This paper is organised as follows. In Chapter 2 we recall the definition and some of the properties of a CQG algebra, and we prove some results on quantum homogeneous spaces needed later on. In the third chapter we introduce a q-deformation $\mathcal{Z}_n$ of the algebra of polynomials on $\mathbb{C}^n$ and we study its structure as a $\mathcal{U}_q(\mathfrak{gl}(n))$-module. Chapter 4 then deals with $\widetilde{\mathcal{Z}}_n$, the q-deformed algebra of polynomials on the sphere in $\mathbb{C}^n$ (this algebra is the same as the algebra $A(K\backslash G)$ of [23], Sect. 4.1). In Section 4.1 we introduce invariant integration on $\widetilde{\mathcal{Z}}_n$ and we realise this algebra as a $*$-subalgebra of $A_q(U(n))$. In the subsequent section we describe the irreducible decomposition of $\widetilde{\mathcal{Z}}_n$ as a $\mathcal{U}_q(\mathfrak{gl}(n))$-module. In Section 4.3 the zonal spherical elements are recovered as q-disk polynomials. Next we treat the irreducible decomposition of $\widetilde{\mathcal{Z}}_n$ as a $\mathcal{U}_q(\mathfrak{gl}(n-1))$-module and we identify the associated spherical elements; they turn out to be expressible through q-disk polynomials as well. Finally, in Section 4.5 the addition theorem for q-disk polynomials is proved. So, although we follow another path, the main results of Sections 3.1 up till 4.3 of this paper are essentially already contained in [23].

It should be noted that the definition of q-disk polynomials as polynomials in two non-commuting variables accounts for the fact that the addition formula is an identity in several non-commuting variables as well. However, in a subsequent paper [7] we present an addition theorem in commuting variables which is in fact equivalent to the one stated here.

The notation in this paper is taken from [8]. Throughout we fix a real parameter $0 < q < 1$ and we write $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$.

2. CQG algebras

In this chapter we establish some notation and we recall some facts on compact quantum groups and quantum homogeneous spaces. Our language will be that of [3]; see also [4] and [19]. The ground field for the vector spaces under consideration will always be the field $\mathbb{C}$ of complex numbers.

2.1. Definitions and generalities

Let A be a Hopf algebra with comultiplication $\Delta\colon A \to A \otimes A$, counit $\varepsilon\colon A \to \mathbb{C}$ and antipode $S : A \to A$ (for the theory of Hopf algebras we refer the reader to [27]). A is called a *Hopf $*$-algebra* if there exists an anti-linear involution $*: A \to A$ which turns A into a $*$-algebra and which is such that Δ and ε are $*$-homomorphisms. It is not difficult to show that if A is a Hopf $*$-algebra, the antipode S is invertible and satisfies $S \circ * \circ S \circ * = \mathrm{id}$. A *right corepresentation* of A is a pair (V, π) of a complex vector space V and a linear map $\pi\colon V \to V \otimes A$, satisfying

$$(\pi \otimes \mathrm{id}) \circ \pi = (\mathrm{id} \otimes \Delta) \circ \pi, \qquad (\mathrm{id} \otimes \varepsilon) \circ \pi = \mathrm{id}. \tag{2.1}$$

We also say that (V, π), or simply V, is a right comodule for A. If V is finite dimensional with basis $\{e_i\}_{i=1}^N$ and if we write $\pi(e_i) = \sum_{k=1}^N e_k \otimes \pi_{ki}$, then (2.1) is equivalent to saying that the π_{ij} satisfy $\Delta(\pi_{ij}) = \sum_{k=1}^N \pi_{ik} \otimes \pi_{kj}$ and $\varepsilon(\pi_{ij}) = \delta_{ij}$. The elements π_{ij} of A are called the *matrix coefficients* of this corepresentation. Furthermore, an element v of V is said to be *(right) A-invariant* if $\pi(v) = v \otimes 1$. When $V = Z$ is a ($*$-)algebra, the corepresentation π is called a *($*$-)coaction* if it is a homomorphism of ($*$-)algebras. A particular example of a right corepresentation is given when V is a subspace of A such that $\Delta(V) \subset V \otimes A$, and $\pi = \Delta$. In this case V is said to be a *(right) coideal* of A.

Assume that A is a Hopf $*$-algebra. A right corepresentation $\pi\colon V \to V \otimes A$ is called *unitarisable* if there exists a hermitean inner product $\langle \cdot, \cdot \rangle$ on V such that

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$\langle \pi(v), \pi(w) \rangle = \langle v, w \rangle 1_A$ for all $v, w \in V$. Here we extended the inner product $\langle \cdot, \cdot \rangle$ to a map from $V \otimes A$ to A by setting $\langle v \otimes a, w \otimes b \rangle = \langle v, w \rangle b^* a$. If V is endowed with this inner product, the corepresentation (V, π) is called *unitary* and the inner product is called A-*invariant*.

Now suppose that (V, π) is a finite dimensional right comodule for A, with orthonormal basis $\{e_i\}_{i=1}^N$ with respect to a given inner product. Denote by $\{\pi_{ij}\}$ the collection of matrix coefficients of this corepresentation. Then the following statements are equivalent:

(1) (V, π) is a unitary corepresentation
(2) $\sum_{k=1}^N \pi_{ki}^* \pi_{kj} = \delta_{ij} 1_A$ for all $1 \leqslant i, j \leqslant N$,
(3) $S(\pi_{ij}) = \pi_{ji}^*$ for all $1 \leqslant i, j \leqslant N$,
(4) $\sum_{k=1}^N \pi_{ik} \pi_{jk}^* = \delta_{ij} 1_A$ for all $1 \leqslant i, j \leqslant N$.

Let us write $\Sigma = \Sigma(A)$ for the set of equivalence classes of finite dimensional irreducible unitary corepresentations of the Hopf $*$-algebra A. Furthermore, for a given $\pi = (\pi_{ij})_{i,j=1}^{d_\pi} \in \Sigma$ we put $A_\pi = \mathrm{span}\{\pi_{ij}\}_{i,j=1}^{d_\pi}$ and $A_{\pi(r)} = \mathrm{span}\{\pi_{rj}\}_{j=1}^{d_\pi}$ $(1 \leqslant r \leqslant d_\pi)$. Then one can prove that the $\{\pi_{ij}\}$ $(\pi \in \Sigma; 1 \leqslant i, j \leqslant d_\pi)$ are linearly independent and $\sum_{\pi \in \Sigma} A_\pi$ is a direct sum (see e.g. [4], [19]; this is actually true in the more general situation of coalgebras).

DEFINITION. A Hopf $*$-algebra A is called a *CQG algebra* if A is spanned by the matrix coefficients of all its finite dimensional (irreducible) unitary corepresentations, i.e. if $A = \sum_{\pi \in \Sigma} A_\pi$.

The direct sum decomposition $A = \sum_{\pi \in \Sigma} A_\pi$ is usually referred to as the Peter–Weyl decomposition. We also remark that, with respect to Δ, we have the following irreducible decomposition of A as a right comodule: $A = \bigoplus_{\pi \in \Sigma} \bigoplus_{r=1}^{d_\pi} A_{\pi(r)}$.

THEOREM 2.1 ([19, Sect. 2.2], [3, Sect. 2.1]). *Let A be a CQG algebra. Then there exists a unique linear functional $h : A \to \mathbb{C}$, called normalised Haar functional, that satisfies*

(1) $h(1_A) = 1$,
(2) $(h \otimes \mathrm{id})\Delta(a) = h(a) 1_A = (\mathrm{id} \otimes h)\Delta(a) \quad (a \in A)$,
(3) $h(a^* a) > 0 \quad (a \in A, a \neq 0)$.

An important ingredient in the proof of this theorem is the following Schur type orthogonality result.

THEOREM 2.2 ([19, Prop. 2.6], [3, Prop. 2.1.14]). *If A is a CQG algebra, then every finite dimensional corepresentation of A is unitarisable. Moreover, for all $\pi \in \Sigma$ there exists a positive-definite constant matrix F_π such that the equalities*

$$h(\pi_{ij} \rho_{kl}^*) = \delta_{\pi\rho} \delta_{ik} \frac{(F_\pi)_{jl}}{\mathrm{tr}(F_\pi)}, \qquad h(\pi_{ij}^* \rho_{kl}) = \delta_{\pi\rho} \delta_{jl} \frac{(F_\pi^{-1})_{ki}}{\mathrm{tr}(F_\pi^{-1})}$$

hold for all $\pi, \rho \in \Sigma$. *Here 'tr' denotes the matrix trace.*

A particular class of CQG algebras is given by the finitely generated ones, called CMQG algebras. A CQG algebra A is a CMQG algebra if and only if there exists a single finite dimensional unitary corepresentation $t = (t_{ij})$ of A such that A, as an algebra, is generated by the matrix elements t_{ij}. This corepresentation is usually referred to as the *fundamental* (or natural) *corepresentation*. As an example of a CMQG algebra we will meet the algebra $\mathcal{A}_q(n) = \mathcal{A}_q(U(n))$ of regular functions on the quantum unitary group.

2.2. Transitive coactions

In this section we recall and establish some results on quantum homogeneous spaces. Our language will be that of [3, Sect. 4.1].

Let a CQG algebra A, with Haar functional h, and a $*$-algebra Z be given. Furthermore, let us assume that there exists a $*$-coaction $\delta\colon Z \to Z \otimes A$ of A on Z. Suppose that this coaction is *transitive*, i.e. suppose that there exists an injective $*$-algebra homomorphism $\Psi : Z \to A$ which intertwines the coactions δ on Z and Δ on A. Then we know from [3, Thm. 4.1.5] that Z possesses a normalised positive definite A-invariant linear functional $h_Z\colon Z \to \mathbb{C}$, by which we mean that h_Z satisfies $(h_Z \otimes \mathrm{id}) \circ \delta(z) = h_Z(z)1_A$ for all $z \in Z$, that $h_Z(1_Z) = 1$ and $h_Z(z^*z) > 0$ if $z \neq 0$. Moreover, the coaction δ is unitary with respect to the inner product given by

$$\langle z, w\rangle = h(w^*z) \qquad (z, w \in Z). \tag{2.2}$$

Upon identifying Z with $\Psi(Z)$, we may assume that Z is a $*$-subalgebra and right coideal of A, and that $\delta = \Delta$ and $h_Z = h$.

Assume that there exists a Hopf $*$-algebra C with unital Hopf $*$-algebra epimorphism $\theta\colon A \to C$ such that for all $z \in Z$ there holds

$$\theta(z) = \varepsilon(z)1_C. \tag{2.3}$$

By means of θ we define a $*$-coaction δ_C from C on Z by putting $\delta_C = (\mathrm{id}\otimes\theta)\circ\Delta$.

PROPOSITION 2.3. *Suppose V is a subcomodule of Z of finite dimension $N \geqslant 1$, so $\Delta\colon V \to V \otimes A$. Then V contains a nonzero C-invariant vector. That is, there exists an element $\zeta_0 \neq 0$ in V with the property $\delta_C(\zeta_0) = \zeta_0 \otimes 1_C$.*

Proof. Let $\{e_i\}_{i=1}^N$ be an orthonormal basis for V with respect to the inner product (2.2). Write $\Delta(e_i) = \sum_{k=1}^N e_k \otimes \pi_{ki}$. Unitarity of Δ will imply that $\Delta \boxtimes \Delta := (\mathrm{id}\otimes\mathrm{id}\otimes m_A)\circ\sigma_{23}\circ(\Delta\otimes\Delta)\colon Z\otimes Z \to Z\otimes Z\otimes A$ is also a unitary coaction with respect to the inner product $\langle v_1\otimes w_1, v_2\otimes w_2\rangle = \langle v_1, v_2\rangle\langle w_1, w_2\rangle$. Here $m_A : A\otimes A \to A$ denotes multiplication in A and σ_{23} is the flip operator interchanging the second and third tensor factors. Furthermore, we have

$$\Delta\circ m_A = (m_A\otimes\mathrm{id})\circ(\Delta\boxtimes\Delta). \tag{2.4}$$

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Now consider the element $\zeta := \sum_{i=1}^{N} e_i \otimes e_i^*$ in $Z \otimes Z$. Unitarity of Δ implies (cf. Sect. 2.1)

$$\begin{aligned}(\Delta \boxtimes \Delta)(\zeta) &= \sum_{i,j,k=1}^{N} e_j \otimes e_k^* \otimes \pi_{ji}\pi_{ki}^* \\ &= \sum_{j,k=1}^{N} \delta_{jk} e_j \otimes e_k^* \otimes 1_A = \zeta \otimes 1_A, \end{aligned} \tag{2.5}$$

i.e. ζ is A-invariant. If we put $\tilde{\zeta} := m(\zeta) = \sum_{i=1}^{N} e_i e_i^* \in Z$, then it follows from (2.4) and (2.5) that $\Delta(\tilde{\zeta}) = (\Delta \circ m_A)(\zeta) = (m_A \otimes \mathrm{id}) \circ (\Delta \boxtimes \Delta)(\zeta) = \tilde{\zeta} \otimes 1_A$, and hence $\tilde{\zeta} = (\mathrm{id} \otimes h) \circ \Delta(\tilde{\zeta})$. By the invariance of the Haar functional (Theorem 2.1(2)) we find $\tilde{\zeta} = h(\tilde{\zeta}) 1_A$, and from this we obtain $\varepsilon(\tilde{\zeta}) = h(\tilde{\zeta}) = \sum_{i=1}^{N} h(e_i e_i^*) = \sum_{i=1}^{N} \langle e_i^*, e_i^* \rangle > 0$. Finally, put $\zeta_0 := (\mathrm{id} \otimes \varepsilon)(\zeta) = \sum_{i=1}^{N} \varepsilon(e_i^*) e_i$. Then obviously $\zeta_0 \in V$. Also $\zeta_0 \neq 0$, since $\varepsilon(\zeta_0) = \varepsilon(\tilde{\zeta}) > 0$. The last thing to show is that ζ_0 is C-invariant. Writing m_C for the multiplication in C, one has

$$\begin{aligned}\delta_C(\zeta_0) &= (\mathrm{id} \otimes \theta) \circ \Delta(\zeta_0) = (\mathrm{id} \otimes m_C) \circ \big((\mathrm{id} \otimes \theta) \circ \Delta(\zeta_0) \otimes 1_C\big) \\ &= (\mathrm{id} \otimes m_C) \circ \big((\mathrm{id} \otimes \theta) \circ (\Delta \otimes \varepsilon)(\zeta) \otimes 1_C\big) \\ &= (\mathrm{id} \otimes m_C) \circ (\mathrm{id} \otimes \theta \otimes \mathrm{id}) \circ (\Delta \otimes (\varepsilon \otimes \theta) \circ \Delta)(\zeta) \\ &= (\mathrm{id} \otimes m_C) \circ (\mathrm{id} \otimes \theta \otimes \varepsilon \otimes \theta) \circ (\Delta \otimes \Delta)(\zeta) \\ &= (\mathrm{id} \otimes m_C) \circ (\mathrm{id} \otimes \varepsilon \otimes \theta \otimes \theta) \circ \sigma_{23} \circ (\Delta \otimes \Delta)(\zeta) \\ &= (\mathrm{id} \otimes \varepsilon \otimes \theta) \circ (\mathrm{id} \otimes \mathrm{id} \otimes m_A) \circ \sigma_{23} \circ (\Delta \otimes \Delta)(\zeta) \\ &= (\mathrm{id} \otimes \varepsilon \otimes \theta) \circ (\Delta \boxtimes \Delta)(\zeta) = (\mathrm{id} \otimes \varepsilon \otimes \theta) \circ (\zeta \otimes 1_A) \\ &= (\mathrm{id} \otimes \varepsilon)(\zeta) \otimes 1_C = \zeta_0 \otimes 1_C, \end{aligned}$$

where in the fourth equality we used (2.3) to find that for all $z \in Z$ there holds $\varepsilon(z) 1_C = \theta(z) = (\varepsilon \otimes \theta) \circ \Delta(z)$. This proves the proposition. □

Remark 2.4. Observe that the element ζ does not depend on the choice of the orthonormal basis of V. Suppose that C is also a CQG-algebra and suppose moreover that the pair (A, C) forms a quantum Gel'fand pair, meaning that in each A_π there is an at most one-dimensional subspace of C-invariant elements. If the comodule structure on V is unitary and irreducible, and if we consider V as a subspace of A as before, then V can be realised as some $A_{\pi(r)}$, say $A_{\pi(1)}$, for certain $\pi \in \Sigma$ (cf. Sect. 2.1). The invariant subspace will then be spanned by π_{11}. From Proposition 2.2 we know that the elements $e_i = \big(\mathrm{tr}(F_\pi^{-1})/(F_\pi^{-1})_{11}\big)^{1/2} \pi_{1i}$ form an orthonormal basis of V. Thus we obtain, again by Proposition 2.2, that $\varepsilon(\zeta_0) = \sum_{i=1}^{N} h(e_i e_i^*) = \mathrm{tr}(F_\pi^{-1})/(F_\pi^{-1})_{11} = 1/h(\pi_{11}^* \pi_{11})$.

Remark 2.5. The element ζ plays a role similar to the kernel function on a compact homogeneous space: let $T_\zeta : Z \to Z$ be the mapping $T_\zeta(z) = (\mathrm{id} \otimes h_Z)(\zeta.(1 \otimes z))$. Then it is easily seen that $T_\zeta(v) = v$ for all $v \in V$ and that $T_\zeta(w) = 0$ if w is orthogonal to V. Indeed, if $v = \sum_{j=1}^N \alpha_j e_j$ then $T_\zeta(v) = \sum_{i,j=1}^N \alpha_j e_i h_Z(e_i^* e_j) = \sum_{i,j=1}^N \alpha_j e_i \langle e_i, e_j\rangle = \sum_{j=1}^N \alpha_j e_j = v$. In the same way it follows that $T_\zeta(w) = 0$ whenever $\langle w, V\rangle = 0$.

We end this section with a small lemma, needed later on.

LEMMA 2.6. *Suppose A, C are two CQG-algebras with normalised Haar functionals h_A, h_C respectively, and such that there exists a unital Hopf $*$-algebra epimorphism $\pi: A \to C$ (so C is a so-called quantum subgroup of A). Suppose furthermore that $\delta: Z \to Z \otimes A$ is a right $*$-coaction of A on some $*$-algebra Z, which is unitary with respect to a given inner product $\langle\cdot,\cdot\rangle$.*

Let $V \subset Z$ be a subcomodule for the corepresentation $\delta_C = (\mathrm{id} \otimes \pi) \circ \delta$, and let a C-invariant element ϕ in Z be given.

Now, if ϕ is orthogonal with respect to $\langle\cdot,\cdot\rangle$ to all C-invariant elements in V, then ϕ is orthogonal to the whole of V.

Proof. By assumption we have that $\delta_C(\phi) = \phi \otimes 1_C$. Define the linear map $T_C: Z \to Z$ by $T_C = (\mathrm{id} \otimes h_C) \circ \delta_C$. Since ϕ is C-invariant, one has $T_C(\phi) = \phi$. Furthermore, if $\psi \in V$ is arbitrary, then $T_C(\psi)$ is C-invariant and contained in V. Now note that the A-invariant inner product $\langle\cdot,\cdot\rangle$ is also C-invariant: $\langle \delta_C(\phi), \delta_C(\psi)\rangle = \langle \phi, \psi\rangle 1_C$ for all $\phi, \psi \in Z$. Consequently, if $\phi \in Z$ is C-invariant and orthogonal to all C-invariant elements in V, and if $\psi \in V$ is arbitrary then

$$\begin{aligned} \langle \phi, \psi\rangle &= h_C(\langle\phi,\psi\rangle 1_C) = h_C(\langle \delta_C(\phi), \delta_C(\psi)\rangle) \\ &= h_C(\langle \phi \otimes 1_C, \delta_C(\psi)\rangle) = \langle \phi, T_C(\psi)\rangle = 0 \end{aligned}$$

since $T_C(\psi)$ is C-invariant. This proves the lemma. □

3. The quantised algebra of polynomials on $\mathbb{C}^n$

In this chapter we introduce the algebra $\mathcal{Z}_n$, which is a q-deformation of the involutive algebra of polynomials on $\mathbb{C}^n$. On this algebra we define a $*$-action of $\mathcal{U}_q(n)$, the quantised universal enveloping algebra of the unitary group $U(n)$, as the ‘differential’ of a certain $*$-coaction of $\mathcal{A}_q(n)$, the quantised algebra of regular functions on $U(n)$.

3.1. DEFINITION AND STRUCTURE OF $\mathcal{Z}_n$

Write $\mathcal{Z}_n$ for the complex $*$-algebra generated by the elements z_i, w_i ($1 \leqslant i \leqslant n$) subject to the relations

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$$
\begin{aligned}
&z_i z_j = q z_j z_i && (1 \leqslant i < j \leqslant n)\\
&w_j w_i = q w_i w_j && (1 \leqslant i < j \leqslant n)\\
&w_i z_j = q z_j w_i && (1 \leqslant i, j \leqslant n, i \neq j)\\
&w_i z_i = z_i w_i + (1-q^2)\textstyle\sum_{k<i} z_k w_k && (1 \leqslant i \leqslant n)
\end{aligned}
\tag{3.1}
$$

and with involution $*: \mathcal{Z}_n \to \mathcal{Z}_n,\ z_i^* = w_i\ (1 \leqslant i \leqslant n)$. For $q = 1$ this algebra can be viewed as the commutative involutive algebra of polynomials in the n coordinates $z_1, \ldots, z_n$ on $\mathbb{C}^n$ and their conjugates. Using [2] one proves

PROPOSITION 3.1. *$\mathcal{Z}_n$ has as a $\mathbb{C}$-linear basis the set $\{z^\lambda w^\mu := z_1^{\lambda_1} \ldots z_n^{\lambda_n} w_n^{\mu_n} \ldots w_1^{\mu_1}\}$ where the multi-indices λ, μ run over $\mathbb{Z}_+^n$.*

Remark 3.2. In the same way one can prove that the set $\{w^\mu z^\lambda := w_1^{\mu_1} \ldots w_n^{\mu_n} z_n^{\lambda_n} \ldots z_1^{\lambda_1}\}$, with $\lambda, \mu \in \mathbb{Z}_+^n$, constitutes a $\mathbb{C}$-basis for $\mathcal{Z}_n$. This also follows from Proposition 3.1 by applying the algebra isomorphism which interchanges z_i and w_i $(1 \leqslant i \leqslant n)$ and sends q to q^{-1}.

LEMMA 3.3. *For $1 \leqslant i, k \leqslant n; m \in \mathbb{Z}_+$ and with $Q_k = \sum_{j=1}^k z_j w_j$ there holds*

$$
\begin{aligned}
&Q_i^* = Q_i, && Q_i Q_k = Q_k Q_i,\\
&z_k w_k = Q_k - Q_{k-1}, && w_k z_k = Q_k - q^2 Q_{k-1},\\
&z_k Q_i = q^{-2} Q_i z_k \quad \textit{and} && w_k Q_i = q^2 Q_i w_k \quad \textit{if } k > i,\\
&z_k Q_i = Q_i z_k \quad \textit{and} && w_k Q_i = Q_i w_k \quad \textit{if } k \leqslant i,\\
&z_k^m w_k^m = Q_k^m \left(\tfrac{Q_{k-1}}{Q_k}; q^{-2}\right)_m, && w_k^m z_k^m = Q_k^m \left(q^2 \tfrac{Q_{k-1}}{Q_k}; q^2\right)_m.
\end{aligned}
$$

Here $(a;q)_m = \prod_{j=0}^{m-1}(1 - aq^j)$ denotes the q-shifted factorial. The proof of this lemma is straightforward from (3.1). As for the centre of $\mathcal{Z}_n$, we have the following result.

PROPOSITION 3.4. *The centre* $\mathrm{Cent}(\mathcal{Z}_n)$ *of $\mathcal{Z}_n$ is equal to the polynomial algebra in the element Q_n:* $\mathrm{Cent}(\mathcal{Z}_n) = \mathbb{C}[Q_n]$.

Proof. It is easy to check that Q_n commutes with all the z_i and w_i, and therefore that $\mathbb{C}[Q_n] \subset \mathrm{Cent}(\mathcal{Z}_n)$. To prove the reverse inclusion, we introduce a total ordering on the basis of $\mathcal{Z}_n$ as follows: with a basis element $z^\lambda w^\mu$ we associate the sequence $\{\lambda, \mu\} = (|\lambda| + |\mu|, \lambda_n, \ldots, \lambda_1, \mu_1, \ldots, \mu_n)$; here $|\lambda| = \lambda_1 + \cdots + \lambda_n$. We declare $z^\lambda w^\mu \succeq z^\rho w^\sigma$ if $\{\lambda, \mu\} \geqslant \{\rho, \sigma\}$ with respect to the lexicographic ordering of elements in $\mathbb{Z}_+^{2n+1}$. Now an induction argument, similar to the one given in [26, Prop. 2.2], finishes the proof. □

Remark 3.5. What we actually constantly use is the following fact. If $\phi \in \mathcal{Z}_n$ is a monomial which contains z_i and w_i exactly λ_i respectively μ_i times $(1 \leqslant i \leqslant n)$, and if $\natural(\phi)$ denotes the highest order term of ϕ with respect to $\succeq$, then ϕ can be written in such a way that $\natural(\phi) = cz^\lambda w^\mu$ with $c \neq 0$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\mu = (\mu_1, \ldots, \mu_n)$.

3.2. A $\mathcal{U}_q(\mathfrak{gl}(n))$-MODULE STRUCTURE ON $\mathcal{Z}_n$

Let us write $\mathcal{A}_q(n) = A_q(U(n))$, with generators $t_{ij} (1 \leqslant i, j \leqslant n), \det_q^{-1}$ and $\mathcal{U}_q(n) = U_q(\mathfrak{gl}(n))$, with generators $q^h (h \in P^* = \sum_{i=1}^n \mathbb{Z}\varepsilon_i), e_k, f_k (1 \leqslant k \leqslant n-1)$, for the quantised coordinate ring of $U(n)$ and the quantised universal enveloping algebra of $\mathfrak{gl}(n)$ respectively. Both algebras are Hopf $*$-algebras. For their definition we refer the reader to [21] and [23]; the structural maps are taken as in [21]. So in particular we define the coproduct Δ on the generators q^h, e_k, f_k of $\mathcal{U}_q(n)$ by

$$\begin{aligned}
&\Delta(q^h) = q^h \otimes q^h, \\
&\Delta(e_k) = q^{\varepsilon_k - \varepsilon_{k+1}} \otimes e_k + e_k \otimes 1, \\
&\Delta(f_k) = 1 \otimes f_k + f_k \otimes q^{-(\varepsilon_k - \varepsilon_{k+1})}.
\end{aligned} \tag{3.2}$$

Furthermore, $\mathcal{A}_q(n)$ and $\mathcal{U}_q(n)$ become Hopf $*$-algebra in duality (see [21, Sect. 1.3]) if we define the pairing on the generators as

$$\langle q^h, t_{ij} \rangle = \delta_{ij} q^{\langle h, \varepsilon_i \rangle}, \quad \langle e_k, t_{ij} \rangle = \delta_{ki}\delta_{k+1,j}, \quad \langle f_k, t_{ij} \rangle = \delta_{k+1,i}\delta_{kj}. \tag{3.3}$$

Next we define $\delta \colon \mathcal{Z}_n \to \mathcal{Z}_n \otimes \mathcal{A}_q(n)$ on the generators of $\mathcal{Z}_n$ by

$$z_i \mapsto \sum_{k=1}^n z_k \otimes t_{ki}, \qquad w_i \mapsto \sum_{k=1}^n w_k \otimes t_{ki}^* \tag{3.4}$$

and extend this map linearly in both factors.

LEMMA 3.6. *The map δ as defined in* (3.4) *extends to a $*$-algebra homomorphism on $\mathcal{Z}_n$ and satisfies* (2.1). *In other words, δ extends to a right $*$-coaction of $\mathcal{A}_q(n)$ on $\mathcal{Z}_n$.*

Proof. Extending δ as a $*$-algebra homomorphism, we only need to verify that it respects the relations (3.1). One can readily check that this is implied by the relations [9, (2.1), (2.13–16)], which are valid in $\mathcal{A}_q(n)$. It is also immediate that δ satisfies (2.1). □

'Differentiating' this right $*$-coaction one obtains a left $*$-action of the quantised universal enveloping algebra $\mathcal{U}_q(n)$ on $\mathcal{Z}_n$;

$$X \cdot \phi = (\mathrm{id} \otimes X) \circ \delta(\phi), \qquad (X \in \mathcal{U}_q(n); \phi \in \mathcal{Z}_n). \tag{3.5}$$

The element X on the right of (3.5) is identified with the linear functional on $\mathcal{A}_q(n)$ which is induced by the pairing between $\mathcal{U}_q(n)$ and $\mathcal{A}_q(n)$. Following [27] we symbolically write $\Delta(X) = \sum_{(X)} X_{(1)} \otimes X_{(2)}$ for $X \in \mathcal{U}_q(n)$.

LEMMA 3.7. (3.5) *defines an algebra action of* $\mathcal{U}_q(n)$ *on* $\mathcal{Z}_n$*, and it satisfies* $X \cdot (\phi\psi) = \sum_{(X)} (X_{(1)} \cdot \phi)(X_{(2)} \cdot \psi)$. *Moreover,* $X \cdot \phi^* = (S(X)^* \cdot \phi)^*$ *for all* $X \in \mathcal{U}_q(n)$ *and all* $\phi, \psi \in \mathcal{Z}_n$.

Note that if $\phi \in \mathcal{Z}_n$ is $\mathcal{A}_q(n)$-invariant this will imply that $X \cdot \phi = \varepsilon(X)\phi$ for all $X \in \mathcal{U}_q(n)$, i.e. ϕ is $\mathcal{U}_q(n)$-invariant.

LEMMA 3.8. *The element* Q_n *is* $\mathcal{A}_q(n)$*-invariant, and hence also* $\mathcal{U}_q(n)$*-invariant.*

Proof. This follows immediately from (3.4), the fact that δ is an algebra homomorphism and the relation $\sum_{k=1}^n t_{ik} t_{jk}^* = \delta_{ij} 1_{\mathcal{A}_q(n)}$ (see [9, (2.12)]). □

The $*$-action of $\mathcal{U}_q(n)$ corresponding to δ is given in the following proposition.

PROPOSITION 3.9. *For* $h \in P^*$, $1 \leqslant k \leqslant n-1$ *and* $\lambda, \mu \in \mathbb{Z}_+^n$ *there holds*

$$\begin{aligned}
q^h \cdot z^\lambda w^\mu &= q^{\langle h, \lambda-\mu\rangle} z^\lambda w^\mu \\
f_k \cdot z^\lambda w^\mu &= -q^{\mu_k+1}[\mu_{k+1}]_{q^{-2}} z^\lambda w^{\mu-\varepsilon_{k+1}+\varepsilon_k} \\
&\quad + q^{\lambda_{k+1}+\mu_k-\mu_{k+1}}[\lambda_k]_{q^{-2}} z^{\lambda-\varepsilon_k+\varepsilon_{k+1}} w^\mu, \\
e_k \cdot z^\lambda w^\mu &= -q^{-1} q^{\mu_{k+1}+\lambda_k-\lambda_{k+1}}[\mu_k]_{q^{-2}} z^\lambda w^{\mu+\varepsilon_{k+1}-\varepsilon_k} \\
&\quad + q^{\lambda_k}[\lambda_{k+1}]_{q^{-2}} z^{\lambda+\varepsilon_k-\varepsilon_{k+1}} w^\mu,
\end{aligned}$$

where $[m]_q = (1-q^m)/(1-q)$ *is the q-number and* $\lambda + \varepsilon_i = (\lambda_1, \ldots, \lambda_i + 1, \ldots, \lambda_n)$.

Proof. Combine (3.5) and Lemma 3.7 with (3.2), and use the relations (3.1). □

Remark 3.10. One can rewrite this result a little when using that for all $m \in \mathbb{Z}_+$ one has the identity $[m]_{q^{-2}} = q^{-2(m-1)}[m]_{q^2}$.

It is actually the left action of $\mathcal{U}_q(n)$ given by Proposition 3.9 that we will consider, rather than the $\mathcal{A}_q(n)$-coaction, since it is easier to handle.

Observe that we have the following decomposition of $\mathcal{Z}_n$;

$$\mathcal{Z}_n = \bigoplus_{l,m \in \mathbb{Z}_+} \mathcal{Z}_n(l,m),$$

where $\mathcal{Z}_n(l,m)$ is the subspace of $\mathcal{Z}_n$ spanned by all elements which are homogeneous of degree l in the z_k and homogeneous of degree m in the w_k. Note that it makes sense to speak of homogeneous elements, since the relations (3.1) are homogeneous. Also note that 0 is homogeneous of any degree (l,m). It follows from Proposition 3.1 that $\mathcal{Z}_n(l,m)$ has a linear basis consisting of all elements $z^\lambda w^\mu$ with the property that $|\lambda| = l$ and $|\mu| = m$; here $|\lambda| = \lambda_1 + \cdots + \lambda_n$. Hence the $\mathcal{Z}_n(l,m)$ $(l, m \in \mathbb{Z}_+)$ are finite dimensional. It is easy to check that they are subcomodules of $\mathcal{Z}_n$ under δ (and hence submodules for the action of $\mathcal{U}_q(n)$). Observe also that $Q_n \mathcal{Z}_n(l-1, m-1) \subset \mathcal{Z}_n(l,m)$ and $\mathcal{Z}_n(l,m)^* = \mathcal{Z}_n(m,l)$.

PROPOSITION 3.11. *Suppose $\phi \in \mathcal{Z}_n(l,m)$ is $\mathcal{U}_q(n)$-invariant. Then;*

(i) *if $l \neq m$, then $\phi = 0$,*
(ii) *if $l = m$, then $\phi = cQ_n^l$ $(c \in \mathbb{C})$.*

Proof. We already know from Lemma 3.8 that Q_n is $\mathcal{U}_q(n)$-invariant. Suppose now that $\phi \in \mathcal{Z}_n(l,m)$ is $\mathcal{U}_q(n)$-invariant, and write $\phi = cz^\lambda w^\mu + \sum_i c_i z^{\lambda^{(i)}} w^{\mu^{(i)}}$ where $c \neq 0$ and for all i there holds $z^\lambda w^\mu \succ z^{\lambda^{(i)}} w^{\mu^{(i)}}$ with respect to the total ordering on monomials in $\mathcal{Z}_n$ (cf. Proposition 3.4 and Remark 3.5). As before let $\natural(\phi)$ be the highest order part of ϕ (so $\natural(\phi) = cz^\lambda w^\mu$). Since $\varepsilon(q^h) = 1$ for all $h \in P^*$, we must have $\natural(q^{\varepsilon_i} \cdot \phi) = \natural(\phi)$. But from Proposition 3.9 we find that $\natural(q^{\varepsilon_i} \cdot \phi) = cq^{\lambda_i - \mu_i} z^\lambda w^\mu$. This implies that $\lambda_i = \mu_i$ for all $1 \leqslant i \leqslant n$, and therefore $l = |\lambda| = |\mu| = m$ if $\phi \neq 0$. This proves part (i). Arguing by contradiction, one proves part (ii) in a similar way (see [6, Prop. 3.15]). □

LEMMA 3.12. *One has $w_i^m z_i = q^{2m} z_i w_i^m + (1 - q^{2m}) w_i^{m-1} Q_i$ for all $1 \leqslant i \leqslant n$ and each $m \in \mathbb{Z}_+$.*

Proof. Observe that $w_i z_i = q^2 z_i w_i + (1 - q^2) Q_i$. Now proceed by induction with respect to m. □

COROLLARY 3.13. *For all $1 \leqslant i \leqslant n$ and each $m \in \mathbb{Z}_+$ one has the identity $(z_i w_i)^m = \sum_{k=0}^m c_k z_i^k w_i^k Q_i^{m-k}$ for certain coefficients $c_k \in \mathbb{Z}[q^2]$.*

Suppose we are given the two algebras $\mathcal{Z}_n$ and $\mathcal{Z}_s$ with generators $z_i,\ w_i$ $(1 \leqslant i \leqslant n)$ and $z_i{}',\ w_i{}'$ $(1 \leqslant i \leqslant s)$ respectively, and assume $s < n$. Then we have the canonical embedding $\iota^{(s,n)}\colon \mathcal{Z}_s \hookrightarrow \mathcal{Z}_n$ which sends the generators $z_i{}', w_i{}'$ of $\mathcal{Z}_s$ to the first s pairs of generators $z_i,\ w_i$ $(1 \leqslant i \leqslant s)$ of $\mathcal{Z}_n$. We also have a restriction map $\rho^{(n,s)}\colon \mathcal{Z}_n \to \mathcal{Z}_s$ which puts z_i and w_i equal to zero for $i = 1, \ldots, n-s$ and maps z_i and w_i to z'_{i-n+s} and w'_{i-n+s} respectively for $i = n-s+1, \ldots, n$. Both maps are $*$-algebra homomorphisms. So in particular we can view $\mathcal{Z}_{n-1}(l,m)$ as sitting in $\mathcal{Z}_n(l,m)$, by means of $\iota^{(n-1,n)}$.

Furthermore, observe that for $1 \leqslant p \leqslant n-1$ we have a natural embedding $\mathcal{U}_q(n-p) \hookrightarrow \mathcal{U}_q(n)$ by identifying $\mathcal{U}_q(n-p)$ with the subalgebra of $\mathcal{U}_q(n)$ generated by the elements q^{ε_i} $(1 \leqslant i \leqslant n-p), e_k, f_k$ $(1 \leqslant k \leqslant n-p-1)$. In this way it is possible to speak of $\mathcal{U}_q(n-p)$-invariant elements in $\mathcal{Z}_n$.

PROPOSITION 3.14. *Suppose $\phi \in \mathcal{Z}_n(l,m)$ is $\mathcal{U}_q(n-1)$-invariant. Then ϕ is of the form $\phi = \sum_{j=0}^{l\wedge m} c_j z_n^{l-j} w_n^{m-j} Q_n^j$, where $l \wedge m = \min(l,m)$. Conversely, any ϕ of this form is $\mathcal{U}_q(n-1)$-invariant. Hence the dimension of $\mathcal{U}_q(n-1)$-invariant elements in $\mathcal{Z}_n(l,m)$ equals $l \wedge m + 1$.*

Proof. Using Proposition 3.1 and the commutation relations for the z_i, w_i, we see that ϕ can be written uniquely as $\phi = \sum_{i=0}^l \sum_{j=0}^m z_n^{l-i} w_n^{m-j} p_{ij}(z_1, \ldots, z_{n-1}, w_1, \ldots, w_{n-1})$ for certain $p_{ij} \in \mathcal{Z}_{n-1}(i,j)$. Then the action of $\mathcal{U}_q(n-1)$ is on the elements p_{ij} (cf. Proposition 3.9). From Proposition 3.11 we obtain that

$p_{ij} = \delta_{ij} d_j Q_{n-1}^j$, and thus $\phi = \sum_{j=0}^{l\wedge m} d_j z_n^{l-j} w_n^{m-j} Q_{n-1}^j$ which already yields the stated dimension. Since Q_n is central we can write $Q_{n-1}^j = (Q_n - z_n w_n)^j = \sum_{k=0}^{j} (-1)^k a_{k,j} (z_n w_n)^k Q_n^{j-k}$, where the $a_{k,j}$ are ordinary binomial coefficients. If we substitute this in ϕ we obtain

$$\phi = \sum_{j=0}^{l\wedge m} \sum_{k=0}^{j} d_j a_{k,j} z_n^{l-j} w_n^{m-j} (z_n w_n)^k Q_n^{j-k}.$$

Now, after applying Corollary 3.13 and changing the summations we obtain that ϕ is of the asserted form. The converse statement in the proposition is obvious. □

Similarly one proves

PROPOSITION 3.15. *Suppose* $\phi \in \mathcal{Z}_n(l, m)$. *Then* ϕ *is* $\mathcal{U}_q(n-2)$*-invariant if and only if* ϕ *is of the form* $\phi = \sum_{j=0}^{l\wedge m} \sum_{r=0}^{l-j} \sum_{s=0}^{m-j} a_{j,r,s} z_n^{l-j-r} w_n^{m-j-s} z_{n-1}^r w_{n-1}^s Q_n^j$.

4. The quantised algebra of polynomials on the sphere in $\mathbb{C}^n$

In this last chapter we construct the quantised algebra of polynomials on the sphere S^{2n-1} from the algebra $\mathcal{Z}_n$ by putting the invariant central element Q_n equal to 1. Furthermore we construct an invariant functional on this algebra, we give its irreducible decomposition into $\mathcal{U}_q(n)$-modules and we recover the $\mathcal{U}_q(n-1)$-invariant elements, the so-called zonal spherical functions, as q-disk polynomials. Finally we prove an addition theorem for these q-disk polynomials.

4.1. Definition of $\widetilde{\mathcal{Z}}_n$ and invariant functional

We know (Proposition 3.4) that Q_n is a central element of $\mathcal{Z}_n$. So it makes sense to consider the following projection map

$$\pi\colon \mathcal{Z}_n \to \mathcal{Z}_n/(Q_n - 1) =: \widetilde{\mathcal{Z}}_n. \tag{4.1}$$

We denote the images of the generators z_i, w_i of $\mathcal{Z}_n$ under π by the same symbols and we define the map δ on those images as in (3.4). This gives a well-defined $*$-coaction of $\mathcal{A}_q(n)$ on $\widetilde{\mathcal{Z}}_n$, since Q_n is a trivial element for the $\mathcal{A}_q(n)$-coaction (Lemma 3.8). In other words, δ factors through the projection π. The algebra $\widetilde{\mathcal{Z}}_n$ plays the role of quantised polynomial algebra on the $(2n-1)$-sphere S^{2n-1} within $\mathbb{C}^n$, and was introduced in [24]. It is the same as the algebra $A(K\backslash G)$ of [23, Sect. 4.1].

We recall that a linear functional $h\colon \widetilde{\mathcal{Z}}_n \to \mathbb{C}$ is said to be $\mathcal{A}_q(n)$-invariant if the identity $(h \otimes \mathrm{id}) \circ \delta(\phi) = h(\phi) 1_{\mathcal{A}_q(n)}$ holds for all $\phi \in \widetilde{\mathcal{Z}}_n$. This will imply (see (3.5)) that $h(X \cdot \phi) = \varepsilon(X) h(\phi)$ for $\phi \in \widetilde{\mathcal{Z}}_n$ and $X \in \mathcal{U}_q(n)$. In other words, h will be $\mathcal{U}_q(n)$-invariant. The functional is called positive definite if $h(\phi^*\phi) > 0$ whenever $\phi \neq 0$.

PROPOSITION 4.1. *On $\widetilde{\mathcal{Z}}_n$ there exists a unique normalised, positive definite, $\mathcal{U}_q(n)$-invariant functional $h_n\colon \widetilde{\mathcal{Z}}_n \to \mathbb{C}$. It is given on the basis elements by*

$$h_n(z^\lambda w^\mu) = \delta_{\lambda\mu} q^{-2((n-1)\lambda_1+(n-2)\lambda_2+\cdots+\lambda_{n-1})} \times \frac{(q^{-2};q^{-2})_{\lambda_1}\dots(q^{-2};q^{-2})_{\lambda_n}(q^{-2};q^{-2})_{n-1}}{(q^{-2};q^{-2})_{|\lambda|+n-1}}, \tag{4.2}$$

where $(a;q)_m$ is the q-shifted factorial (*Lemma* 3.3) *and $|\lambda| = \lambda_1 + \cdots + \lambda_n$.*

The proof of this proposition is analogous to [23, Prop. 4.5], and uses Proposition 3.9.

Remark 4.2. One can write (4.2) equivalently as

$$h_n(z^\lambda w^\mu) = \delta_{\lambda\mu} q^{|\lambda|^2+\sum_{i=1}^n (2(i-1)\lambda_i-\lambda_i^2)} \times \frac{(q^2;q^2)_{\lambda_1}\dots(q^2;q^2)_{\lambda_n}(q^2;q^2)_{n-1}}{(q^2;q^2)_{|\lambda|+n-1}}, \tag{4.3}$$

when using that $(a^{-1};q^{-1})_m = (-1)^m a^{-m} q^{-\frac{1}{2}m(m-1)}(a;q)_m$.

Recall the q-*integral* for functions on $[0,c]$ (see [8]);

$$\int_0^c f(x)d_qx = c(1-q)\sum_{k=0}^\infty f(cq^k)q^k.$$

It satisfies

$$\int_0^c f\left(\frac{x}{c}\right) d_qx = c\int_0^1 f(x)d_qx. \tag{4.4}$$

The next lemma now follows from a direct calculation.

LEMMA 4.3. *For all $\alpha, \beta \in \mathbb{Z}_+$ and any continuous function f there holds*

$$\int_0^1 f(q^{-\beta}x)x^\alpha(x;q^{-1})_\beta d_qx = q^{\beta(\alpha+1)}\int_0^1 f(x)x^\alpha(xq;q)_\beta d_qx.$$

Consequently we find that $\int_0^1 x^\alpha(x;q^{-1})_\beta d_qx = q^{\beta(\alpha+1)}(1-q)(q;q)_\alpha(q;q)_\beta/(q;q)_{\alpha+\beta+1}$ since the integral on the right-hand side in Lemma 4.3 is the q-beta integral in case $f=1$ (see [8, (1.11.7)]);

$$\int_0^1 x^\alpha(xq;q)_\beta d_qx = (1-q)\frac{(q;q)_\alpha(q;q)_\beta}{(q;q)_{\alpha+\beta+1}}.$$

The following result was already found in [23, Thm. 4.6].

PROPOSITION 4.4. *In* $\widetilde{\mathcal{Z}}_n$ *we have that* $\operatorname{span}\{z^\lambda w^\lambda | \lambda \in \mathbb{Z}_+\} = \mathbb{C}[Q_1, \ldots, Q_{n-1}]$, *and for each* $\phi = \phi(Q_1, \ldots, Q_{n-1}) \in \mathbb{C}[Q_1, \ldots, Q_{n-1}]$ *the value of the invariant functional is given by the following multiple* q*-integral;*

$$\begin{aligned} h_n(\phi) &= \frac{(q^2;q^2)_{n-1}}{(1-q^2)^{n-1}} \times \int_0^1 \int_0^{Q_{n-1}} \cdots \int_0^{Q_2} \phi(Q_1, \ldots, Q_{n-1}) \\ &\quad \times d_{q^2}Q_1 \ldots d_{q^2}Q_{n-2} d_{q^2}Q_{n-1}. \end{aligned} \tag{4.5}$$

Proof. The first statement follows from Lemma 3.3. Hence we only have to verify that (4.5) is true for any monomial $z^\lambda w^\lambda$. From Lemma 3.3 we obtain the identity $z^\lambda w^\lambda = Q_1^{\lambda_1} Q_2^{\lambda_2} \ldots Q_{n-1}^{\lambda_{n-1}} (Q_1/Q_2; q^{-2})_{\lambda_2} (Q_2/Q_3; q^{-2})_{\lambda_3} \ldots (Q_{n-1}; q^{-2})_{\lambda_n}$. Now substitute this into the right-hand side of (4.5) and use (4.4). Then, by successive use of Lemma 4.3, one checks that for $\phi = z^\lambda w^\lambda$ (4.5) agrees with (4.3). □

PROPOSITION 4.5. *The* $*$*-algebra homomorphism* $\Psi\colon \widetilde{\mathcal{Z}}_n \to \mathcal{A}_q(n),\ z_k \mapsto t_{nk}$ *is well-defined, intertwines* δ *and* Δ *and is injective. So we can apply the results of Section 2.2 with* $A = \mathcal{A}_q(n)$ *and* $Z = \widetilde{\mathcal{Z}}_n$.

Proof. It is straightforward to verify that Ψ is well-defined and intertwines δ and Δ. To show injectivity we first view Ψ as a map from $\mathcal{Z}_n$ to $A(\mathrm{Mat}_q(n)) \otimes \mathbb{C}[\det_q^{-1}]$ (we use the notation of [23, Sect. 1.1.]), where we assume $\det_q^{-1}$ to be a central element but we do not assume the identities $\det_q \det_q^{-1} = 1 = \det_q^{-1} \det_q$ to hold. Suppose ϕ is a monomial in the elements t_{ij} $(1 \leqslant i,j \leqslant n)$ which contains a_{ij} factors t_{ij} $(1 \leqslant i,j \leqslant n)$. We can arrange things in such a way that ϕ has leading term $t^A = t_{11}^{a_{11}} \ldots t_{1n}^{a_{1n}} t_{21}^{a_{21}} \ldots t_{2n}^{a_{2n}} \ldots t_{nn}^{a_{nn}}$ (so we use the following total ordering on the generators t_{ij}: $t_{ij} \prec t_{kl}$ if $i < k$, or if $i = k$ and $j < l$). From [9, Thm. 3.1] we know that monomials in the t_{ij} corresponding to different matrices $A = (a_{ij})$ are linearly independent in $A(\mathrm{Mat}_q(n))$. Hence, since we know a linear basis for $\mathcal{Z}_n$, we must show that the matrix $A(\lambda, \mu)$ corresponding to $\Psi(z^\lambda w^\mu)$ is different for different choices of the pair (λ, μ). This is straightforwardly checked (since it suffices to look at the highest order term of $\Psi(z^\lambda w^\mu)$). Finally, using the identity $\sum_{k=1}^n t_{nk}(-q)^{k-n} D_{\widehat{n}\widehat{k}} = \det_q$ in $A(\mathrm{Mat}_q(n))$ ([9, (2.10)], [23, (1.15.b)]), we find that $\Psi(Q_n - 1) = \det_q \det_q^{-1} - 1$. This shows that Ψ extends to an injective homomorphism from $\widetilde{\mathcal{Z}}_n$ to $\mathcal{A}_q(n)$. □

Remark 4.6. From this it follows that the algebra $A(K\backslash G)$ of [23, Sect. 4.1] has no relations additional to the relations (4.9.a–d) (loc. cit.). This was already observed in [28, Thm. 4.4].

4.2. IRREDUCIBLE DECOMPOSITION

With the invariant functional h_n of Proposition 4.1 one can define an invariant inner product on $\widetilde{\mathcal{Z}}_n$ as follows

$$\begin{aligned} &\langle \cdot, \cdot \rangle : \widetilde{\mathcal{Z}}_n \times \widetilde{\mathcal{Z}}_n \to \mathbb{C} \\ &\langle \phi, \psi \rangle := h_n(\psi^* \phi). \end{aligned} \tag{4.6}$$

This non-degenerate bilinear form satisfies $\langle X \cdot \phi, \psi \rangle = \langle \phi, X^* \cdot \psi \rangle$ for all $X \in \mathcal{U}_q(n)$ and all $\phi, \psi \in \widetilde{\mathcal{Z}}_n$. So in particular $\langle q^h \cdot \phi, \psi \rangle = \langle \phi, q^h \cdot \psi \rangle$ for all $h \in P^*$ and all $\phi, \psi \in \widetilde{\mathcal{Z}}_n$. Put $\widetilde{\mathcal{Z}}_n(l,m) := \pi(\mathcal{Z}_n(l,m))$, with π the projection (4.1), and let $\widetilde{\mathcal{H}}_n(l,m)$ be the orthogonal complement of $\widetilde{\mathcal{Z}}_n(l-1,m-1)$ in $\widetilde{\mathcal{Z}}_n(l,m)$ with respect to the inner product (4.6) (recall that we have $Q_n\mathcal{Z}_n(l-1,m-1) \subset \mathcal{Z}_n(l,m)$, whence $\widetilde{\mathcal{Z}}_n(l-1,m-1) \subset \widetilde{\mathcal{Z}}_n(l,m)$). So there is the orthogonal direct sum decomposition

$$\widetilde{\mathcal{Z}}_n(l,m) = \widetilde{\mathcal{Z}}_n(l-1,m-1) \oplus \widetilde{\mathcal{H}}_n(l,m).$$

LEMMA 4.7. *The projection map* $\pi\colon \mathcal{Z}_n(l,m) \to \widetilde{\mathcal{Z}}_n(l,m)$ *is injective.*

Proof. This is a consequence of Proposition 3.1 and the fact that the element $Q_n - 1$ is not homogeneous. □

From this lemma it follows that $\pi : \mathcal{Z}_n(l,m) \to \widetilde{\mathcal{Z}}_n(l,m)$ is an isomorphism. Hence we obtain from Proposition 3.1 that

$$\dim \widetilde{\mathcal{Z}}_n(l,m) = \dim \mathcal{Z}_n(l,m) = \binom{l+n-1}{n-1}\binom{m+n-1}{n-1} \tag{4.7}$$

and

$$\begin{aligned} d_n(l,m) &:= \dim \widetilde{\mathcal{H}}_n(l,m) = \dim \widetilde{\mathcal{Z}}_n(l,m) - \dim \widetilde{\mathcal{Z}}_n(l-1,m-1) \\ &= \frac{(l+m+n-1)(l+n-2)!(m+n-2)!}{l!m!(n-1)!(n-2)!}. \end{aligned} \tag{4.8}$$

Moreover, we have the decomposition

$$\widetilde{\mathcal{Z}}_n = \sum_{l,m \in \mathbb{Z}_+} \widetilde{\mathcal{Z}}_n(l,m). \tag{4.9}$$

If we write $\mathcal{H}_n(l,m)$ for the inverse image of $\widetilde{\mathcal{H}}_n(l,m)$ under the projection $\pi\colon \mathcal{Z}_n(l,m) \to \widetilde{\mathcal{Z}}_n(l,m)$, then $\mathcal{Z}_n(l,m) = Q_n\mathcal{Z}_n(l-1,m-1) \oplus \mathcal{H}_n(l,m)$ as a direct sum.

PROPOSITION 4.8. *There exists the following orthogonal decomposition into inequivalent irreducible* $\mathcal{U}_q(n)$*-modules;*

$$\widetilde{\mathcal{Z}}_n(l,m) = \bigoplus_{k=0}^{l\wedge m} \widetilde{\mathcal{H}}_n(l-k,m-k). \tag{4.10}$$

Here $l \wedge m = \min(l,m)$.

Proof. It is clear, by the definition of the spaces $\widetilde{\mathcal{H}}_n(r,s)$, that $\widetilde{\mathcal{Z}}_n(l,m)$ allows the orthogonal direct sum decomposition (4.10). Irreducibility of $\widetilde{\mathcal{H}}_n(l-k,m-k)$ follows from Proposition 2.3: each nontrivial $\mathcal{U}_q(n)$-invariant subspace of a given $\widetilde{\mathcal{H}}_n(l-k,m-k)$ should contain at least one $\mathcal{U}_q(n-1)$-invariant element. But there are, according to Proposition 3.14, only $l \wedge m + 1$ linearly independent invariant elements in the space $\widetilde{\mathcal{Z}}_n(l,m)$. Hence none of the spaces $\widetilde{\mathcal{H}}_n(l-k,m-k)$ contains a nontrivial invariant subspace.

To prove inequivalence, assume that $\widetilde{\mathcal{H}}_n(l-k,m-k) \simeq \widetilde{\mathcal{H}}_n(l-k',m-k')$ for $k \neq k'$. So in particular $d_n(l-k,m-k) = d_n(l-k',m-k') =: N$. Take orthonormal bases $\{e_i\}_{i=1}^N$ and $\{f_j\}_{j=1}^N$ in the respective spaces and construct the elements $\zeta_0 = \sum_{i=1}^N \varepsilon(e_i^*)e_i$ and $\zeta_0{}' = \sum_{j=1}^N \varepsilon(f_j^*)f_j$ as in the proof of Proposition 2.3. By linear independence it follows that not all of the $\varepsilon(e_i^*)$ and $\varepsilon(f_j^*)$ can be zero. Furthermore, put $\eta = \sum_{k=1}^N e_k \otimes f_k$. Again we will have that $(\Delta \boxtimes \Delta)(\eta) = \eta \otimes 1$ and $\Delta(\widetilde{\eta}) = \widetilde{\eta} \otimes 1$ where $\widetilde{\eta} = \sum_{k=1}^N e_k f_k$ (cf. the proof of Proposition 2.3). From this and the orthogonality of the spaces $\widetilde{\mathcal{H}}_n(l-k,m-k)$ and $\widetilde{\mathcal{H}}_n(l-k',m-k')$, we obtain $\varepsilon(\widetilde{\eta}) = h_n(\widetilde{\eta}) = 0$. Finally, put $\eta_0 = \sum_{k=1}^N \varepsilon(f_k^*)e_k \in \widetilde{\mathcal{H}}_n(l-k,m-k)$. Then $\eta_0 \neq 0$, since not all of the $\varepsilon(f_k^*)$ are zero and the e_i are linearly independent. Moreover $\varepsilon(\eta_0) = \varepsilon(\widetilde{\eta}) = 0$. But this means that we have two linearly independent invariant elements within $\widetilde{\mathcal{H}}_n(l-k,m-k)$, namely ζ_0 and η_0 (linearly independent since $\varepsilon(\zeta_0) > 0$ and $\varepsilon(\eta_0) = 0$). This gives a contradiction. Hence the two spaces $\widetilde{\mathcal{H}}_n(l-k,m-k)$ and $\widetilde{\mathcal{H}}_n(l-k',m-k')$ cannot be equivalent. □

Remark 4.9. Using exactly the same argument as in the proof of the previous proposition, one shows that the modules $\widetilde{\mathcal{H}}_n(l,m)$ are inequivalent for different choices of the pair (l,m). From the proof of Proposition 4.8 we immediately obtain that each $\widetilde{\mathcal{H}}_n(l,m)$ contains a unique, up to constants, $\mathcal{U}_q(n-1)$-invariant element. It is called a *(zonal) spherical function*, or spherical element.

LEMMA 4.10. *If* $(l,m) \neq (l',m')$, *then* $\widetilde{\mathcal{H}}_n(l,m) \perp \widetilde{\mathcal{H}}_n(l',m')$.

Proof. Suppose first that $l'-l = m'-m$, and say this is non-negative. This means that $l' = l+k, m' = m+k$ for some $k \geqslant 0$. But then $\widetilde{\mathcal{H}}_n(l,m)$ and $\widetilde{\mathcal{H}}_n(l',m')$ are contained in the same space $\widetilde{\mathcal{Z}}_n(l',m')$, and hence are orthogonal by Proposition 4.8. If, on the other hand, $l'-l \neq m'-m$, choose $\phi \in \widetilde{\mathcal{H}}_n(l,m)$ and $\psi \in \widetilde{\mathcal{H}}_n(l',m')$. By invariance of the inner product (4.6) we have $q^{l-m}\langle\phi,\psi\rangle = \langle q^{\varepsilon_1+\cdots+\varepsilon_n}\cdot\phi,\psi\rangle = \langle\phi, q^{\varepsilon_1+\cdots+\varepsilon_n}\cdot\psi\rangle = q^{l'-m'}\langle\phi,\psi\rangle$. Since

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we assumed that $l'-l \neq m'-m$, and since q is not a root of unity, this proves that $\langle \phi, \psi \rangle = 0$. □

COROLLARY 4.11. *There is the orthogonal, irreducible decomposition into inequivalent $\mathcal{U}_q(n)$-modules*

$$\widetilde{\mathcal{Z}}_n = \bigoplus_{l,m\in\mathbb{Z}_+} \widetilde{\mathcal{H}}_n(l,m).$$

Proof. This now follows from the decomposition (4.9) together with Proposition 4.8, Lemma 4.10 and Remark 4.9. □

4.3. Zonal spherical functions

Let us write $\psi(l,m)$ for a spherical element contained in $\widetilde{\mathcal{H}}_n(l,m)$, which is unique up to constants (see Remark 4.9). Now suppose that $(l,m) \neq (l',m')$, and assume that $l-m = l'-m' = \beta \geqslant 0$. From Proposition 3.14 and Lemma 3.3 we know that $\psi(l,m) = z_n^{l-m} \sum_{j=0}^{m} c_j z_n^{m-j} w_n^{m-j} = z_n^{\beta} p_m(Q_{n-1})$ and $\psi(l',m') = z_n^{l'-m'} \sum_{j=0}^{m'} c_j z_n^{m'-j} w_n^{m'-j} = z_n^{\beta} p_{m'}(Q_{n-1})$ for certain polynomials $p_m, p_{m'}$ of degree m and m' respectively. As a consequence of the orthogonality of the spaces $\widetilde{\mathcal{H}}_n(l,m)$ and $\widetilde{\mathcal{H}}_n(l',m')$ we find

$$\begin{aligned} 0 &= \langle \psi(l',m'), \psi(l,m) \rangle = h_n(p_m(Q_{n-1})^* w_n^{\beta} z_n^{\beta} p_{m'}(Q_{n-1})) \\ &= h_n(p_m(Q_{n-1})^* p_{m'}(Q_{n-1})(q^2Q_{n-1};q^2)_{\beta}) \end{aligned}$$

(cf. Lemma 3.3). Observe that

$$\int_0^{Q_{n-1}} \cdots \int_0^{Q_2} d_{q^2}Q_1 \ldots d_{q^2}Q_{n-2} = \frac{(1-q^2)^{n-2}}{(q^2;q^2)_{n-2}} Q_{n-1}^{n-2}. \tag{4.11}$$

So we obtain from Proposition 4.4

$$0 = \int_0^1 \overline{p_m(Q_{n-1})} p_{m'}(Q_{n-1}) Q_{n-1}^{n-2} (q^2Q_{n-1};q^2)_{\beta} d_{q^2}Q_{n-1} \quad (m \neq m').$$

But letting m and m' run over $\mathbb{Z}_+$, these are exactly the orthogonality relations [8, (7.3.3)] for the little q-Jacobi polynomials $p_m^{(n-2,\beta)}(Q_{n-1};q^2)$. In other words, there exist constants $c_m \in \mathbb{C}$ such that $p_m(Q_{n-1}) = c_m p_m^{(n-2,\beta)}(Q_{n-1};q^2)$ $(m \in \mathbb{Z}_+)$. Thus we obtain that in case $l \geqslant m$ a general spherical element in $\widetilde{\mathcal{H}}_n(l,m)$ is given by a constant multiple of $z_n^{l-m} p_m^{(n-2,l-m)}(Q_{n-1};q^2)$. Similar calculations are made in case $m-l \geqslant 0$. One finds, using (4.11), Lemma 3.3 and Lemma 4.3, that the spherical elements of $\widetilde{\mathcal{H}}_n(l,m)$ for $l \leqslant m$ are constant multiples of $p_l^{(n-2,m-l)}(Q_{n-1};q^2) w_n^{m-l}$. Summarising we have

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THEOREM 4.12. *For arbitrary $l, m \in \mathbb{Z}_+$ the $\mathcal{U}_q(n-1)$-invariant elements (i.e. the zonal spherical elements) in $\widetilde{\mathcal{H}}_n(l,m)$ are constant multiples of the q-disk polynomials*

$$R_{l,m}^{(n-2)}(z_n, w_n; q^2) = \begin{cases} z_n^{l-m} p_m^{(n-2,l-m)}(Q_{n-1}; q^2) & (l \geqslant m) \\ p_l^{(n-2,m-l)}(Q_{n-1}; q^2) w_n^{m-l} & (l \leqslant m) \end{cases},$$

where $p_k^{(\alpha,\beta)}(x;q) = p_k(x; q^\alpha, q^\beta; q)$ is the little q-Jacobi polynomial [8, (7.3.1)].

This result was already obtained in [23, Thm. 4.7].

Finally we calculate the norms of these spherical elements, since we will need them later on. First let us assume that $l - m = \beta \geqslant 0$. Recall that

$$\begin{aligned} &\int_0^1 R_{l,m}^{(n-2)}(z_n, w_n; q^2)^* R_{l,m}^{(n-2)}(z_n, w_n; q^2) Q_{n-1}^{n-2}\, d_{q^2} Q_{n-1} \\ &= \int_0^1 p_m^{(n-2,\beta)}(Q_{n-1}; q^2) p_m^{(n-2,\beta)}(Q_{n-1}; q^2) Q_{n-1}^{n-2} (q^2 Q_{n-1}; q^2)_\beta d_{q^2} Q_{n-1} \\ &= \frac{(1-q^2) q^{2m(\alpha+1)}}{1 - q^{2(\alpha+\beta+2m+1)}} \frac{(q^2;q^2)_m (q^2;q^2)_{\beta+m}}{(q^{2(\alpha+1)};q^2)_m (q^{2(\alpha+1)};q^2)_{\beta+m}}. \end{aligned}$$

Thus, for $l \geqslant m$,

$$\begin{aligned} &\langle R_{l,m}^{(n-2)}(z_n, w_n; q^2), R_{l,m}^{(n-2)}(z_n, w_n; q^2) \rangle \\ &= \frac{(1-q^{2(n-1)}) q^{2m(n-1)}}{1 - q^{2(n+l+m-1)}} \frac{(q^2;q^2)_l (q^2;q^2)_m}{(q^{2(n-1)};q^2)_l (q^{2(n-1)};q^2)_m}. \end{aligned}$$

The case $m - l = \beta \geqslant 0$ is treated similarly and gives the same answer. This proves the following proposition.

PROPOSITION 4.13. *For $l, m \in \mathbb{Z}_+$ and $\alpha = n-2 \in \mathbb{Z}_+$, the square of the norm of the q-disk polynomial $R_{l,m}^{(\alpha)}(z_n, w_n; q^2)$ is given by*

$$\|R_{l,m}^{(\alpha)}(z_n, w_n; q^2)\|^2 = h_n(R_{l,m}^{(\alpha)}(z_n, w_n; q^2)^* R_{l,m}^{(\alpha)}(z_n, w_n; q^2)) = c_{l,m}^{(\alpha)}$$

in which

$$c_{l,m}^{(\alpha)} = \frac{(1-q^{2(\alpha+1)}) q^{2m(\alpha+1)}}{1 - q^{2(\alpha+l+m+1)}} \frac{(q^2;q^2)_l (q^2;q^2)_m}{(q^{2(\alpha+1)};q^2)_l (q^{2(\alpha+1)};q^2)_m}. \tag{4.12}$$

Note that, unlike in the classical case, $c_{l,m}^{(\alpha)}$ is not symmetric in l and m.

4.4. Associated Spherical Functions

Suppose we are given the $*$-algebra $\mathcal{Z}_{n-1}$ with generators z_i', w_i' $(1 \leqslant i \leqslant n-1)$ and with corresponding projection $\pi' : \mathcal{Z}_{n-1} \to \widetilde{\mathcal{Z}}_{n-1}$. Recall that for $r, s \in \mathbb{Z}_+$

we have the embedding $\iota^{(n-1,n)} : \mathcal{Z}_{n-1}(r,s) \hookrightarrow \mathcal{Z}_n(r,s)$ (Sect. 3.2). Then, by use of the map $\pi \circ \iota^{(n-1,n)} \circ (\pi')^{-1}$, one can identify $\phi = \phi(z', w') \in \widetilde{\mathcal{Z}}_{n-1}(r,s)$ with $Q_{n-1}^{(r+s)/2}\phi(zQ_{n-1}^{-(1/2)}, wQ_{n-1}^{-(1/2)}) \in \widetilde{\mathcal{Z}}_n(r,s)$.

Given $l, m \in \mathbb{Z}_+$ and $0 \leqslant r \leqslant l; 0 \leqslant s \leqslant m$, define the following elements in $\mathcal{Z}_n(l,m)$;

$$\begin{aligned}\psi(l,m;r,s) \;&=\; Q_n^{(l-r+m-s)/2} R_{l-r,m-s}^{(n-2+r+s)}(z_n Q_n^{-(1/2)}, w_n Q_n^{-(1/2)}; q^2) \\ &\quad \times Q_{n-1}^{(r+s)/2} R_{r,s}^{(n-3)}(z_{n-1}Q_{n-1}^{-(1/2)}, w_{n-1}Q_{n-1}^{-(1/2)}; q^2).\end{aligned}$$

Denote their restrictions in $\widetilde{\mathcal{Z}}_n(l,m)$ via π by the same symbols.

PROPOSITION 4.14. *In $\widetilde{\mathcal{Z}}_n$ there holds $\langle \psi(l',m';r',s'), \psi(l,m;r,s)\rangle = 0$ whenever one has $(l',m',r',s') \neq (l,m,r,s)$. Moreover,*

$$\|\psi(l,m;r,s)\|^2 = \frac{1-q^{2(\alpha+1)}}{1-q^{2(\alpha+r+s+1)}} c_{l-r,m-s}^{(\alpha+r+s)} c_{r,s}^{(\alpha-1)}$$

where $\alpha = n-2$.

Proof. This is done by direct calculation. □

PROPOSITION 4.15. *For fixed $l, m \in \mathbb{Z}_+$, $\psi(l,m;r,s) \in \widetilde{\mathcal{H}}_n(l,m)$ for all $0 \leqslant r \leqslant l$ and all $0 \leqslant s \leqslant m$. Moreover, an element $F \in \widetilde{\mathcal{H}}_n(l,m)$ is $\mathcal{U}_q(n-2)$-invariant if and only if $F \in$ span$\{\psi(l,m;r,s) | 0 \leqslant r \leqslant l, 0 \leqslant s \leqslant m\}$.*

Proof. For $0 \leqslant j \leqslant l \wedge m; 0 \leqslant r \leqslant l-j; 0 \leqslant s \leqslant m-j$ consider the elements $Q_n^j \psi(l-j, m-j; r, s)$ in $\mathcal{Z}_n(l,m)$. Their restrictions to $\widetilde{\mathcal{Z}}_n$ are mutually orthogonal by the previous proposition, hence they are linearly independent in $\mathcal{Z}_n(l,m)$. Since they are all $\mathcal{U}_q(n-2)$-invariant they will span the entire space of $\mathcal{U}_q(n-2)$-invariant elements within $\mathcal{Z}_n(l,m)$ because of their number, cf. Proposition 3.15. In the same way all elements $Q_n^j \psi(l-j, m-j; r, s)$ with $1 \leqslant j \leqslant l \wedge m$ and $0 \leqslant r \leqslant l-j; 0 \leqslant s \leqslant m-j$ will span the subspace of $\mathcal{U}_q(n-2)$-invariant elements in $Q_n \mathcal{Z}_n(l-1, m-1)$. Now, using the orthogonality of the $\psi(l,m;r,s)$ together with Lemma 2.6, we conclude that the $\psi(l,m;r,s)$ with $0 \leqslant r \leqslant l, 0 \leqslant s \leqslant m$ are orthogonal to the whole of $\widetilde{\mathcal{Z}}_n(l-1, m-1)$, hence belong to $\widetilde{\mathcal{H}}_n(l,m)$. Because of their number the second part of the proposition is also clear. □

For given $l, m \in \mathbb{Z}_+$ and $0 \leqslant r \leqslant l$, $0 \leqslant s \leqslant m$ put

$$\begin{aligned}\mathcal{H}_n(l,m;r,s) \;&=\; Q_n^{(l-r-m+s)/2} R_{l-r,m-s}^{(n-2+r+s)} \times (z_n Q_n^{-(1/2)}, w_n Q_n^{-(1/2)}; q^2) \\ &\quad \times \mathcal{H}_{n-1}(r,s).\end{aligned}$$

Then clearly $\mathcal{H}_n(l,m;r,s) \subset \mathcal{Z}_n(l,m)$ $(0 \leqslant r \leqslant l; 0 \leqslant s \leqslant m)$. There even holds the following result.

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LEMMA 4.16. *For all* $0 \leqslant r \leqslant l$ *and all* $0 \leqslant s \leqslant m$ *we have the inclusion*

$$\mathcal{H}_n(l,m;r,s) \subset \mathcal{H}_n(l,m).$$

Proof. As $\mathcal{U}_q(n-1)$-modules there is the isomorphism $\mathcal{H}_n(l,m;r,s) \cong \mathcal{H}_{n-1}(r,s)$. Since $\mathcal{H}_{n-1}(r,s)$ is irreducible as a $\mathcal{U}_q(n-1)$-module, by the fact that it is isomorphic to the module $\widetilde{\mathcal{H}}_{n-1}(r,s)$, and since $\psi(l,m;r,s) \in \mathcal{H}_n(l,m;r,s)$, we get $\mathcal{U}_q(n-1)\cdot\psi(l,m;r,s) = \mathcal{H}_n(l,m;r,s)$ for all $0 \leqslant r \leqslant l$ and all $0 \leqslant s \leqslant m$. On the other hand we know that $\psi(l,m;r,s) \in \mathcal{H}_n(l,m)$ (Proposition 4.15), hence $\mathcal{U}_q(n-1)\cdot\psi(l,m;r,s) \subset \mathcal{H}_n(l,m)$. Thus we see that $\mathcal{H}_n(l,m;r,s) \subset \mathcal{H}_n(l,m)$ for all $0 \leqslant r \leqslant l$ and all $0 \leqslant s \leqslant m$. □

PROPOSITION 4.17. *We have the following direct sum decomposition into irreducible, inequivalent* $\mathcal{U}_q(n-1)$*-modules;*

$$\mathcal{H}_n(l,m) = \bigoplus_{r=0}^{l}\bigoplus_{s=0}^{m}\mathcal{H}_n(l,m;r,s). \tag{4.13}$$

Proof. From the previous lemma we obtain that the direct sum on the right-hand side is contained in $\mathcal{H}_n(l,m)$. Counting dimensions gives the equality. □

Write $\widetilde{\mathcal{H}}_n(l,m;r,s) = \pi(\mathcal{H}_n(l,m;r,s))$ for the image of $\mathcal{H}_n(l,m;r,s)$ under the projection π of (4.1).

PROPOSITION 4.18. *There exists the following orthogonal decomposition into irreducible, inequivalent* $\mathcal{U}_q(n-1)$*-modules*

$$\widetilde{\mathcal{H}}_n(l,m) = \bigoplus_{r=0}^{l}\bigoplus_{s=0}^{m}\widetilde{\mathcal{H}}_n(l,m;r,s). \tag{4.14}$$

This proposition follows from the previous proposition and the following one.

PROPOSITION 4.19. *If the set* $\{\pi'(g_i(r,s))\}$*, with* $i = 1,\ldots,d_{n-1}(r,s)$*, forms an orthonormal basis for* $\widetilde{\mathcal{H}}_{n-1}(r,s)$ *with respect to the inner product defined by* h_{n-1}*, then the set*

$$\{((1-q^{2(n-1)})\,c_{l-r,m-s}^{(n+r+s-2)}/(1-q^{2(n+r+s-1)}))^{-(1/2)}$$
$$R_{l-r,m-s}^{(n-2+r+s)}(z_n,w_n;q^2)\pi(g_i(r,s))\},$$

with $0 \leqslant r \leqslant l; 0 \leqslant s \leqslant m$ *and* $i = 1,\ldots,d_{n-1}(r,s)$*, forms an orthonormal basis for* $\widetilde{\mathcal{H}}_n(l,m)$ *with respect to the inner product on* $\widetilde{\mathcal{Z}}_n$ *defined by* h_n*.*

Proof. The proof of this is along the same lines as the proof of Proposition 4.14. □

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Remark 4.20. The elements in Proposition 4.19 are called *associated spherical elements* in $\mathcal{H}_n(l,m)$.

4.5. Addition formula for q-disk polynomials

We are now at the stage where we can prove the addition theorem for q-disk polynomials. For this we use the concrete realisation of $\widetilde{\mathcal{Z}}_n$ as a $*$-subalgebra of $\mathcal{A}_q(n)$ which was established in Proposition 4.5.

So let us identify $z_i = t_{ni},\ w_i = t^*_{ni}$. Under this correspondence the coaction δ is merely the comultiplication Δ of $\mathcal{A}_q(n)$. Write τ for the anti-linear and involutive algebra automorphism $* \circ S$ of $\mathcal{A}_q(n)$. We know that $\tau(t_{ij}) = t_{ji}$, since $t^*_{ij} = S(t_{ji}) = (-q)^{j-i} D_{\widehat{i}\widehat{j}} \det_q^{-1}$, where D_{IJ} denotes the quantum minor-determinant corresponding to the two subsets $I, J \subset \{1,\dots,n\}$ (see [23, (1.8)]). So in particular we get $\tau(z_n) = z_n$. Moreover, recall from [23, (3.2)] that

$$(D_{IJ})^* = S(D_{JI}) = \frac{\mathrm{sgn}_q(J;J^c)}{\mathrm{sgn}_q(I;I^c)} D_{I^cJ^c} \det{}_q^{-1} \tag{4.15}$$

in which I^c denotes the complement of I in $\{1,\dots,n\}$, and

$$\mathrm{sgn}_q(I;J) = \begin{cases} 0 & I\cap J \neq \emptyset \\ (-q)^{l(I;J)} & I\cap J = \emptyset \end{cases},$$

where $l(I;J) = \#\{(i,j) \in I\times J \mid i>j\}$. Using (4.15) we see that $S(t^*_{nn}) = t_{nn}$, whence $\tau(w_n) = w_n$. So we conclude that $\tau(Q_{n-1}) = Q_{n-1}$ in $\widetilde{\mathcal{Z}}_n$.

As was observed in Remark 2.4 we can exhibit $\widetilde{\mathcal{H}}_n(l,m)$ as the row space $A_{\pi(1)}$ for some irreducible unitary matrix corepresentation π of $\mathcal{A}_q(n)$, such that $\pi_{11} = R^{(n-2)}_{l,m}(z_n,w_n;q^2)$ (since it is easily seen that $\varepsilon(R^{(n-2)}_{l,m}(z_n,w_n;q^2)) = 1$). The basis elements $\{\pi_{1i}\}$ of $A_{\pi(1)}$ then correspond to the elements given in Proposition 4.19 with $r+s\neq 0$. Again by virtue of Remark 2.4, and by the fact that $(\mathrm{id}\otimes\tau)\circ\Delta(\pi_{11}) = \sum_k \pi_{1k}\otimes\pi_{1k}$, one can write

$$\begin{aligned}
&(\mathrm{id}\otimes\tau)\circ\Delta R^{(n-2)}_{l,m}(z_n,w_n;q^2)\\
&\quad = h_n(R^{(n-2)}_{l,m}(z_n,w_n;q^2)^* R^{(n-2)}_{l,m}(z_n,w_n;q^2))\\
&\qquad \times \sum_{r=0}^{l}\sum_{s=0}^{m}\sum_{i=1}^{d_{n-1}(r,s)} (a^{-(1/2)}_{l,m,r,s} R^{(n-2+r+s)}_{l-r,m-s}(z_n,w_n;q^2)\pi(g_i(r,s)))\\
&\qquad \otimes a^{-(1/2)}_{l,m,r,s} R^{(n-2+r+s)}_{l-r,m-s}(z_n,w_n;q^2)\pi(g_i(r,s)),
\end{aligned}$$

where $a_{l,m,r,s} = (1-q^{2(n-1)})(1-q^{2(n+r+s-1)})^{-1} c^{(n+r+s-2)}_{l-r,m-s}$. Here we choose the bases $\{g_i(r,s)\}$ of $\mathcal{H}_{n-1}(r,s)$ in such a way that for all $r,s\in\mathbb{Z}_+$ there holds

$g_1(r,s) = (c_{r,s}^{(n-3)})^{-(1/2)} Q_{n-1}^{(r+s)/2} R_{r,s}^{(n-3)}(z_{n-1}Q_{n-1}^{-(1/2)}, w_{n-1}Q_{n-1}^{-(1/2)}; q^2)$. Let us pull the above identity in $\widetilde{\mathcal{H}}_n(l,m) \otimes \widetilde{\mathcal{H}}_n(l,m)$ back to $\mathcal{H}_n(l,m) \otimes \mathcal{H}_n(l,m)$. We obtain

$$\begin{aligned}
&(Q_n \otimes Q_n)^{(l+m)/2} R_{l,m}^{(n-2)} \left(\frac{(\mathrm{id} \otimes \tau)\Delta(z_n)}{(Q_n \otimes Q_n)^{1/2}}, \frac{(\mathrm{id} \otimes \tau)\Delta(w_n)}{(Q_n \otimes Q_n)^{1/2}}; q^2 \right) \\
&\quad = c_{l,m}^{(n-2)} \sum_{r=0}^{l} \sum_{s=0}^{m} \sum_{i=1}^{d_{n-1}(r,s)} a_{l,m,r,s}^{-1} \\
&\qquad \times Q_n^{(l-r+m-s)/2} R_{l-r,m-s}^{(n-2+r+s)}(z_n Q_n^{-(1/2)}, w_n Q_n^{-(1/2)}; q^2) g_i(r,s) \\
&\qquad \otimes Q_n^{(l-r+m-s)/2} R_{l-r,m-s}^{(n-2+r+s)}(z_n Q_n^{-(1/2)}, w_n Q_n^{-(1/2)}; q^2) g_i(r,s).
\end{aligned} \tag{4.16}$$

Now recall the projection $\rho^{(n,2)}: \mathcal{Z}_n \to \mathcal{Z}_2$ which puts $z_1, \ldots, z_{n-2}, w_1, \ldots, w_{n-2}$ equal to zero (cf. Sect. 3.2). Let us write $\gamma, \delta, -q^{-1}\beta, \alpha$ for the respective generators $z_1{}', z_2{}', w_1{}'$ and $w_2{}'$ of $\mathcal{Z}_2$. We also write D for the generator of the centre of $\mathcal{Z}_2$: $D = \alpha^*\alpha + \gamma\gamma^* = \delta\alpha - q^{-1}\beta\gamma$. Another way of writing D is $D = \alpha\alpha^* + q^2\gamma\gamma^* = \alpha\delta - q\beta\gamma$. Having this, we can write the projection $\rho^{(n,2)}$ as

$$\begin{aligned}
&\rho^{(n,2)}: \mathcal{Z}_n \to \mathcal{Z}_2 \\
&z_{n-1} \to \gamma, \qquad w_{n-1} \to \gamma^* = -q^{-1}\beta, \\
&z_n \to \delta, \qquad w_n \to \delta^* = \alpha. \\
&z_i, w_i \to 0 \quad (i = 1, \ldots, n-2).
\end{aligned} \tag{4.17}$$

LEMMA 4.21. *For* $0 \leqslant r \leqslant l; 0 \leqslant s \leqslant m$ *pick a basis* $\{g_i(r,s)\}$ *of* $\mathcal{H}_{n-1}(r,s)$ *such that in each case* $g_1(r,s) = (c_{r,s}^{(n-3)})^{-(1/2)} Q_{n-1}^{(r+s)/2} R_{r,s}^{(n-3)}(z_{n-1}Q_{n-1}^{-(1/2)}, w_{n-1}Q_{n-1}^{-(1/2)}; q^2)$. *Then* $\rho^{(n,2)}(g_i(r,s)) = \delta_{i1}(c_{r,s}^{(n-3)})^{-(1/2)} \gamma^r (\gamma^*)^s$.

Proof. From Proposition 4.17 we obtain the decomposition

$$\mathcal{H}_{n-1}(r,s) = \bigoplus_{u=0}^{r} \bigoplus_{v=0}^{s} \mathcal{H}_{n-1}(r,s;u,v).$$

This immediately yields that $\rho^{(n,2)}|_{\mathcal{H}_{n-1}(r,s;u,v)} \neq 0$ if and only if $(u,v) = (0,0)$. Since $\mathcal{H}_{n-1}(r,s;0,0)$ is the one-dimensional space spanned by $g_1(r,s)$, the lemma now follows from an easy computation. □

LEMMA 4.22. *The following equalities hold;*

$$(\mathrm{id} \otimes \tau) \circ \Delta(z_n) = \sum_{k=1}^{n} z_k \otimes z_k, \quad (\mathrm{id} \otimes \tau) \circ \Delta(w_n) = \sum_{k=1}^{n} q^{2(n-k)} w_k \otimes w_k.$$

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Proof. The first equality follows directly from (3.4) and the fact that $\tau(t_{ij}) = t_{ji}$. As for the second one, use (4.15) to obtain $\tau(t^*_{kn}) = q^{2(n-k)} w_k$. Together with (3.4) this yields the stated result. □

As an immediate consequence of this lemma and (4.17) we find

COROLLARY 4.23. *The following identities are valid in $\mathcal{Z}_n \otimes \mathcal{Z}_2$;*

$$\begin{aligned}(\mathrm{id}\otimes\rho^{(n,2)})\circ(\mathrm{id}\otimes\tau)\circ\Delta(z_n) &= z_{n-1}\otimes\gamma + z_n\otimes\delta\\ (\mathrm{id}\otimes\rho^{(n,2)})\circ(\mathrm{id}\otimes\tau)\circ\Delta(w_n) &= q^2 w_{n-1}\otimes\gamma^* + w_n\otimes\delta^*\\ &= -q w_{n-1}\otimes\beta + w_n\otimes\alpha.\end{aligned}$$

Consider the $*$-algebras $\mathcal{X}$ and $\mathcal{Y}$ generated by the elements

$$\begin{aligned}&\mathcal{X}: Q = Q_n,\quad X_1 = z_{n-1},\quad X_1^* = w_{n-1},\quad X_2 = z_n,\quad X_2^* = w_n\\ &\mathcal{Y}: D = D,\ Y_1 = \gamma,\quad Y_1^* = \gamma^*,\quad Y_2 = \delta,\quad Y_2^* = \delta^*\end{aligned}\tag{4.18}$$

and with $*$-structures

$$\begin{aligned}&Q^* = Q,\qquad (X_1)^* = X_1^*,\qquad (X_2)^* = X_2^*\\ &D^* = D,\qquad (Y_1)^* = Y_1^*,\qquad (Y_2)^* = Y_2^*\end{aligned}\tag{4.19}$$

(so we merely changed notations). It is straightforward from (3.1) that the following relations are satisfied.

LEMMA 4.24. *In $\mathcal{X}$ and $\mathcal{Y}$ respectively, one has*

$$\begin{aligned}&X_1X_2 = qX_2X_1,\qquad Y_1Y_2 = qY_2Y_1,\\ &X_1^*X_2 = qX_2X_1^*,\qquad Y_1^*Y_2 = qY_2Y_1^*,\\ &X_2^*X_2 = q^2X_2X_2^* + (1-q^2)Q,\qquad Y_1Y_1^* = Y_1^*Y_1,\\ &X_1^*X_1 = q^2X_1X_1^*,\quad D = Y_1Y_1^* + Y_2Y_2^* = q^2Y_1^*Y_1 + Y_2^*Y_2,\\ &\quad +(1-q^2)(Q - X_2X_2^*),\qquad Q\ \textit{central}.\end{aligned}\tag{4.20}$$

Remark 4.25. Centrality of Q in $\mathcal{X}$ does not follow automatically from the first four relations above, but is imposed on the algebra $\mathcal{X}$. However, D is clearly central.

Write $\mathcal{B} = \mathcal{X}\otimes\mathcal{Y}$ and identify X_1 with $X_1\otimes 1$, Y_1 with $1\otimes Y_1$ and so on. We will prove that the relations in Lemma 4.24 are in fact the only nontrivial relations between the generators of $\mathcal{B}$.

LEMMA 4.26. *Write $Q' = Q - X_1X_1^* - X_2X_2^*$. A linear basis for $\mathcal{X}$ is given by the set of monomials $\{X_1^rX_2^s(X_2^*)^t(X_1^*)^u(Q')^v | r,s,t,u,v\in\mathbb{Z}_+\}$.*

Proof. Rewrite the relations in $\mathcal{X}$ in terms of X_1, X_2, X_1^*, X_2^* and Q';

$$X_1X_2 = qX_2X_1, \qquad X_2^*X_2 = X_2X_2^* + (1-q^2)(Q' + X_1X_1^*)$$
$$X_1^*X_2 = qX_2X_1^*, \qquad X_1^*X_1 = X_1X_1^* + (1-q^2)Q'.$$

From this it readily follows that $\mathcal{X}$ is spanned by the monomials given in the lemma. Hence we only need to show linear independence of this set. For this it suffices to show linear independence of the highest order terms, which, as elements of $\mathcal{Z}_n$, equal

$$\natural(X_1^rX_2^s(X_2^*)^t(X_1^*)^u(Q')^v) = \natural(z_{n-1}^r z_n^s w_n^t w_{n-1}^u ({Q_n}')^v)$$
$$= z_{n-1}^r z_n^s w_n^t w_{n-1}^u z_{n-2}^v w_{n-2}^v.$$

By virtue of Proposition 3.1 these highest order terms are linearly independent as elements of $\mathcal{Z}_n$, hence also as elements of $\mathcal{X}$. □

PROPOSITION 4.27. *A linear basis for $\mathcal{B}$ is given by the set of monomials of the form*

$$(X_1\otimes 1)^r(X_2\otimes 1)^s(X_2^*\otimes 1)^t(X_1^*\otimes 1)^u(Q'\otimes 1)^v$$
$$\times(1\otimes Y_1)^k(1\otimes Y_1^*)^l(1\otimes Y_2)^m(1\otimes Y_2^*)^p,$$

where $r, s, t, u, v, k, l, m, p \in \mathbb{Z}_+$.

This follows directly from Lemma 4.26 and Proposition 3.1. Now are able to prove

PROPOSITION 4.28. *The relations (4.20) are the only nontrivial relations among the generators of $\mathcal{B}$.*

Proof. Write $\mathcal{E}$ for the $*$-algebra with abstract generators $X_1, X_1^*, X_2, X_2^*, Q$ and $Y_1, Y_1^*, Y_2, Y_2^*, D$ and with relations (4.20). Furthermore, impose that all of the first five generators commute with all of the last five ones. The $*$-structure on $\mathcal{E}$ is given by (4.19). From (4.20) we see that $\mathcal{E}$ is spanned by the elements $X_1^rX_2^s(X_2^*)^t(X_1^*)^u(Q')^vY_1^k(Y_1^*)^lY_2^m(Y_2^*)^p$ with $r, s, t, u, v, k, l, m, p \in \mathbb{Z}_+$; as before we write $Q' = Q - X_1X_1^* - X_2X_2^*$. There is a unique surjective $*$-algebra homomorphism $\Theta : \mathcal{E} \to \mathcal{B}$ sending $X_1, X_2, X_2^*, X_1^*, Q', Y_1, Y_1^*, Y_2$ and Y_2^* to $z_{n-1}\otimes 1, z_n\otimes 1, w_n\otimes 1, w_{n-1}\otimes 1, {Q_n}'\otimes 1, 1\otimes\gamma, 1\otimes\gamma^*, 1\otimes\delta$ and $1\otimes\delta^*$ respectively. It now easily follows from Proposition 4.27 that this is actually an isomorphism. □

With the notation as in (4.18), and with the aid of Lemma 4.21 and Corollary 4.23, we can write down the effect of the mapping $\mathrm{id}\otimes\rho^{(n,2)}$ on (4.16). It reads;

$$R_{l,m}^{(n-2)}(X_1\otimes Y_1 + X_2\otimes Y_2, q^2X_1^*\otimes Y_1^* + X_2^*\otimes Y_2^*, Q\otimes D; q^2)$$

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$$
\begin{aligned}
&= \sum_{r=0}^{l} \sum_{s=0}^{m} c_{l,m;r,s}^{(n-2)} R_{l-r,m-s}^{(n-2+r+s)}(X_2, X_2^*, Q; q^2) \\
&\quad \times R_{r,s}^{(n-3)}(X_1, X_1^*, Q - X_2X_2^*; q^2) \\
&\quad \otimes R_{l-r,m-s}^{(n-2+r+s)}(Y_2, Y_2^*, D; q^2) Y_1^r (Y_1^*)^s
\end{aligned} \tag{4.21}
$$

in which, for $\alpha = n - 2$,

$$
c_{l,m;r,s}^{(\alpha)} = c_{l,m}^{(\alpha)} a_{l,m,r,s}^{-1} (c_{r,s}^{(\alpha-1)})^{-1} = \frac{1 - q^{2(\alpha+r+s+1)}}{1 - q^{2(\alpha+1)}} \frac{c_{l,m}^{(\alpha)}}{c_{l-r,m-s}^{(\alpha+r+s)} c_{r,s}^{(\alpha-1)}} \tag{4.22}
$$

and $c_{l,m}^{(\alpha)}$ is as in (4.12). Here we employed the following

NOTATION 4.29. *For $\alpha > -1$ and $l, m \in \mathbb{Z}_+$ we put*

$$
R_{l,m}^{(\alpha)}(A, B, C; q) = \begin{cases} C^m A^{l-m} p_m^{(\alpha,l-m)}\left(\frac{C-AB}{C}; q\right) & (l \geqslant m) \\ C^l p_l^{(\alpha,m-l)}\left(\frac{C-AB}{C}; q\right) B^{m-l} & (l \leqslant m) \end{cases} \tag{4.23}
$$

in terms of the little q-Jacobi polynomials (cf. Sect. 1). We use (4.23) for non-commuting variables A, B, C, with $BA = qAB + (1-q)C$ where we assume that C commutes with A and B, so that (4.23) is polynomial in A, B and C.

Observe that $R_{l,m}^{(\alpha)}(A, B, 1; q^2) \equiv R_{l,m}^{(\alpha)}(A, B; q^2)$.

Let us have a closer look at the polynomials $R_{l,m}^{(\alpha)}(A, B, C; q)$;

$$
R_{l,m}^{(\alpha)}(A, B, C; q) = \begin{cases} C^m A^{l-m} \sum_{k=0}^{m} \frac{(q^{-m};q)_k (q^{\alpha+l+1};q)_k}{(q^{\alpha+1};q)_k (q;q)_k} \left(q \frac{C-AB}{C}\right)^k \\ C^l \sum_{k=0}^{l} \frac{(q^{-l};q)_k (q^{\alpha+m+1};q)_k}{(q^{\alpha+1};q)_k (q;q)_k} \left(q \frac{C-AB}{C}\right)^k B^{m-l} \end{cases}
$$

in the respective cases $l \geqslant m$ and $l \leqslant m$. So these polynomials are rational in q^α. Hence (see also (4.12)) both sides of (4.21) are rational functions of $q^{2\alpha}$. Multiplying with a suitable factor we will obtain from (4.21) an identity which is polynomial in $q^{2\alpha}$ and which holds for $\alpha = 1, 2, \ldots$. But then obviously the identity is true for all $\alpha > 0$.

Finally, let $\sigma : \mathcal{Y} \to \mathcal{Y}$ be the automorphism that sends Y_1 to $-qY_1^*$, Y_1^* to $-q^{-1}Y_1$, and that fixes Y_2 and Y_2^* (this comes down to interchanging β and γ in $\mathcal{Z}_2$). If we now apply $\mathrm{id} \otimes \sigma$ to (4.21) we end up with the final form of the addition formula.

THEOREM 4.30. *Suppose we are given the abstract complex $*$-algebras $\mathcal{X}$ and $\mathcal{Y}$ with generators $X_1, X_2, X_1^*, X_2^*, Q$, respectively $Y_1, Y_2, Y_1^*, Y_2^*, D$, relations (4.20) and $*$-structures (4.19).*

Then, for arbitrary $\alpha > 0$ and arbitrary $l, m \in \mathbb{Z}_+$, we have the following addition formula for q-disk polynomials;

$$\begin{aligned} &R^{(\alpha)}_{l,m}(-qX_1 \otimes Y_1^* + X_2 \otimes Y_2, -qX_1^* \otimes Y_1 + X_2^* \otimes Y_2^*, Q \otimes D; q^2) \\ &\quad = \sum_{r=0}^{l} \sum_{s=0}^{m} c^{(\alpha)}_{l,m;r,s} R^{(\alpha+r+s)}_{l-r,m-s}(X_2, X_2^*, Q; q^2) \\ &\qquad \times R^{(\alpha-1)}_{r,s}(X_1, X_1^*, Q - X_2 X_2^*; q^2) \\ &\qquad \otimes (-q)^{r-s} R^{(\alpha+r+s)}_{l-r,m-s}(Y_2, Y_2^*, D; q^2) Y_1^s (Y_1^*)^r \end{aligned} \tag{4.24}$$

where we use the notations (4.22) *and* (4.23).

Remark 4.31. For $\alpha = n - 2$ this is in fact an identity in $\mathcal{Z}_n \otimes \mathcal{Z}_2$, which we can rewrite as an identity in $\widetilde{\mathcal{Z}}_n \otimes \widetilde{\mathcal{Z}}_2$ by putting $Q = D = 1$ in (4.24). For general $\alpha > 0$ we can do something similar; the relations among the generators are then given by (4.20) but with Q and D equal to 1.

Remark 4.32. It is possible to generalize the findings in this paper by considering a one-parameter extension of the algebra $\mathcal{Z}_n$. The zonal and associated spherical functions are expressed in that case in terms of certain big q-Jacobi polynomials, which are one-parameter extensions of the little q-Jacobi polynomials found here. Consequently we obtain a generalization by one parameter of the addition formula Theorem 4.30 and of the main results in [7]. Details are written down in an as of yet unpublished paper [6].

Acknowledgements

The author is very grateful to Prof. T. H. Koornwinder for his valuable comments and suggestions and to Dr. H. T. Koelink for carefully reading an earlier version of the manuscript.

References

1. Andrews, G. E. and Askey, R.: Enumeration of partitions: The role of Eulerian series and q-orthogonal polynomials, *Higher Combinatorics*, M. Aigner (ed.), Reidel, Dordrecht, 1977, pp. 3–26.
2. Bergman , G. M.: The diamond lemma for ring theory, *Adv. Math.* 29 (1978), 178–218.
3. Dijkhuizen, M. S.: *On compact quantum groups and quantum homogeneous spaces*, PhD thesis, University of Amsterdam, 1994.
4. Dijkhuizen, M. S. and Koornwinder, T. H.: CQG algebras: a direct algebraic approach to compact quantum groups, *Lett. Math. Phys* 32 (1994), 315–330.
5. Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F. G.: *Higher transcendental functions, Vol 2*, McGraw-Hill Book Company, 1953.
6. Floris, P. G. A.: Addition theorems for big q-Jacobi polynomials.

7. Floris, P. G. A. and Koelink, H. T.: A commuting q-analogue of the addition formula for disk polynomials, *Constr. Approx*. 13(4) (1997), to appear.
8. Gasper, G. and Rahman, M.: *Basic Hypergeometric Series, Encyclopedia of Mathematics and its Applications* 35, Cambridge University Press, Cambridge, 1990.
9. Koelink, H. T.: On $*$-representations of the Hopf $*$-algebra associated with the quantum group $U_q(n)$, *Compositio Math.* 77 (1991), 199–231.
10. Koelink, H. T.: The addition formula for continuous q-Legendre polynomials and associated spherical elements on the $\mathrm{SU}(2)$ quantum group related to Akey-Wilson polynomials, *SIAM J. Math. Anal*. 25(1) (1994), 197–217 .
11. Koelink, H. T.: *Addition formulas for q-special functions*, in Special functions, q-series and related topics (eds. M. E. H. Ismail, D. R. Masson, M. Rahman), *Fields Institute Comm*. 14, AMS, 1997, 109–129.
12. Koelink, H. T.: Askey-Wilson polynomials and the quantum $\mathrm{SU}(2)$ group: survey and applications, *Acta Appl. Math.* 44 (1996), 295–352.
13. Koornwinder, T. H.: The addition formula for Jacobi polynomials I, Summary of the results, *Nederl. Akad. Wetensch. Proc. Ser. A* 75 (1972), 188–191.
14. Koornwinder, T. H.: *The addition formula for Jacobi polynomials, Parts II and III* , Reports TW 133/72 and 135/72, Math. Centrum, Amsterdam, 1972.
15. Koornwinder, T. H.: Orthogonal polynomials in connection with quantum groups, in *Orthogonal polynomials: theory and practice*, P. Nevai (ed.), NATO ASI Series C#294, Kluwer, 1990, pp. 257–292.
16. Koornwinder, T. H.: The addition formula for little q-Legendre polynomials and the $\mathrm{SU}(2)$ quantum group, *SIAM J. Math. Anal.* 22 (1991), 295–301.
17. Koornwinder, T. H.: Positive convolution structures associated with quantum groups, in *Probability measures on groups X*, H. Heyer, (ed.), Plenum , 1991, pp. 249–286.
18. Koornwinder, T. H.: Askey-Wilson polynomials as zonal spherical functions on the $SU(2)$ quantum group, *SIAM J. Math. Anal.* 24(3) (1993), 795–813.
19. Koornwinder, T. H.: Compact Quantum Groups and q-Special Functions, in *Representations of Lie groups and quantum groups*, V. Baldoni and M. Picardello (eds), Pitman Research Notes in Mathematics Series 311, Longman Scientific and Technical, 1994.
20. Noumi, M.: Quantum groups and q-orthogonal polynomials. Towards a realization of Askey - Wilson polynomials on $\mathrm{SU}_q(2)$, in *Special Functions*, M. Kashiwara and T. Miwa (eds), ICM-90 Satellite Conference Proceedings, Springer, 1991, pp. 260–288.
21. Noumi, M.: Macdonald's symmetric polynomials as zonal spherical functions on some quantum homogeneous spaces, *Adv. Math.* 123 (1996), 16–77.
22. Noumi, M. and Mimachi, K.: Askey-Wilson polynomials and the quantum group $\mathrm{SU}_q(2)$ *Proc. Japan Acad. Ser. A Math. Sci.* 66 (1990), 146–149.
23. Noumi, M., Yamada, H. and Mimachi, K.: Finite dimensional representations of the quantum group $\mathrm{GL}_q(n,\mathbb{C})$ and the zonal spherical functions on $U_q(n-1)\backslash U_q(n)$, *Japanese Jnl. Math.* 19(1) (1993), 31–80.
24. Reshetikhin, N. Yu., Takhtadjan, L. A. and Faddeev, L. D.: Quantization of Lie groups and Lie algebras, *Algebra and Analysis* 1 (1989), 178–206; *English translation in Leningrad Math. Jnl.* 1 (1990), 193–225.
25. Šapiro , R. L.: Special functions connected with representations of the group $SU(n)$ of class I relative to $\mathrm{SU}(n-1)$ $(n \geqslant 3)$, *Izv. Vysš. Učebn. Zaved. Mathematika* 71 (1968), 9–20 (Russian); *AMS Translation Series 2* 113 (1979), 201–211 (English).
26. Sugitani, T.: Harmonic analysis on quantum spheres associated with representations of $U_q(\mathfrak{so}_N)$ and q-Jacobi polynomials, *Compositio Math.* 99 (1995), 249–281.
27. Sweedler, M. E.: *Hopf algebras*, W.A. Benjamin, New York, 1969.
28. Vaksman, L. L. and Soibelman, Ya. S.: The algebra of functions on the quantum group $\mathrm{SU}(n+1)$ and odd-dimensional quantum spheres, *Leningrad Math. Jnl.* 2 (1991), 1023–1042.
29. Vilenkin, N. J. and Klimyk, A. U.: *Representation of Lie groups and Special Functions*, 3 volumes, Kluwer, Dordrecht, 1991–1993.