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BAIRE'S CATEGORY IN EXISTENCE PROBLEMS FOR NON-CONVEX LOWER SEMICONTINUOUS EVOLUTION DIFFERENTIAL INCLUSIONS

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Abstract The existence of mild solutions to the non-convex Cauchy problem

$$\dot{x}(t) \in Ax(t) + \partial F(t, x(t)), \quad x(t_0) = a,$$

is investigated. Here A is the infinitesimal generator of a C_0-semigroup in a reflexive and separable Banach space $\mathbb{E}$, F is a Pompeiu–Hausdorff lower semicontinuous multifunction whose values are closed convex and bounded sets with non-empty interior contained in $\mathbb{E}$, and $\partial F(t, x(t))$ denotes the boundary of $F(t, x(t))$. Our approach is based on the Baire category method, with appropriate modifications which are actually necessary because, under our assumptions, the underlying metric space that naturally enters in the Baire method, i.e. the solution set of the convexified Cauchy problem (C_F), can fail to be a complete metric space.

Keywords: Baire method; Banach spaces; infinitesimal generator; C_0-semigroup; non-convex evolution inclusions; Pompeiu–Hausdorff lower semicontinuity

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1. Introduction

In recent years non-convex differential inclusions have been investigated by several authors (see [**1–4**, **14**, **17**, **18**, **21–23**, **28–30**] and the references therein).

In this paper we study the existence of solutions for non-convex differential inclusions. For this kind of problem the method of approach based on the Baire category has proven to be useful, provided the corresponding convex differential inclusion has an h-continuous (i.e. in the Pompeiu–Hausdorff metric h) right-hand side. In this case the solution set of the convex differential inclusion is a complete metric space and the basic idea is to prove that *most* elements (in the sense of the Baire category) of this space are actually solutions of the original non-convex differential inclusion.

Such a procedure may fail, whenever the corresponding convex differential inclusion is only h-lower semicontinuous because, in this case, its solution set is not necessarily closed (see Example 3.7). A situation of this type is investigated in this paper. More precisely, we consider a non-convex evolution differential inclusion in a Banach space $\mathbb{E}$, of the form

$$\dot{x}(t) \in Ax(t) + \partial F(t, x(t)), \quad x(t_0) = a \in \mathbb{E}. \tag{C}$$

Here A is the infinitesimal generator of a C_0-semigroup on $\mathbb{E}$, F is a bounded h-lower semicontinuous multifunction defined on $[t_0, t_1] \times \mathbb{E}$, whose values $F(t, x)$ are closed, convex and bounded subsets of $\mathbb{E}$ with non-empty interior, and $\partial F(t, x)$ denotes the boundary of $F(t, x)$. If the space $\mathbb{E}$ is reflexive and separable, then we shall prove (Theorem 4.3) that the Cauchy problem (C) has mild solutions defined on $[t_0, t_1]$. Our method of approach is based on the Baire category. This method was introduced in [**8**, **10**, **11**], starting from a generic type result due to Cellina [**5**]. For further information on the Baire method and differential inclusions see [**6**, **12**, **17**, **25**, **30**].

It is worth noting that the non-convex term $\partial F(t, x)$ in (C) can have a very irregular behaviour since an h-lower semicontinuous multifunction F can be discontinuous on a dense set of points (see Example 4.4). Furthermore, observe that without the assumption that $F(t, x)$ has non-empty interior the existence of solutions may fail, in view of Godunov's counter-example [**15**].

In conclusion, we note that, in addition to the Baire category method, another method of approach, essentially based on Gromov convex integration theory [**16**], has proven to be useful in the investigation of existence problems for some important classes of non-convex differential inclusions [**19**, **20**, **26**]. However, it is not clear whether this second approach can be suitably adapted for non-continuous differential inclusions and, if it can, whether it is eventually more effective, making it possible to avoid some technicalities of the Baire category method.

2. Notation, terminology and auxiliary results

Let (M, d) be a metric space. If $A \subset M$, by $\operatorname{int} A$, $\bar{A}$ and ∂A we denote the interior, the closure and the boundary of A, respectively. For $x \in M$ and $r > 0$ we set $B(x, r) = \{z \in M \mid d(z, x) < r\}$ and $B[x, r] = \{z \in M \mid d(z, x) \leqslant r\}$. For $x \in M$ and $A \subset M$, $A \neq \emptyset$, we set $d(x, A) = \inf\{d(x, a) \mid a \in A\}$. If X, Y are non-empty (closed) bounded subsets of M we denote by $h(X, Y)$ the Pompeiu–Hausdorff distance, i.e.

$$h(X, Y) = \max\Big\{\sup_{x \in X} d(x, Y), \sup_{y \in Y} d(y, X)\Big\}.$$

We denote by B (respectively, U) the unit closed ball centred at 0 contained in a Banach space $\mathbb{E}$ (respectively, $\mathbb{R}$). A closed (respectively, open) segment with endpoints $a, b \in \mathbb{E}$, $a \neq b$, is denoted by $[a, b]$ (respectively, (a, b)) and, if $a \in \mathbb{R}$, we set $J_{a,\delta} = (a - \delta, a + \delta)$, $\delta > 0$.

For $J \subset \mathbb{R}$, we denote by $|J|$ the Lebesgue measure of J and by χ_J the characteristic function of J.

Throughout this paper $\mathbb{E}$ is a reflexive and separable real Banach space with norm $\|\cdot\|$, and

$$\mathcal{B}(\mathbb{E}) = \{X \subset \mathbb{E} \mid X \text{ is closed convex bounded with } \operatorname{int} X \neq \emptyset\}.$$

$\mathcal{B}(\mathbb{E})$ is endowed with the Pompeiu–Hausdorff metric h. If X is a non-empty subset of $\mathbb{E}$ we set $\|X\| = \sup\{\|x\| \mid x \in X\}$. A map $\Phi\colon M \to \mathcal{B}(\mathbb{E})$ is said *h-lower semicontinuous (h-l.s.c.) (respectively, h-continuous) at x_0* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $x \in B(x_0, \delta)$ implies $\Phi(x_0) \subset \Phi(x) + \varepsilon B$ (respectively, $h(\Phi(x_0), \Phi(x)) < \varepsilon$).

For $\Phi\colon X \to Y$ and $A \subset X$, by $\Phi|_A$ we denote the restriction of Φ to A.

Let $I = [t_0, t_1]$. A finite family $\Delta = \{A_i\}_{i=1}^{M}$ of intervals $A_i \subset I$ given by $A_i = [a_{i-1}, a_i)$, $i = 1, \dots, M-1$, $A_M = [a_{M-1}, a_M]$, where $t_0 = a_0 < a_1 < \cdots < a_M = t_1$ is called a *partition* of I. If the intervals A_i, $i = 1, \dots, M$, have equal length $\lambda(\Delta)$, this length is called the *step* of the partition.

Given a map $F\colon I \times \mathbb{E} \to \mathcal{B}(\mathbb{E})$ and $a \in \mathbb{E}$, consider the following *non-convex* and *convex* Cauchy problems:

$$\dot{x}(t) \in Ax(t) + \partial F(t, x(t)), \quad x(t_0) = a, \tag{$C_{\partial F}$}$$

$$\dot{x}(t) \in Ax(t) + F(t, x(t)), \quad x(t_0) = a. \tag{C_F}$$

We assume the following:

(h_1) A is the infinitesimal generator of a C_0-semigroup $T(t)$, $t \geqslant 0$, on $\mathbb{E}$;

(h_2) F is h-l.s.c. on $I \times \mathbb{E}$;

(h_3) $\|F(t, x)\| < r$ for every $(t, x) \in I \times \mathbb{E}$, r a positive constant.

As is known [**24**], there exist constants $\alpha \geqslant 0$ and $L \geqslant 1$ such that

$$\|T(t)\| \leqslant L\mathrm{e}^{\alpha t}, \quad t \geqslant 0. \tag{2.1}$$

A function $x\colon I \to \mathbb{E}$ is said to be a *mild solution* of the Cauchy problem ($C_{\partial F}$) (respectively, (C_F)) if x is continuous and there exists a Bochner integrable function $u_x\colon I \to \mathbb{E}$ such that

$$x(t) = T(t - t_0)a + \int_{t_0}^{t} T(t - s)u_x(s)\,\mathrm{d}s, \quad t \in I,$$
$$u_x(t) \in \partial F(t, x(t)) \quad (\text{respectively}, u_x(t) \in F(t, x(t))), \quad t \in I \text{ a.e.}$$

Since the function u_x associated to x is unique in the $L^\infty(I, \mathbb{E})$ sense [**13**], we agree to say that u_x corresponds to x and we call u_x the *pseudo-derivative* of x.

We denote by $C(I, \mathbb{E})$ the Banach space of all continuous functions $x\colon I \to \mathbb{E}$ equipped with the norm of uniform convergence $\|x\|_I = \max\{\|x(t)\| \mid t \in I\}$. The meaning of $L^p(I, \mathbb{E})$, $1 \leqslant p \leqslant +\infty$, is the standard one.

We set

$$\mathcal{M}_{\partial F} = \{x\colon I \to \mathbb{E} \mid x \text{ is a mild solution of } (C_{\partial F})\},$$
$$\mathcal{M}_F = \{x\colon I \to \mathbb{E} \mid x \text{ is a mild solution of } (C_F)\}.$$

Under the assumptions (h_1)–(h_3), F admits locally Lipschitzian selections, and thus $\mathcal{M}_F$ is non-empty. This is no longer true, in general, if the assumption that $\operatorname{int} F(t, x) \neq \emptyset$,

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$(t,x) \in I \times \mathbb{E}$, is removed. To see this it suffices to take $A = 0$ and F a continuous single-valued map as in the Godunov counter-example [**15**].

In the following, we shall use some auxiliary results, stated as Propositions 2.1–2.3, whose proofs are included for completeness.

Proposition 2.1. *Let $F\colon M \to \mathcal{B}(\mathbb{E})$ be h-l.s.c. Then there exists a sequence $\{G_n\}$ of h-continuous multifunctions $G_n\colon M \to \mathcal{B}(\mathbb{E})$ and a sequence $\{\theta_n\}$ of continuous functions $\theta_n\colon M \to (0,+\infty)$ such that for each $x \in M$ the following properties hold:*

(a_1) $G_n(x) + \theta_n(x)B \subset F(x)$, $n \in \mathbb{N}$;

(a_2) $G_n(x) + \theta_n(x)B \subset G_{n+1}(x)$, $n \in \mathbb{N}$;

(a_3) $\lim_{n\to\infty} h(G_n(x), F(x)) = 0$.

Proof. By virtue of [**7**, Theorem 3.6], there exists a sequence $\{\Gamma_n\}$ of h-continuous multifunctions $\Gamma_n\colon M \to \mathcal{B}(\mathbb{E})$ satisfying, for every $x \in M$, the following properties:

(i) $\Gamma_n(x) \subset F(x)$, $n \in \mathbb{N}$,

(ii) $\Gamma_n(x) \subset \Gamma_{n+1}(x)$, $n \in \mathbb{N}$,

(iii) $\lim_{n\to\infty} h(\Gamma_n(x), F(x)) = 0$.

Let $\Gamma\colon M \to \mathcal{B}(\mathbb{E})$ be an h-continuous map satisfying $\Gamma(x) \subset \operatorname{int}\Gamma_1(x)$, $x \in M$ [**9**, Proposition 3.7], and let $f\colon M \to \mathbb{E}$ be a continuous selection of Γ. Clearly, there is a continuous $\theta\colon M \to (0,\infty)$ such that

$$f(x) + \theta(x)B \subset \Gamma_1(x), \quad x \in M.$$

For $n \in \mathbb{N}$ define $G_n\colon M \to \mathcal{B}(\mathbb{E})$ by

$$G_n(x) = \left(1 - \frac{1}{n+1}\right)\Gamma_n(x) + \frac{1}{n+1} f(x), \quad x \in M.$$

We claim that, for each $n \in \mathbb{N}$,

$$G_n(x) + \frac{\theta(x)}{(n+1)(n+2)} B \subset G_{n+1}(x), \quad x \in M. \tag{2.2}$$

In fact, since $\Gamma_n(x) \subset \Gamma_{n+1}(x)$ and $f(x) + \theta(x)B \subset \Gamma_{n+1}(x)$, we have

$$\begin{aligned}\left(1 - \frac{1}{n+1}\right)\Gamma_n(x) + \frac{f(x)}{(n+1)(n+2)} + \frac{\theta(x)}{(n+1)(n+2)}B \\ \subset \left(1 - \frac{1}{n+1}\right)\Gamma_{n+1}(x) + \frac{1}{(n+1)(n+2)}\Gamma_{n+1}(x)\end{aligned}$$

and thus

$$\left(1 - \frac{1}{n+1}\right)\Gamma_n(x) + \frac{f(x)}{n+1} + \frac{\theta(x)}{(n+1)(n+2)}B \subset \left(1 - \frac{1}{n+2}\right)\Gamma_{n+1}(x) + \frac{f(x)}{n+2}.$$

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Therefore, (2.2) holds and hence (a_2) is satisfied with $\theta_n(x) = \theta(x)/(n+1)(n+2)$. Furthermore, (a_3) is valid since, for each $x \in M$, we have

$$h(G_n(x), F(x)) \leqslant \left(1 - \frac{1}{n+1}\right) h(\Gamma_n(x), F(x)) + \frac{1}{n+1} h(f(x), F(x)).$$

As (a_1) is obvious, the proof is complete. □

Proposition 2.2. *Let $F\colon I \to \mathcal{B}(\mathbb{E})$ be h-l.s.c. Then for every $\varepsilon > 0$ there exists a compact $J \subset I$, with $|I \setminus J| < \varepsilon$, such that $F|_J$ is h-continuous.*

Proof. Let $\{G_n\}$ be as in Proposition 2.1 (with $M = I$). For each $n \in \mathbb{N}$ the real-valued function $h(G_n(\cdot), F(\cdot))$ is lower semicontinuous on I. In fact, let $t_0 \in I$ and $\sigma > 0$ be arbitrary. Since F is h-l.s.c. and G_n is h-continuous there exists $\delta > 0$ such that $|t - t_0| < \delta$, $t \in I$, implies

$$F(t_0) \subset F(t) + \sigma B, \qquad h(G_n(t), G_n(t_0)) < \sigma. \tag{2.3}$$

Moreover, setting $h_{t_0} = h(G_n(t_0), F(t_0))$ and $h_t = h(G_n(t), F(t))$, we have

$$F(t_0) \subset G_n(t_0) + (h_{t_0} + \sigma)B, \qquad F(t) \subset G_n(t) + (h_t + \sigma)B. \tag{2.4}$$

By virtue of (2.3) and (2.4) we obtain

$$F(t_0) \subset F(t) + \sigma B \subset G_n(t) + (h_t + 2\sigma)B \subset G_n(t_0) + (h_t + 3\sigma)B.$$

Since, on the other hand, $G_n(t_0) \subset F(t_0)$, we infer that $h_{t_0} \leqslant h_t + 3\sigma$, and thus $h(G_n(\cdot), F(\cdot))$ is lower semicontinuous on I.

By Luzin's Theorem there exists a compact set $J \subset I$, with $|I \setminus J| < \varepsilon$, such that for each $n \in \mathbb{N}$ the function $h(G_n(\cdot), F(\cdot))|_J$ is continuous.

It remains to be shown that $F|_J$ is h-continuous. Let $t_0 \in J$ and $\sigma > 0$ be arbitrary. Fix $\bar{n} \in \mathbb{N}$ so that $h(G_{\bar{n}}(t_0), F(t_0)) < \sigma$. Since $G_{\bar{n}}(\cdot)|_J$ and $h(G_{\bar{n}}(\cdot), F(\cdot))|_J$ are continuous at t_0, there exists $\delta > 0$ such that $|t - t_0| < \delta$, $t \in J$, implies

$$h(G_{\bar{n}}(t), F(t)) < h(G_{\bar{n}}(t_0), F(t_0)) + \sigma, \qquad h(G_{\bar{n}}(t), G_{\bar{n}}(t_0)) < \sigma. \tag{2.5}$$

In view of (2.5), for any $t \in J$, with $|t - t_0| < \delta$, we have

$$\begin{aligned} h(F(t), F(t_0)) &\leqslant h(F(t), G_{\bar{n}}(t)) + h(G_{\bar{n}}(t), G_{\bar{n}}(t_0)) + h(G_{\bar{n}}(t_0), F(t_0)) \\ &< 2h(G_{\bar{n}}(t_0), F(t_0)) + 2\sigma < 4\sigma. \end{aligned}$$

Hence $F|_J$ is h-continuous. This completes the proof. □

Proposition 2.3. *Let $F, G \in \mathcal{B}(\mathbb{E})$ satisfy $G \subset \operatorname{int} F \subset rB$, and let $\rho > 2r$. Then for each $x \in F$ we have the following:*

(b_1) $x \in \partial G \Leftrightarrow d(x, \partial(G + \rho B)) = \rho$;

(b_2) $x \in \operatorname{int} G \Leftrightarrow d(x, \partial(G + \rho B)) > \rho$;

(b_3) $x \in F \setminus G \Leftrightarrow d(x, \partial(G + \rho B)) < \rho$.

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Moreover, the function $\phi\colon F \to [0,+\infty)$ *given by* $\phi(x) = d(x, \partial(G+\rho B))$, $x \in F$, *is concave and continuous.*

Proof. Clearly, under our assumptions, we have that $G + \rho B \in \mathcal{B}(\mathbb{E})$.

(b_1) Let us prove $\Rightarrow$. Suppose $d(x, \partial(G+\rho B)) < \rho$. Then the ball $x + \rho B$ contains some point $y \notin G + \rho B$ and thus $x + \rho B$ is not contained in $G + \rho B$, which implies that $x \notin G$: a contradiction. Suppose $d(x, \partial(G+\rho B)) = \rho' > \rho$. As $x \in F \subset G + \rho B$, we have $x + \rho' B \subset G + \rho B$, which implies that $x + (\rho' - \rho)B \subset G$: a contradiction. Therefore, $d(x, \partial(G+\rho B)) = \rho$.

Let us prove $\Leftarrow$. Suppose $x \in \operatorname{int} G$. Then, for some $\delta > 0$, we have $x + \delta B \subset G$ and thus $x + (\rho+\delta)B \subset G + \rho B$, which implies $d(x, \partial(G+\rho B)) \geqslant \rho + \delta$: a contradiction. Suppose $x \in F \setminus G$. As $x \in F \subset G + \rho B$ and $d(x, \partial(G+\rho B)) = \rho$, it follows that $x + \rho B \subset G + \rho B$, which implies that $x \in G$: a contradiction. Therefore, $x \in \partial G$, and (b_1) holds. Similarly, one can show $(b_2), (b_3)$. Since $F \subset G + \rho B$, we have that

$$\phi(x) = \sup\{s \geqslant 0 \mid x + sB \subset G + \rho B\},$$

and thus ϕ is concave. The continuity is obvious. This completes the proof. □

3. The convex Cauchy problem

In this section we consider the convex Cauchy problem (C_F) and, for this problem, we construct a suitable space of mild solutions. Its closure, say $\mathcal{M}$, might contain elements that are not mild solutions of (C_F). Yet it will be proved that most elements of $\mathcal{M}$ are actually solutions of (C_F).

Let A and $F\colon I \times \mathbb{E} \to \mathcal{B}(\mathbb{E})$ satisfy the assumptions (h_1)–(h_3). By Proposition 2.1 there exist a sequence $\{G_n\}$ of h-continuous maps $G_n\colon I \times \mathbb{E} \to \mathcal{B}(\mathbb{E})$ and a sequence $\{\theta_n\}$ of continuous functions $\theta_n\colon I \times \mathbb{E} \to (0,+\infty)$ such that, for each $(t,x) \in I \times \mathbb{E}$, the following properties hold:

(c_1) $G_n(t,x) + \theta_n(t,x)B \subset F(t,x)$, $n \in \mathbb{N}$;

(c_2) $G_n(t,x) + \theta_n(t,x)B \subset G_{n+1}(t,x)$, $n \in \mathbb{N}$,

(c_3) $\lim_{n\to\infty} h(G_n(t,x), F(t,x)) = 0$.

For $n \in \mathbb{N}$, consider the Cauchy problem

$$\dot{x}(t) \in Ax(t) + G_n(t, x(t)), \quad x(t_0) = a. \qquad (C_{G_n})$$

Let $x\colon I \to \mathbb{E}$ be a mild solution of (C_{G_n}), with pseudo-derivative u_x, satisfying the following two conditions:

(j) (respectively, (j′)) there exists a partition $\Delta = \{A_i\}_{i=1}^M$ of I (respectively, partition $\Delta = \{A_i\}_{i=1}^M$ of I with step $\lambda(\Delta)$), given by $A_i = [a_{i-1}, a_i)$, $i = 1, \dots, M-1$, $A_M = [a_{M-1}, a_M]$, where $t_0 = a_0 < a_1 < \dots < a_M = t_1$, such that

$$u_x(t) = \sum_{i=1}^{M} u_i \chi_{A_i}(t), \quad t \in I;$$

(jj) there exists $\theta > 0$ such that

$$B(u_x(t), \theta) \subset G_n(t, x(t)), \quad t \in I.$$

Now set

$$\mathcal{M}^0_{G_n} = \{x\colon I \to \mathbb{E} \mid x \text{ is a mild solution of } (C_{G_n}) \text{ satisfying (j), (jj)}\}$$

and

$$\mathcal{M} = \overline{\bigcup_{n=1}^{\infty} \mathcal{M}^0_{G_n}},$$

where the closure is in $C(I, \mathbb{E})$. We equip $\mathcal{M}$ with the metric of $C(I, \mathbb{E})$, that is, $\|x_1 - x_2\|_I = \max_{t \in I} \|x_1(t) - x_2(t)\|$, $x_1, x_2 \in \mathcal{M}$.

Proposition 3.1. *For every $x \in \mathcal{M}^0_{G_n}$, $n \in \mathbb{N}$, and $\varepsilon > 0$, there exists a $y \in \mathcal{M}^0_{G_n}$, with pseudo-derivative u_y, satisfying (j′) and (jj) and such that $\|y - x\|_I < \varepsilon$.*

Proof. Consider an arbitrary $x \in \mathcal{M}^0_{G_n}$, with pseudo-derivative u_x and corresponding partition $\Delta = \{A_i\}_{i=1}^M$ as in (j), and let $\varepsilon > 0$. As G_n is h-continuous and u_x satisfies (j) and (jj), then, by taking a partition $\Delta' = \{B_k\}_{k=1}^N$ of I with step sufficiently small and modifying, if necessary, the values of u_x only on those intervals $B_k \in \Delta'$ that contain a point a_i of the partition Δ, one can easily construct a $y \in \mathcal{M}^0_{G_n}$ with pseudo-derivative u_y, so that $\|y - x\|_I < \varepsilon$ and, moreover, (j′) and (jj) are satisfied. This completes the proof. □

Proposition 3.2. *Each $x \in \mathcal{M}$ is given by*

$$x(t) = T(t - t_0)a + \int_{t_0}^{t} T(t - s)u_x(s)\,\mathrm{d}s, \quad t \in I,$$

where u_x, the pseudo-derivative of x, is unique in the $L^\infty(I, \mathbb{E})$ sense and satisfies $\|u_x(t)\| \leqslant r$, $t \in I$ a.e. (r is the constant in (h_3).)

Proof. Let $x \in \mathcal{M}$. By definition of $\mathcal{M}$ there exists a sequence $\{x_n\}$ of mild solutions $x_n \in \bigcup_{n=1}^{\infty} \mathcal{M}^0_{G_n}$, with pseudo-derivative u_{x_n}, such that $x_n \to x$ as $n \to \infty$. Since $\|u_{x_n}(t)\| \leqslant r, t \in I$ a.e., the sequence $\{u_{x_n}\}$ is contained in a closed ball of $L^2(I, \mathbb{E})$. As $L^2(I, \mathbb{E})$ is reflexive, there exists a subsequence, say $\{u_{x_n}\}$, which converges weakly in $L^2(I, \mathbb{E})$ (and thus in $L^1(I, \mathbb{E})$) to some $\omega \in L^2(I, \mathbb{E})$. Then, by virtue of Mazur's Theorem, one has $\|\omega(t)\| \leqslant r$, $t \in I$ a.e., and, moreover,

$$x(t) = T(t - t_0)a + \int_{t_0}^{t} T(t - s)\omega(s)\,\mathrm{d}s, \quad t \in I.$$

From the latter, setting $u_x = \omega$, the statement follows, completing the proof. □

It is worth noting that an $x \in \mathcal{M}$ is in $\mathcal{M}_F$ if and only if x has pseudo-derivative u_x satisfying $u_x(t) \in F(t, x(t))$, $t \in I$ a.e., and thus it can happen that the set $\mathcal{M} \setminus \mathcal{M}_F$ is non-empty. However, most (in the sense of the Baire's category) $x \in \mathcal{M}$ are actually in $\mathcal{M}_F$, as is shown by Theorem 3.6.

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Proposition 3.3. *$\mathcal{M}$ is a complete metric space.*

Proof. Let us prove that $\mathcal{M}$ is non-empty, the completeness being obvious. To this end, it suffices to show that $\mathcal{M}^0_{G_2} \neq \emptyset$. Let $g\colon I \times \mathbb{E} \to \mathbb{E}$ be a locally Lipschitzian selection of $G_1\colon I \times \mathbb{E} \to \mathcal{B}(\mathbb{E})$. Clearly, for each $(t,x) \in I \times \mathbb{E}$, we have

$$g(t,x) + \theta_1(t,x)B \subset G_2(t,x),$$

for $G_1(t,x) + \theta_1(t,x)B \subset G_2(t,x)$. Since g is locally Lipschitzian and bounded, the Cauchy problem

$$\dot{x}(t) = Ax(t) + g(t,x(t)), \quad x(t_0) = a,$$

has a mild solution $x\colon I \to \mathbb{E}$ given by

$$x(t) = T(t-t_0)a + \int_{t_0}^{t} T(t-s)g(s,x(s))\,\mathrm{d}s, \quad t \in I.$$

Moreover, for some $0 < \theta < \min_{t\in I} \theta_1(t,x(t))$, we have

$$g(t,x(t)) + 2\theta B \subset G_2(t,x(t)), \quad t \in I.$$

Since g and G_2 are continuous on the graph of x, a compact set, by Lebesgue's covering lemma there exists a $\delta > 0$ such that $t \in I$ and $\|z - x(t)\| < \delta$ imply

$$g(t,z) + \theta B \subset G_2(t,z), \tag{3.1}$$

$$\|g(t,z) - g(t,x(t))\| < \tfrac{1}{4}\theta. \tag{3.2}$$

Using the notation in (j), let $\Delta = \{A_i\}_{i=1}^{M}$ be a partition of I with step $\lambda(\Delta)$, and set

$$\omega(t) = \sum_{i=1}^{M} u_i \chi_{A_i}(t), \quad \text{where } u_i = g(a_{i-1}, x(a_{i-1})),\ t \in I,$$

$$z(t) = T(t-t_0)a + \int_{t_0}^{t} T(t-s)\omega(s)\,\mathrm{d}s, \quad t \in I.$$

Now fix the step $\lambda(\Delta)$ small enough to have

$$\|\omega(t) - g(t,x(t))\| < \tfrac{1}{4}\theta, \quad t \in I, \tag{3.3}$$

$$\|z(t) - x(t)\| < \delta, \quad t \in I. \tag{3.4}$$

From (3.1) and (3.2), by virtue of (3.4), it follows that

$$g(t,z(t)) + \theta B \subset G_2(t,z(t)), \quad t \in I, \tag{3.5}$$

$$\|g(t,z(t)) - g(t,x(t))\| < \tfrac{1}{4}\theta, \quad t \in I. \tag{3.6}$$

Moreover, (3.3) and (3.6) imply $\omega(t) \in g(t,z(t)) + \frac{1}{2}\theta B$ and hence, by (3.5),

$$\omega(t) + \tfrac{1}{2}\theta B \subset G_2(t,z(t)), \quad t \in I.$$

Therefore, z is a mild solution of (C_{G_2}), with pseudo-derivative $u_z = \omega$, satisfying (j) and (jj), and so $z \in \mathcal{M}^0_{G_2}$. This completes the proof. □

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By hypothesis, the Banach space $\mathbb{E}$ is reflexive and separable, and hence its dual $\mathbb{E}^*$ is also reflexive and separable. Consequently, $L^2(I, \mathbb{E}^*)$ is separable, and thus it contains a sequence $\{f_k\}_{k=1}^{\infty}$ that is dense in $L^2(I, \mathbb{E}^*)$. Denote by $\|\cdot\|_2$ the norm of $L^2(I, \mathbb{E}^*)$, and by $\langle\cdot\,,\cdot\rangle$ the pairing between $\mathbb{E}$ and $\mathbb{E}^*$.

For $k \in \mathbb{N}$ and $\alpha > 0$ set

$$\mathcal{M}_\alpha^k = \left\{ x \in \mathcal{M} \;\middle|\; d\biggl(\int_I \langle f_k(t), u_x(t)\rangle \,\mathrm{d}t, \int_I \langle f_k(t), F(t, x(t))\rangle \,\mathrm{d}t \biggr) < \alpha \right\},$$

where u_x corresponds to x according to Proposition 3.2, and

$$\int_I \langle f_k(t), F(t, x(t))\rangle \,\mathrm{d}t = \left\{ \int_I \langle f_k(t), y(t)\rangle \,\mathrm{d}t \;\middle|\; y(t) \in F(t, x(t)) \text{ a.e., } y \text{ measurable} \right\}.$$

Lemma 3.4. *$\mathcal{M}_\alpha^k$ is dense in $\mathcal{M}$.*

Proof. Let $z \in \mathcal{M}$ and $\varepsilon > 0$ be arbitrary. Take $x \in \mathcal{M}_{G_n}^0$, for some $n \in \mathbb{N}$ so that $\|x - z\|_I < \varepsilon$, and let u_x correspond. Since $u_x(t) \in G_n(t, x(t)) \subset F(t, x(t))$, we have

$$d\biggl(\int_I \langle f_k(t), u_x(t)\rangle \,\mathrm{d}t, \int_I \langle f_k(t), F(t, x(t))\rangle \,\mathrm{d}t \biggr) = 0.$$

Hence $x \in \mathcal{M}_\alpha^k$, completing the proof. □

Lemma 3.5. *$\mathcal{M}_\alpha^k$ is open in $\mathcal{M}$.*

Proof. Let $\{x_n\} \subset \mathcal{M} \setminus \mathcal{M}_\alpha^k$ be a sequence converging to an $x \in \mathcal{M}$ and let $\{u_{x_n}\} \subset L^2(I, \mathbb{E})$ correspond, according to Proposition 3.2. Since $\{u_{x_n}\}$ is uniformly bounded and $L^2(I, \mathbb{E})$ is reflexive, passing to a subsequence (without changing notation), we have that $u_{x_n} \to u_x$ weakly in $L^2(I, \mathbb{E})$.

Let $\varepsilon > 0$ be arbitrary. Fix σ so that

$$0 < \sigma < \frac{\varepsilon}{1 + 2(r + \sqrt{|I|})\|f_k\|_2}, \tag{3.7}$$

where $|I| = t_2 - t_1$ and r is the constant in assumption (h_3).

We claim that there exists $n_0 \in \mathbb{N}$ such that $n > n_0$ implies

$$\int_I \langle f_k(t), F(t, x(t))\rangle \,\mathrm{d}t \subset \int_I \langle f_k(t), F(t, x_n(t))\rangle \,\mathrm{d}t + \varepsilon U. \tag{3.8}$$

Indeed, by virtue of Proposition 2.2, there exists a compact $J \subset I$, with $|I \setminus J| < \sigma^2$, such that $F(\cdot\,, x(\cdot))|_J$ is h-continuous. Let $\tau \in J$ be arbitrary. Since $F(\cdot\,, x(\cdot))|_J$ is h-continuous at τ and F is h-l.s.c. at $(\tau, x(\tau))$, there exist $\delta_\tau > 0$ and $n_\tau \in \mathbb{N}$ such that $t \in J_{\tau,\delta_\tau} \cap J$ and $n > n_\tau$ imply

$$h(F(t, x(t)), F(\tau, x(\tau))) < \sigma, \quad F(\tau, x(\tau)) \subset F(t, x_n(t)) + \sigma B,$$

and hence

$$F(t, x(t)) \subset F(t, x_n(t)) + 2\sigma B. \tag{3.9}$$

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Now $\{J_{\tau,\delta_\tau}\}_{\tau\in J}$ is an open covering of J, and thus it admits a finite subcovering, say $\{J_{\tau_i,\delta_{\tau_i}}\}_{i=1}^N$. Setting $n_0=\max\{n_{\tau_1},\dots,n_{\tau_N}\}$ it follows that (3.9) holds for any $n>n_0$ and all $t\in J$. Thus, for every $n>n_0$ we have

$$\int_J \langle f_k(t), F(t,x(t))\rangle\,\mathrm{d}t \subset \int_J \langle f_k(t), F(t,x_n(t))\rangle\,\mathrm{d}t + 2\sigma\sqrt{|I|}\|f_k\|_2 U. \tag{3.10}$$

On the other hand, in view of assumption (h_3), as $|I\setminus J|<\sigma^2$, one can easily show that

$$\int_I \langle f_k(t), F(t,x(t))\rangle\,\mathrm{d}t \subset \int_J \langle f_k(t), F(t,x(t))\rangle\,\mathrm{d}t + \sigma r\|f_k\|_2 U \tag{3.11}$$

and

$$\int_J \langle f_k(t), F(t,x_n(t))\rangle\,\mathrm{d}t \subset \int_I \langle f_k(t), F(t,x_n(t))\rangle\,\mathrm{d}t + \sigma r\|f_k\|_2 U, \quad n\in\mathbb{N}. \tag{3.12}$$

From (3.11), by virtue of (3.10) and (3.12), for every $n>n_0$ we have

$$\int_I \langle f_k(t), F(t,x(t))\rangle\,\mathrm{d}t \subset \int_I \langle f_k(t), F(t,x_n(t))\rangle\,\mathrm{d}t + 2\sigma(r+\sqrt{|I|})\|f_k\|_2 U,$$

which yields (3.8), as σ satisfies (3.7).

We are now ready to finish the proof of the lemma. Indeed, as $u_{x_n}\to u_x$ weakly in $L^2(I,\mathbb{E})$, there exists $n_1>n_0$ such that $n>n_1$ implies

$$\left|\int_I \langle f_k(t), u_{x_n}(t)\rangle\,\mathrm{d}t - \int_I \langle f_k(t), u_x(t)\rangle\,\mathrm{d}t\right| < \varepsilon. \tag{3.13}$$

For a fixed $n>n_1$, by virtue of (3.13) and (3.8) we have

$$\begin{aligned}
d\biggl(\int_I \langle f_k(t), u_x(t)\rangle\,\mathrm{d}t, &\int_I \langle f_k(t), F(t,x(t))\rangle\,\mathrm{d}t\biggr)\\
&\geqslant d\biggl(\int_I \langle f_k(t), u_{x_n}(t)\rangle\,\mathrm{d}t, \int_I \langle f_k(t), F(t,x(t))\rangle\,\mathrm{d}t\biggr) - \varepsilon\\
&\geqslant d\biggl(\int_I \langle f_k(t), u_{x_n}(t)\rangle\,\mathrm{d}t, \int_I \langle f_k(t), F(t,x_n(t))\rangle\,\mathrm{d}t + \varepsilon U\biggr) - \varepsilon\\
&\geqslant d\biggl(\int_I \langle f_k(t), u_{x_n}(t)\rangle\,\mathrm{d}t, \int_I \langle f_k(t), F(t,x_n(t))\rangle\,\mathrm{d}t\biggr) - \varepsilon - \varepsilon\\
&\geqslant \alpha - 2\varepsilon,
\end{aligned}$$

since $x_n\in\mathcal{M}\setminus\mathcal{M}_\alpha^k$. As $\varepsilon>0$ is arbitrary, it follows that $x\in\mathcal{M}\setminus\mathcal{M}_\alpha^k$. This completes the proof. □

In a complete metric space, the complement of any set of Baire first category is called a *residual set*.

Theorem 3.6. *The set $\tilde{\mathcal{M}}=\mathcal{M}\cap\mathcal{M}_F$ is residual in $\mathcal{M}$.*

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Proof. By virtue of Lemmas 3.4 and 3.5, the set

$$\mathcal{M}^* = \bigcap_{k=1}^{\infty} \bigcap_{h=1}^{\infty} \mathcal{M}^k_{\alpha_h}, \quad \alpha_h = 1/h$$

is residual in $\mathcal{M}$, which is a complete metric space by Proposition 3.3. Therefore, to prove the theorem it suffices to show that

$$\mathcal{M}^* \subset \mathcal{M}_F, \tag{3.14}$$

since in this case we have $\mathcal{M}^* \subset \tilde{\mathcal{M}} \subset \mathcal{M}$, which implies that $\tilde{\mathcal{M}}$ is residual in $\mathcal{M}$. Suppose, on the contrary, that (3.14) does not hold, and take an $x \in \mathcal{M}^*$ such that $x \notin \mathcal{M}_F$. Clearly, $x \in \mathcal{M} \setminus \mathcal{M}_F$ and thus its pseudo-derivative u_x, which exists by Proposition 3.2, satisfies

$$u_x(t) \notin F(t, x(t)), \quad t \in J_0,$$

for some measurable set $J_0 \subset I$, with $|J_0| > 0$. By virtue of Luzin's Theorem and Proposition 2.2, there exists a compact $J \subset J_0$, with $|J| > 0$, such that $u_x|_J$ and $F(\cdot\,, x(\cdot))|_J$ are continuous. Let $\tau \in J$ be a point of density of J. By the Hahn–Banach Theorem there exists an $x^* \in \mathbb{E}^*$ separating $u_x(\tau)$ and $F(\tau, x(\tau))$, i.e.

$$\langle x^*, u_x(\tau)\rangle \geqslant c + \varepsilon, \qquad \langle x^*, F(\tau, x(\tau))\rangle \leqslant c - \varepsilon,$$

for some constants c and ε, $\varepsilon > 0$. Hence, for every $t \in J_{\tau,\delta} \cap J$ with $\delta > 0$ sufficiently small, we have

$$\langle x^*, u_x(t)\rangle \geqslant c + \tfrac{1}{2}\varepsilon, \qquad \langle x^*, F(t, x(t))\rangle \leqslant c - \tfrac{1}{2}\varepsilon. \tag{3.15}$$

Let $f \in L^2(I, \mathbb{E}^*)$ be given by

$$f(t) = \begin{cases} x^*, & t \in J_{\tau,\delta} \cap J, \\ 0, & t \in I \setminus (J_{\tau,\delta} \cap J). \end{cases}$$

By virtue of (3.15), which is valid for $t \in J_{\tau,\delta} \cap J$, a set of strictly positive measure, we have

$$d\biggl(\int_I \langle f(t), u_x(t)\rangle \,\mathrm{d}t, \int_I \langle f(t), F(t, x(t))\rangle \,\mathrm{d}t \biggr) = \lambda > 0$$

and hence

$$d\biggl(\int_I \langle f_k(t), u_x(t)\rangle \,\mathrm{d}t, \int_I \langle f_k(t), F(t, x(t))\rangle \,\mathrm{d}t \biggr) > \tfrac{1}{2}\lambda, \tag{3.16}$$

for some f_k sufficiently close to f in the $L^2(I, \mathbb{E}^*)$ norm. In view of (3.16) it follows that $x \notin \mathcal{M}^k_{\alpha_h}$, if $\alpha_h < \frac{1}{2}\lambda$, and hence $x \notin \mathcal{M}^*$: a contradiction. Consequently, (3.14) is valid. This completes the proof. □

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In the following example we present an h-l.s.c. differential inclusion of the form

$$\dot{x}(t) \in F(x(t)), \qquad x(0) = 0, \tag{P}$$

whose solution set $\mathcal{M}_F$ has the property that $\bar{\mathcal{M}}_F \setminus \mathcal{M}_F$ is dense in $\bar{\mathcal{M}}_F$ (closure in the metric of uniform convergence).

Example 3.7. Consider the Cauchy problem (P), where $F\colon \mathbb{R} \to \mathcal{B}(\mathbb{R})$ is given by

$$F(x) = \begin{cases} [0, 1], & x \neq 0, \\ [\frac{1}{2}, 1], & x = 0, \end{cases}$$

and denote by $\mathcal{M}_F$ the space of all solutions of (P) defined on $[0, 1]$.

The set $\mathcal{M}_F$ is not closed, since, for each $n \in \mathbb{N}$, $x_n(t) = t/n$ is in $\mathcal{M}_F$, while its limit $x_0(t) \equiv 0$ is in $\bar{\mathcal{M}}_F \setminus \mathcal{M}_F$. Actually, $\bar{\mathcal{M}}_F \setminus \mathcal{M}_F$ is dense in $\bar{\mathcal{M}}_F$. To see this, let $y_0 \in \bar{\mathcal{M}}_F$ and $0 < \varepsilon < \frac{1}{2}$ be arbitrary. Take $y \in \mathcal{M}_F$ so that $\|y - y_0\|_I < \varepsilon$. Clearly, y is non-decreasing on $[0, 1]$ and satisfies $0 \leqslant y(t) \leqslant t$, $t \in [0, 1]$, since $0 \leqslant \dot{y}(t) \leqslant 1$, $t \in [0, 1]$ a.e. If $y(\tau) = 0$ for some $0 < \tau \leqslant 1$, then $y(t) = 0$ for all $t \in [0, \tau]$, which implies that $0 \in F(0)$: a contradiction. Hence $0 < y(t) \leqslant 1$ for each $t \in (0, 1]$. Now define $x\colon [0, 1] \to \mathbb{R}$ by

$$x(t) = \begin{cases} 0, & t \in [0, \varepsilon], \\ y(t - \varepsilon), & t \in (\varepsilon, 1], \end{cases}$$

and observe that $x \notin \mathcal{M}_F$ and $\|x - y\|_I \leqslant \varepsilon$. Moreover, $x \in \bar{\mathcal{M}}_F$. To see this, consider a strictly decreasing sequence $\{t_n\} \subset (\varepsilon, 2\varepsilon]$ converging to ε and define $z_n\colon [0, 1] \to \mathbb{R}$ by

$$z_n(t) = \begin{cases} \dfrac{x(t_n)}{t_n} t, & t \in [0, t_n], \\ x(t), & t \in (t_n, 1]. \end{cases}$$

Clearly, $x \in \bar{\mathcal{M}}_F$ for $z_n \in \mathcal{M}_F$ and $\{z_n\}$ converges to x. Hence $x \in \bar{\mathcal{M}}_F \setminus \mathcal{M}_F$. Since, in addition, $\|x - y_0\|_I \leqslant \|x - y\|_I + \|y - y_0\|_I < 2\varepsilon$, it follows that $\bar{\mathcal{M}}_F \setminus \mathcal{M}_F$ is dense in $\bar{\mathcal{M}}_F$.

4. The non-convex Cauchy problem

In this section, we use the results of § 3 to establish the existence of mild solutions to the following non-convex Cauchy problem

$$\dot{x}(t) \in Ax(t) + \partial F(t, x(t)), \qquad x(t_0) = a. \tag{$C_{\partial F}$}$$

Let A and $F\colon I \times \mathbb{E} \to \mathcal{B}(\mathbb{E})$ satisfy the assumptions (h_1)–(h_3). Using the notation of § 3, let $\{G_n\}$ be a sequence of h-continuous maps $G_n\colon I \times \mathbb{E} \to \mathcal{B}(\mathbb{E})$ and let $\{\theta_n\}$ be a sequence of continuous functions $\theta_n\colon I \times \mathbb{E} \to (0, \infty)$ satisfying, for each $(t, x) \in I \times \mathbb{E}$, the properties (c_1)–(c_3).

Fix $\rho > 2r$, where r is the constant in (h_3). Since $G_n(t,x) \subset F(t,x) \subset rB$ we have

$$F(t,x) \subset G_n(t,x) + \rho B \quad \text{for every } (t,x) \in I \times \mathbb{E},\ n \in \mathbb{N}.$$

Observe that the set

$$\tilde{\mathcal{M}} = \mathcal{M} \cap \mathcal{M}_F, \quad \text{where } \mathcal{M} = \overline{\bigcup_{n=1}^{\infty} \mathcal{M}^0_{G_n}},$$

is residual in $\mathcal{M}$, by Theorem 3.6.

For $n \in \mathbb{N}$, set

$$\mathcal{M}_n = \biggl\{ x \in \tilde{\mathcal{M}} \biggm| \frac{1}{|I|} \int_I d(u_x(t), \partial(G_n(t,x(t)) + \rho B))\,\mathrm{d}t < \rho \biggr\}.$$

Lemma 4.1. *$\mathcal{M}_n$ is open in $\tilde{\mathcal{M}}$.*

Proof. It is sufficient to show that if $\{x_k\} \subset \tilde{\mathcal{M}} \setminus \mathcal{M}_n$ is a sequence which converges uniformly to $x \in \tilde{\mathcal{M}}$, then $x \in \tilde{\mathcal{M}} \setminus \mathcal{M}_n$. Consider the sequence $\{u_{x_k}\}$, where u_{x_k} corresponds to x_k. Clearly, $\|u_{x_k}(t)\| \leqslant r$, $t \in I$ a.e., and thus $\{u_{x_k}\}$ is contained in a closed ball of $L^2(I,\mathbb{E})$. As $L^2(I,\mathbb{E})$ is reflexive there exists a subsequence, say $\{u_{x_k}\}$, which converges weakly in $L^2(I,\mathbb{E})$ (and so in $L^1(I,\mathbb{E})$) to some $\omega \in L^2(I,\mathbb{E})$. By Mazur's Theorem, there exists a sequence of convex combinations

$$\biggl\{ \sum_{i=n_k}^{n_{k+1}-1} \lambda_i^k u_{x_i} \biggr\}, \tag{4.1}$$

where $n_1 < n_2 < \cdots$, $\lambda_i^k \geqslant 0$, $\sum_{i=n_k}^{n_{k+1}-1} \lambda_i^k = 1$, which converges to ω in $L^1(I,\mathbb{E})$. For each $i \in \mathbb{N}$ we have

$$x_i(t) = T(t-t_0)a + \int_{t_0}^{t} T(t-s)u_{x_i}(s)\,\mathrm{d}s, \quad t \in I,$$

and hence

$$\sum_{i=n_k}^{n_{k+1}-1} \lambda_i^k x_i(t) = T(t-t_0)a + \int_{t_0}^{t} T(t-s)\biggl(\sum_{i=n_k}^{n_{k+1}-1} \lambda_i^k u_{x_i}(s) \biggr)\,\mathrm{d}s, \quad t \in I,$$

from which, letting $k \to \infty$, it follows that

$$x(t) = T(t-t_0)a + \int_{t_0}^{t} T(t-s)\omega(s)\,\mathrm{d}s, \quad t \in I. \tag{4.2}$$

On the other hand, since $x \in \tilde{\mathcal{M}} \subset \mathcal{M}_F$, we have

$$x(t) = T(t-t_0)a + \int_{t_0}^{t} T(t-s)u_x(s)\,\mathrm{d}s, \quad t \in I,$$

which combined with (4.2) implies that $\omega = u_x$.

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In order to prove that $x \in \tilde{\mathcal{M}} \setminus \mathcal{M}_n$ it remains to show that

$$\frac{1}{|I|}\int_I d(\omega(t), \partial(G_n(t,x(t)) + \rho B))\,\mathrm{d}t \geqslant \rho.$$

Since the multifunction $(t,z) \to \partial(G_n(t,z) + \rho B)$ is h-continuous on $I \times \mathbb{E}$ and the set $\Gamma = \{(t,x(t)) \mid t \in I\} \subset I \times \mathbb{E}$ is compact, there exists a $\delta > 0$ such that

$$h(\partial(G_n(t,z) + \rho B), \partial(G_n(t,x(t)) + \rho B)) < \varepsilon/|I|$$

for every $t \in I$ and $\|z - x(t)\| < \delta$. As $x_k \to x$ uniformly on I, it follows that for i large enough, say $i \geqslant \nu$, and $t \in I$ we have

$$h(\partial(G_n(t,x_i(t)) + \rho B), \partial(G_n(t,x(t)) + \rho B)) < \varepsilon/|I|. \tag{4.3}$$

On the other hand, the sequence of functions (4.1) converges to ω in $L^1(I,\mathbb{E})$, and thus, passing to a subsequence, without change of notation, we can assume that it converges to ω a.e. in I. Fix $k_0 > \nu$ so that $k \geqslant k_0$ implies

$$\int_I \biggl\| \sum_{i=n_k}^{n_{k+1}-1} \lambda_i^k u_{x_i}(t) - \omega(t) \biggr\| \,\mathrm{d}t < \varepsilon|I|. \tag{4.4}$$

In view of (4.4), taking into account the fact that, by Proposition 2.3, the function $u \to d(u, \partial(G_n(t,x(t)) + \rho B))$ is concave on $F(t,x(t))$, for each $k \geqslant k_0$ we have

$$\begin{aligned}
&\int_I d(\omega(t), \partial(G_n(t,x(t)) + \rho B))\,\mathrm{d}t \\
&\quad\geqslant \int_I d\biggl(\sum_{i=n_k}^{n_{k+1}-1} \lambda_i^k u_{x_i}(t), \partial(G_n(t,x(t)) + \rho B)\biggr)\,\mathrm{d}t - \int_I \biggl\| \sum_{i=n_k}^{n_{k+1}-1} \lambda_i^k u_{x_i}(t) - \omega(t) \biggr\|\,\mathrm{d}t \\
&\quad\geqslant \biggl\| \sum_{i=n_k}^{n_{k+1}-1} \lambda_i^k \int_I d(u_{x_i}(t), \partial(G_n(t,x(t)) + \rho B))\,\mathrm{d}t \biggr\| - \varepsilon|I| \\
&\quad\geqslant \biggl\| \sum_{i=n_k}^{n_{k+1}-1} \lambda_i^k \int_I d(u_{x_i}(t), \partial(G_n(t,x_i(t)) + \rho B))\,\mathrm{d}t \biggr\| - \varepsilon|I| \\
&\qquad\quad - \biggl\| \sum_{i=n_k}^{n_{k+1}-1} \lambda_i^k \int_I h(\partial(G_n(t,x_i(t)) + \rho B), \partial(G_n(t,x(t)) + \rho B))\,\mathrm{d}t \biggr\| \\
&\quad\geqslant (\rho - 2\varepsilon)|I|,
\end{aligned} \tag{4.5}$$

where the last inequality holds because the functions x_i are in $\tilde{\mathcal{M}} \setminus \mathcal{M}_n$ and satisfy (4.3), as $i \geqslant n_k \geqslant k_0 > \nu$. From (4.5), since $\varepsilon > 0$ is arbitrary, it follows that $x \in \tilde{\mathcal{M}} \setminus \mathcal{M}_n$. This completes the proof. □

Lemma 4.2. *$\mathcal{M}_n$ is dense in $\tilde{\mathcal{M}}$.*

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Proof. Let $x \in \tilde{\mathcal{M}}$ and $\varepsilon > 0$ be arbitrary. Take $\tilde{x} \in \mathcal{M}^0_{G_m}$, for some $m > n$, so that $\|\tilde{x} - x\|_I < \frac{1}{4}\varepsilon$. This is certainly possible since

$$\tilde{\mathcal{M}} \subset \mathcal{M} = \overline{\bigcup_{k=1}^{\infty} \mathcal{M}^0_{G_k}}$$

and $\mathcal{M}^0_{G_k} \subset \mathcal{M}^0_{G_{k+1}}$ for every $k \in \mathbb{N}$. Now $\tilde{x} \in \mathcal{M}^0_{G_m}$, and hence by Proposition 3.1 there exists a $y \in \mathcal{M}^0_{G_m}$ with pseudo-derivative u_y, so that (j′) and (jj) are satisfied and $\|\tilde{x} - y\|_I < \frac{1}{4}\varepsilon$. Consequently,

$$\|y - x\|_I < \tfrac{1}{2}\varepsilon. \tag{4.6}$$

Furthermore, we have

$$y(t) = T(t - t_0)a + \int_{t_0}^{t} T(t - s)u_y(s)\,\mathrm{d}s, \quad t \in I, \tag{4.7}$$

$$u_y(t) = \sum_{i=1}^{M} u_i \chi_{A_i}(t), \quad t \in I, \tag{4.8}$$

where $\Delta = \{A_i\}_{i=1}^M$ is a partition of I, with step $\lambda(\Delta) = |I|/M$, given by

$$A_i = [a_{i-1}, a_i), \quad i = 1, \dots, M - 1, \qquad A_M = [a_{M-1}, a_M]$$

and

$$a_i = t_0 + \frac{i}{M}(t_1 - t_0), \quad i = 0, \dots, M.$$

Moreover, for some $\theta_0 > 0$, we have

$$u_i + 2\theta_0 B \subset G_m(t, y(t)), \quad t \in \bar{A}_i,\ i = 1, \dots, M.$$

Our aim is to find a $z \in \mathcal{M}_n$ such that $\|z - y\|_I < \frac{1}{2}\varepsilon$. To this end we first construct a suitable function $\omega\colon I \to \mathbb{E}$ satisfying $\omega(t) \in \operatorname{int} G_{m+1}(t, y(t)) \setminus G_m(t, y(t)), t \in I$. Then, setting

$$z(t) = T(t - t_0)a + \int_{t_0}^{t} T(t - s)\omega(s)\,\mathrm{d}s, \quad t \in I,$$

we show that $z \in \mathcal{M}_n$ and $\|z - y\|_I < \frac{1}{2}\varepsilon$, and thus $\|z - x\|_I < \varepsilon$ by (4.6).

Step 1. Construction of ω and z.

Consider an arbitrary interval $A_i \in \Delta$ and let $\bar{A}_i = [a_{i-1}, a_i]$. For $e \in \mathbb{E} \setminus \{0\}$, set $r(\lambda) = u_i + \lambda e$, $\lambda \in \mathbb{R}$. For $t \in \bar{A}_i$ denote by $e^1_m(t)$, $e^1_{m+1}(t)$ (respectively, $e^2_m(t)$, $e^2_{m+1}(t)$) the points at which $r(\lambda)$, with $\lambda > 0$ (respectively, $\lambda < 0$), meets $\partial G_m(t, y(t)), \partial G_{m+1}(t, y(t))$. Each of the functions $e^h_m(\cdot), e^h_{m+1}(\cdot), h = 1, 2,$ is continuous, for $\partial G_m(\cdot\,, y(\cdot))$ and $\partial G_{m+1}(\cdot\,, y(\cdot))$ are continuous [**9**]. For $h = 1, 2$ set

$$f^h_{m+1}(t) = \tfrac{1}{2}(e^h_m(t) + e^h_{m+1}(t)), \quad t \in \bar{A}_i.$$

Evidently,

$$f^h_{m+1}(t) \in \operatorname{int} G_{m+1}(t, y(t)), \quad t \in \bar{A}_i,\ h = 1, 2,$$

and thus, in view of the continuity of the maps involved, for some $0 < \theta_1 < \theta_0$, we have

$$B[f^h_{m+1}(t), 3\theta_1] \subset G_{m+1}(t, y(t)) \setminus G_m(t, y(t)), \quad t \in \bar{A}_i,\ h = 1, 2. \tag{4.9}$$

There exists a $\delta_i > 0$ such that $t \in \bar{A}_i$ and $z \in B(y(t), \delta_i)$ imply

$$B[f^h_{m+1}(t), 2\theta_1] \subset G_{m+1}(t, z) \setminus G_m(t, z), \quad h = 1, 2. \tag{4.10}$$

Indeed, let $\tau \in \bar{A}_i$ be arbitrary. In view of (4.9), taking into account the h-continuity of G_m and G_{m+1} at $(\tau, y(\tau))$ and the continuity of f^1_{m+1} and f^2_{m+1} at τ, it follows that there is a δ_τ such that (4.10) holds for every $(t, z) \in (J_{\tau,\delta_\tau} \cap \bar{A}_i) \times B(y(\tau), \delta_\tau)$. Now $\{(J_{\tau,\delta_\tau} \cap \bar{A}_i) \times B(y(\tau), \delta_\tau)\}_{\tau \in \bar{A}_i}$ is an open covering of $\Gamma = \{(t, y(t)) \mid t \in \bar{A}_i\}$, a compact set, and thus it contains a finite subcovering, say

$$\{(J_{\tau_k,\delta_{\tau_k}} \cap \bar{A}_i) \times B(y(\tau_k), \delta_{\tau_k})\}_{k=1}^m.$$

Let δ_i be a Lebesgue number of this subcovering. Then (4.10) is satisfied, if $t \in \bar{A}_i$ and $z \in B(y(t), \delta_i)$, since for some $1 \leqslant k \leqslant m$ we have

$$(t, z) \in (J_{t,\delta_i} \cap \bar{A}_i) \times B(y(t), \delta_i) \subset (J_{\tau_k,\delta_{\tau_k}} \cap \bar{A}_i) \times B(y(\tau_k), \delta_{\tau_k}).$$

Now denote by $\Delta_i = \{B_{i,j}\}_{j=1}^{N_i}$ a partition of A_i, with step $\lambda(A_i) = |A_i|/N_i$, given by

$$B_{i,j} = [b_{i,j-1}, b_{i,j}), \quad j = 1, \dots, N_i - 1, \qquad B_{i,N_i} = [b_{i,N_i-1}, b_{i,N_i}),$$

where

$$b_{i,j} = a_{i-1} + \frac{j}{N_i}(a_i - a_{i-1}), \quad j = 0, 1, \dots, N_i.$$

In view of (4.9) and the uniform continuity of f^1_{m+1} and f^2_{m+1} on $\bar{A}_i$, there exists a partition of A_i, say $\Delta_i = \{B_{i,j}\}_{j=1}^{N_i}$, such that, setting

$$\phi^h_{m+1}(t) = \sum_{j=1}^{N_i} v^h_{i,j} \chi_{B_{i,j}}(t), \quad h = 1, 2, \tag{4.11}$$

where $v^h_{i,j} = f^h_{m+1}(b_{i,j-1})$, $j = 1, \dots, N_i$, $h = 1, 2$, we have

$$B[\phi^h_{m+1}(t), \theta_1] \subset G_{m+1}(t, z) \setminus G_m(t, z), \quad t \in \bar{A}_i,\ z \in B(y(t), \delta_i),\ h = 1, 2. \tag{4.12}$$

Fix σ as follows

$$0 < \sigma < \frac{\delta}{2L e^{\alpha|I|}(r + |I|)}, \quad \text{where } \delta = \min\{\tfrac{1}{2}\varepsilon, \delta_1, \dots, \delta_M\}, \tag{4.13}$$

and α, L are the constants in (2.1).

For each $B_{i,j} \in \Delta_i$ denote by $\Delta_{i,j} = \{C_{i,j,k}\}_{k=1}^{P_{i,j}}$ a partition of $B_{i,j}$, with step $\lambda(\Delta_{i,j}) = |B_{i,j}|/P_{i,j}$, given by

$$C_{i,j,k} = [c_{i,j,k-1}, c_{i,j,k}), \quad k = 1, \dots, P_{i,j} - 1, \qquad C_{i,j,P_{i,j}} = [c_{i,j,P_{i,j}-1}, c_{i,j,P_{i,j}}],$$

where

$$c_{i,j,k} = b_{i,j-1} + \frac{k}{P_{i,j}}(b_{i,j} - b_{i,j-1}), \quad k = 0, 1, \dots, P_{i,j}.$$

Now fix $\Delta_{i,j}$ with step

$$\lambda(\Delta_{i,j}) < \sigma \tag{4.14}$$

small enough so that, for any $k = 1, \dots, P_{i,j}$, $t \in C_{i,j,k}$ and $c_{i,j,k-1} \leqslant s \leqslant t$, we have

$$\|T(t-s)v_{i,j}^h - v_{i,j}^h\| < \sigma, \quad h = 1, 2, \tag{4.15}$$

and

$$\|T(t-s)u_i - u_i\| < \sigma. \tag{4.16}$$

Observe that the family $\mathcal{F}$ of all intervals $C_{i,j,k}$, i.e.

$$\mathcal{F} = \{C_{i,j,k} \mid 1 \leqslant i \leqslant M,\ 1 \leqslant j \leqslant N_i,\ 1 \leqslant k \leqslant P_{i,j}\},$$

is a partition of I, since $\Delta = \{A_i\}_{i=1}^M$, $\Delta_i = \{B_{i,j}\}_{j=1}^{N_i}$ and $\Delta_{i,j} = \{C_{i,j,k}\}_{k=1}^{P_{i,j}}$ are partitions of I, A_i and $B_{i,j}$, respectively. In view of (4.12) and the definition of ϕ_{m+1}^1 and ϕ_{m+1}^2, for each $t \in C_{i,j,k}$ we have

$$u_i \in (\phi_{m+1}^1(t), \phi_{m+1}^2(t)) = (v_{i,j}^1, v_{i,j}^2),$$

and hence there exist $\lambda_{i,j}^1, \lambda_{i,j}^2 \in (0,1)$ with $\lambda_{i,j}^1 + \lambda_{i,j}^2 = 1$ such that

$$u_i = \lambda_{i,j}^1 v_{i,j}^1 + \lambda_{i,j}^2 v_{i,j}^2. \tag{4.17}$$

Divide accordingly each interval $C_{i,j,k}$ into two intervals $C_{i,j,k}^1 = [c_{i,j,k-1}, \gamma)$ and $C_{i,j,k}^2 = [\gamma, c_{i,j,k})$ (or $C_{i,j,k}^2 = [\gamma, c_{i,j,k}]$ if $C_{i,j,k}$ is closed) so that

$$|C_{i,j,k}^h| = \lambda_{i,j}^h |C_{i,j,k}|, \quad h = 1, 2. \tag{4.18}$$

Now define $\omega\colon I \to \mathbb{E}$ and $z\colon I \to \mathbb{E}$ as follows:

$$\omega(t) = \sum_{i=1}^{M} \sum_{j=1}^{N_i} \sum_{k=1}^{P_{i,j}} (v_{i,j}^1\, \chi_{C_{i,j,k}^1}(t) + v_{i,j}^2\, \chi_{C_{i,j,k}^2}(t)), \quad t \in I, \tag{4.19}$$

$$z(t) = T(t-t_0)a + \int_{t_0}^{t} T(t-s)\omega(s)\,\mathrm{d}s, \quad t \in I. \tag{4.20}$$

Step 2. We have that z is a mild solution of $(C_{G_{m+1}})$, with pseudo-derivative $u_z = \omega$, so that $z \in \mathcal{M}_{G_{m+1}}^0$; moreover, $z \in \tilde{\mathcal{M}}$.

We first show that

$$\|z(t) - y(t)\| < \delta \quad \text{for every } t \in I. \tag{4.21}$$

Let $t \in (t_0, t_1]$ be arbitrary ((4.21) trivially holds for $t = t_0$). Take $t \in C_{p,q,r}$ for some $1 \leqslant p \leqslant M$, $1 \leqslant q \leqslant N_p$, $1 \leqslant r \leqslant P_{p,q}$. To fix the ideas we suppose that

$$t \in C_{p,q,r} = [c_{p,q,r-1}, c_{p,q,r}) \quad \text{and} \quad t_0 < c_{p,q,r-1}$$

(in the other cases the argument is similar). From (4.7) and (4.20) it follows that

$$z(t) - y(t) = \int_{t_0}^{t} T(t-s)(\omega(s) - u_y(s))\,\mathrm{d}s = J_1 + J_2, \tag{4.22}$$

where

$$J_1 = \int_{t_0}^{c_{p,q,r-1}} T(t-s)(\omega(s) - u_y(s))\,\mathrm{d}s,$$
$$J_2 = \int_{c_{p,q,r-1}}^{t} T(t-s)(\omega(s) - u_y(s))\,\mathrm{d}s.$$

Let $\mathcal{G}$ be the set of all intervals $C_{i,j,k} \in \mathcal{F}$ whose union is equal to $[t_0, c_{p,q,r-1})$.

Let us evaluate J_1. Clearly,

$$\begin{aligned} J_1 &= \sum_{C_{i,j,k}\in\mathcal{G}} \int_{C_{i,j,k}} T(t-s)(\omega(s) - u_y(s))\,\mathrm{d}s \\ &= \sum_{C_{i,j,k}\in\mathcal{G}} T(t - c_{i,j,k}) \int_{C_{i,j,k}} T(c_{i,j,k} - s)(\omega(s) - u_y(s))\,\mathrm{d}s, \end{aligned}$$

since $c_{i,j,k} \leqslant t$ for each $C_{i,j,k} \in \mathcal{G}$, and thus, by (2.1),

$$\|J_1\| \leqslant \sum_{C_{i,j,k}\in\mathcal{G}} L\mathrm{e}^{\alpha|I|} \bigg\| \int_{C_{i,j,k}} T(c_{i,j,k} - s)(\omega(s) - u_y(s))\,\mathrm{d}s \bigg\|. \tag{4.23}$$

Denote by Λ the integral on the right-hand side of (4.23). In view of (4.19) and (4.8), $\omega(s) = v_{i,j}^1 \chi_{C_{i,j,k}^1}(s) + v_{i,j}^2 \chi_{C_{i,j,k}^2}(s)$ and $u_y(s) = u_i$, $s \in C_{i,j,k}$. Thus, we have

$$\begin{aligned} \Lambda &= \int_{C_{i,j,k}^1} T(c_{i,j,k} - s) v_{i,j}^1\,\mathrm{d}s + \int_{C_{i,j,k}^2} T(c_{i,j,k} - s) v_{i,j}^2\,\mathrm{d}s - \int_{C_{i,j,k}} T(c_{i,j,k} - s) u_y(s)\,\mathrm{d}s \\ &= \int_{C_{i,j,k}^1} (T(c_{i,j,k} - s) v_{i,j}^1 - v_{i,j}^1)\,\mathrm{d}s + \int_{C_{i,j,k}^2} (T(c_{i,j,k} - s) v_{i,j}^2 - v_{i,j}^2)\,\mathrm{d}s \\ &\quad - \int_{C_{i,j,k}} (T(c_{i,j,k} - s) u_i - u_i)\,\mathrm{d}s + \int_{C_{i,j,k}} (v_{i,j}^1 \chi_{C_{i,j,k}^1}(s) + v_{i,j}^2 \chi_{C_{i,j,k}^2}(s) - u_i)\,\mathrm{d}s. \end{aligned}$$

Denote by Λ_1, Λ_2, Λ_3, Λ_4 the first, second, third and fourth integrals above, respectively. By virtue of (4.15) and (4.16) we have

$$\|\Lambda_1\| < \sigma |C_{i,j,k}^1|, \qquad \|\Lambda_2\| < \sigma |C_{i,j,k}^2|, \qquad \|\Lambda_3\| < \sigma |C_{i,j,k}|.$$

Moreover, $\Lambda_4 = 0$ because, in view of (4.17) and (4.18),

$$\Lambda_4 = v_{i,j}^1 |C_{i,j,k}^1| + v_{i,j}^2 |C_{i,j,k}^2| - u_i |C_{i,j,k}| = (\lambda_{i,j}^1 v_{i,j}^1 + \lambda_{i,j}^2 v_{i,j}^2 - u_i)|C_{i,j,k}| = 0.$$

Therefore, $\|\Lambda\| < 2\sigma |C_{i,j,k}|$. From this and (4.23) it follows that

$$\|J_1\| < 2\sigma L\mathrm{e}^{\alpha|I|} \sum_{C_{i,j,k}\in\mathcal{G}} |C_{i,j,k}| = 2\sigma L\mathrm{e}^{\alpha|I|}(c_{p,q,r-1} - t_0) < 2\sigma L\mathrm{e}^{\alpha|I|}|I|. \tag{4.24}$$

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Let us evaluate J_2. We have

$$\|J_2\| \leqslant \int_{c_{p,q,r-1}}^{t} \|T(t-s)\| \, \|\omega(s) - u_y(s)\| \, \mathrm{d}s \leqslant L\mathrm{e}^{\alpha|I|} 2r|C_{p,q,r}| < 2\sigma r L\mathrm{e}^{\alpha|I|}, \quad (4.25)$$

since $|C_{p,q,r}| < \sigma$, by (4.14).

From (4.22), (4.24) and (4.25), since $t \in I$ is arbitrary, it follows that

$$\|z(t) - y(t)\| < 2\sigma L\mathrm{e}^{\alpha|I|}(r + |I|), \quad t \in I,$$

and thus (4.21) is valid, as σ satisfies (4.13).

We are now ready to prove that $z \in \mathcal{M}^0_{G_{m+1}}$. It suffices to show that ω satisfies the conditions (j) and (jj) stated in § 3. By virtue of (4.19), ω verifies (j). Furthermore, (jj) holds if we show that

$$B(\omega(t), \theta_1) \subset G_{m+1}(t, z(t)), \quad t \in I. \quad (4.26)$$

Let $t \in I$ be arbitrary. Then $t \in C_{i,j,k}$, for some $1 \leqslant i \leqslant M$, $1 \leqslant j \leqslant N_i$, $1 \leqslant k \leqslant P_{i,j}$ and thus, by (4.19),

$$\omega(t) = v^1_{i,j}\chi_{C^1_{i,j,k}}(t) + v^2_{i,j}\chi_{C^2_{i,j,k}}(t).$$

As $C^1_{i,j,k}$ and $C^2_{i,j,k}$ are contained in $B_{i,j}$, in view of (4.19) we have

$$\omega(t) = v^h_{i,j} = \phi^h_{m+1}(t) \quad \text{for } t \in C^h_{i,j,k},\ h = 1, 2.$$

Moreover, $\|z(t) - y(t)\| < \delta$ by (4.21). Therefore, (4.12) implies

$$B(\omega(t), \theta_1) \subset G_{m+1}(t, z(t)) \setminus G_m(t, z(t)), \quad (4.27)$$

and hence (4.26) holds as $t \in I$ is arbitrary. Thus, z is a solution of $(C_{G_{m+1}})$, with pseudo-derivative $u_z = \omega$, satisfying (j) and (jj), and hence $z \in \mathcal{M}^0_{G_{m+1}}$. Furthermore, as $\mathcal{M}^0_{G_{m+1}} \subset \mathcal{M}_F$, we have that

$$z \in \overline{\left(\bigcup_{n=1}^{\infty} \mathcal{M}^0_{G_n}\right)} \cap \mathcal{M}_F = \tilde{\mathcal{M}},$$

and Step 2 is proved.

Step 3. We have $z \in \mathcal{M}_n$ and $\|z - x\|_I < \varepsilon$.

As $z \in \tilde{\mathcal{M}}$ and $u_z = \omega$, to prove that $z \in \mathcal{M}_n$ it suffices to show that

$$\frac{1}{|I|} \int_I d(\omega(t), \partial(G_n(t, z(t)) + \rho B)) \, \mathrm{d}t < \rho. \quad (4.28)$$

Indeed, since $m > n$ we have $G_n(t, x) \subset G_m(t, x) \subset G_{m+1}(t, x) \subset F(t, x)$, for all $(t, x) \in I \times \mathbb{E}$ and thus, by (4.27),

$$B(\omega(t), \theta_1) \subset F(t, z(t)) \setminus G_n(t, z(t)), \quad t \in I.$$

Hence, by Proposition 2.3, we have

$$d(\omega(t), \partial(G_n(t, z(t)) + \rho B)) < \rho, \quad t \in I,$$

and thus (4.28) holds. Therefore, $z \in \mathcal{M}_n$. On the other hand, by virtue of (4.6) and (4.21), we have

$$\|z - x\|_I \leqslant \|z - y\|_I + \|y - x\|_I < \delta + \tfrac{1}{2}\varepsilon < \varepsilon,$$

as $\delta < \frac{1}{2}\varepsilon$ by (4.13), and Step 3 is proved.

Since $x \in \tilde{\mathcal{M}}$ and $\varepsilon > 0$ are arbitrary and $z \in \mathcal{M}_n$, it follows that the set $\mathcal{M}_n$ is dense in $\tilde{\mathcal{M}}$. This completes the proof. □

We are now in a position to prove our main result concerning the existence of mild solutions to the following non-convex Cauchy problem:

$$\dot{x}(t) \in Ax(t) + \partial F(t, x(t)), \qquad x(t_0) = a. \tag{$C_{\partial F}$}$$

Theorem 4.3. *Let A and $F\colon I \times \mathbb{E} \to \mathcal{B}(\mathbb{E})$ satisfy the assumptions (h_1)–(h_3). Then the Cauchy problem $(C_{\partial F})$ has a mild solution $x\colon I \to \mathbb{E}$.*

Proof. Set

$$\mathcal{M}^* = \bigcap_{n=1}^{\infty} \mathcal{M}_n$$

and observe that

$$\mathcal{M}^* \subset \tilde{\mathcal{M}} \subset \mathcal{M},$$

where $\mathcal{M}$ is a non-empty complete metric space, by Proposition 3.3.

We first prove that $\mathcal{M}^*$ is dense in $\mathcal{M}$. Since

$$\mathcal{M} = \mathcal{M}^* \cup (\mathcal{M} \setminus \mathcal{M}^*),$$

it suffices to show that $\mathcal{M} \setminus \mathcal{M}^*$ is of the Baire first category in $\mathcal{M}$. It is evident that

$$\mathcal{M} \setminus \mathcal{M}^* = (\mathcal{M} \setminus \tilde{\mathcal{M}}) \cup \bigcup_{n=1}^{\infty} (\tilde{\mathcal{M}} \setminus \mathcal{M}_n).$$

Now $\mathcal{M} \setminus \tilde{\mathcal{M}}$ is of the Baire first category in $\mathcal{M}$, by Theorem 3.6. Furthermore, in view of Lemmas 4.1 and 4.2, each set $\tilde{\mathcal{M}} \setminus \mathcal{M}_n$ is nowhere dense in $\tilde{\mathcal{M}}$ and thus also in $\mathcal{M}$ [**27**, p. 129]. Consequently, $\mathcal{M} \setminus \mathcal{M}^*$ is of the Baire first category in $\mathcal{M}$ and hence, by the Baire Category Theorem, the set $\mathcal{M}^*$ is dense in $\mathcal{M}$ and, in particular, $\mathcal{M}^* \neq \emptyset$.

We now prove that $\mathcal{M}^* \subset \mathcal{M}_{\partial F}$. Let $x \in \mathcal{M}^*$. Clearly, $x \in \mathcal{M}_F$, for $\mathcal{M}^* \subset \tilde{\mathcal{M}} \subset \mathcal{M}_F$. Suppose, on the contrary, that $x \notin \mathcal{M}_{\partial F}$. This implies that the set

$$J_0 = \{t \in I \mid d(u_x(t), \partial F(t, x(t))) > 0\}$$

has measure $|J_0| > 0$ and hence, for some $\eta > 0$, also the set

$$J = \{t \in I \mid d(u_x(t), \partial F(t, x(t))) > \eta\}$$

has measure $|J| > 0$. It is evident that $d(u_x(t), \partial(F(t,x(t)) + \rho B)) \geqslant \rho$, for $t \in I$ a.e. Furthermore, for $t \in J$ a.e. we have $u_x(t) + \eta B \subset F(t,x(t))$ and hence $u_x(t) + (\eta+\rho)B \subset F(t,x(t)) + \rho B$, which implies that

$$d(u_x(t), \partial(F(t,x(t)) + \rho B)) \geqslant \rho + \eta, \quad t \in J \text{ a.e.}$$

Now, we have

$$\begin{aligned} \int_I d(u_x(t), \partial(F(t,x(t)) + \rho B))\,\mathrm{d}t &= \int_{I\setminus J} d(u_x(t), \partial(F(t,x(t)) + \rho B))\,\mathrm{d}t \\ &\quad + \int_J d(u_x(t), \partial(F(t,x(t)) + \rho B))\,\mathrm{d}t \\ &\geqslant \rho|I \setminus J| + (\rho+\eta)|J| \\ &= \rho|I| + \eta|J|. \end{aligned} \tag{4.29}$$

On the other hand, for any $t \in I$, as $n \to \infty$ we have $h(G_n(t,x(t))+\rho B, F(t,x(t))+\rho B) \to 0$ and thus, by [**9**], $h(\partial(G_n(t,x(t)) + \rho B), \partial(F(t,x(t)) + \rho B)) \to 0$. Consequently,

$$\lim_{n\to\infty} \int_I d(u_x(t), \partial(G_n(t,x(t)) + \rho B))\,\mathrm{d}t = \int_I d(u_x(t), \partial(F(t,x(t)) + \rho B))\,\mathrm{d}t.$$

Hence, for n large enough, say $n \geqslant n_0$, we have

$$\int_I d(u_x(t), \partial(G_n(t,x(t)) + \rho B))\,\mathrm{d}t > \int_I d(u_x(t), \partial(F(t,x(t)) + \rho B))\,\mathrm{d}t - \tfrac{1}{2}\eta|J|. \tag{4.30}$$

By combining (4.30) with (4.29) we obtain

$$\frac{1}{|I|}\int_I d(u_x(t), \partial(G_n(t,x(t)) + \rho B))\,\mathrm{d}t > \rho + \frac{\eta|J|}{2|I|}, \quad n \geqslant n_0.$$

This implies that $x \notin \mathcal{M}_n$: a contradiction, since $x \in \mathcal{M}^*$. Therefore, $x \in \mathcal{M}_{\partial F}$, i.e. x is a mild solution of $(C_{\partial F})$. This completes the proof. □

The following example shows that an h-l.s.c. multifunction can be discontinuous on a dense set of points.

Example 4.4. Let $X = \{x_n\}_{n=1}^{\infty}$ be a set dense in $\mathbb{E}$. Let $\phi\colon \mathbb{E} \to \mathcal{B}(\mathbb{E})$ be given by

$$\phi(x) = \begin{cases} \{0\} & \text{if } x = 0, \\ B & \text{if } x \in \mathbb{E} \setminus \{0\}. \end{cases}$$

Now define $F\colon \mathbb{E} \to \mathcal{B}(\mathbb{E})$ by

$$F(x) = \sum_{n=1}^{\infty} \frac{\phi(x - x_n)}{2^n}, \quad x \in \mathbb{E}.$$

Let us show that F is h-l.s.c. We have $F(x) = B$ if $x \in \mathbb{E} \setminus X$, and $F(x_k) = (1-2^{-k})B$ if $x_k \in X$. Let $\bar{x} \in \mathbb{E}$ and $\varepsilon > 0$ be arbitrary.

If $\bar{x} \in X$, say $\bar{x} = x_k$, then $F(\bar{x}) \subset F(x)$ for each $x \in \mathbb{E} \setminus X$ and $F(\bar{x}) \subset F(x_n)$ for all $n \geqslant k$. Hence, setting $\delta = \min_{1 \leqslant n \leqslant k}\{\|x_n - \bar{x}\|\}$, we have

$$F(\bar{x}) \subset F(x), \quad \text{for every } x \in B(\bar{x}, \delta). \tag{4.31}$$

Let $\bar{x} \in \mathbb{E} \setminus X$, and fix k_0 so that $2^{-k_0} < \varepsilon$. Then for every $n \geqslant k_0$ we have $F(x_n) = (1-2^{-n})B \supset (1-\varepsilon)B$, which implies $F(x_n) + \varepsilon B \supset B$, i.e. $F(x_n) + \varepsilon B \supset F(\bar{x})$. Clearly, $F(\bar{x}) = F(x)$ for each $x \in \mathbb{E} \setminus X$. Hence, setting $\delta = \min_{1 \leqslant n \leqslant k_0} \|x_n - \bar{x}\|$, we have

$$F(\bar{x}) \subset F(x) + \varepsilon B \quad \text{for every } x \in B(\bar{x}, \delta). \tag{4.32}$$

In view of (4.31) and (4.32) it follows that F is h-l.s.c. Moreover, F is not h-continuous at each $x_k \in X$, because $F(x_k) = (1-2^{-k})B$, while $F(x) = B$ for every $x \in \mathbb{E} \setminus X$.

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