ON A PROPERTY OF THE BOUNDARY CORRESPONDENCE UNDER QUASICONFORMAL MAPPINGS

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Let $w = f(z)$ be a quasiconformal mapping, in the sense of Pfluger [5]-Ahlfors [1], with maximal dilatation K, which will be simply referred to a K-QC mapping. As is well known, any K-QC mapping $w = f(z)$ of $\mathrm{Im}\, z > 0$ onto $\mathrm{Im}\, w > 0$ can be extended to a homeomorphism from $\mathrm{Im}\, z \geqq 0$ onto $\mathrm{Im}\, w \geqq 0$ and hence it transforms any set of logarithmic capacity zero on $\mathrm{Im}\, z = 0$ into a set with the same property on $\mathrm{Im}\, w = 0$.

According to Beurling-Ahlfors [2], there exist a set E of linear measure zero on $\mathrm{Im}\, z = 0$ and a K-QC mapping $w = f(z)$ of $\mathrm{Im}\, z > 0$ onto $\mathrm{Im}\, w > 0$ such that the image set $f(E)$ of E under $w = f(z)$ is of positive linear measure.

The purpose of this note is to prove the following theorem which is of some interest in contrast with the above theorem of Beurling-Ahlfors.

THEOREM. *There exists on the real axis a closed set E which is of linear measure zero and of positive logarithmic capacity and whose image set $f(E)$ under any K-QC mapping $w = f(z)$ of* $\mathrm{Im}\, z > 0$ *onto* $\mathrm{Im}\, w > 0$ *is of linear measure zero.*

1. Take a closed segment S_1 with length l_1 on the real axis and delete from S_1 an open segment T_1 with length $\frac{l_1}{p_1}$ $(p_1 > 1)$ such that the set $S_2 = S_1 - T_1$ consists of two closed segments $S_2^{(j)}$ $(j = 1, 2)$ with equal length l_2. In general, we delete from the set S_{m-1} open segments $T_{m-1}^{(j)}$ $(j = 1, 2, \ldots, 2^{m-2})$ such that each $T_{m-1}^{(j)}$ has length $\frac{l_{m-1}}{p_{m-1}}$ $(p_{m-1} > 1)$ and the set $S_m = S_{m-1} - \bigcup_{j=1}^{2^{m-2}} T_{m-1}^{(j)}$ consists of closed segments $S_m^{(j)}$ $(j = 1, 2, \ldots, 2^{m-1})$ with equal length l_m. It is obvious that the total length of the set S_m is $2^{m-1} l_m = 2^{m-2} l_{m-1} \left(1 - \frac{1}{p_{m-1}}\right) = l_1 \prod_{n=1}^{m-1} \left(1 - \frac{1}{p_n}\right)$, $S_m \subset S_{m-1}$ and $\bigcap_{m=1}^{\infty} S_m$ is a non-empty perfect closed

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set. We denote the Cantor set $\bigcap_{m=1}^{\infty} S_m$ by $E(p_1, p_2, \ldots)$.

If we put $p_n = \frac{e^n}{e^n - 1}$ (>1), $n = 1, 2, \ldots$, then it holds the following relations:

$$(\alpha) \qquad \prod_{n=1}^{\infty}\left(1-\frac{1}{p_n}\right)=0,$$

$$(\beta) \qquad \sum_{n=1}^{\infty}\frac{\log\left\{1\Big/\left(1-\frac{1}{p_n}\right)\right\}}{2^n}<\infty,$$

$$(\gamma) \qquad 2p_{n+1}>p_n-1,$$

because $\sum_{n=1}^{\infty}\frac{1}{p_n}=\sum_{n=1}^{\infty}\left(1-\frac{1}{e^n}\right)=\infty$, $\sum_{n=1}^{\infty}\left\{\log 1\Big/\left(1-\frac{1}{p_n}\right)\right\}\Big/2^n=\sum_{n=1}^{\infty}n/2^n<\infty$ and $2p_{n+1}-p_n+1=\{(2e^n-3)\,e^{n+1}+1\}/(e^{n+1}-1)\,(e^n-1)>0$. Hence we can see that the Cantor set $E(p_1, p_2, \ldots)$, where $p_n=\frac{e^n}{e^n-1}$, is of linear measure zero by (α) and is of positive logarithmic capacity by (β) (cf. R. Nevanlinna [4]).

Now, take the systems $R_n=\{R_n^{(j)}\}_{j=1}^{2^n}(n=1, 2, \ldots)$ constructed by Kuroda [3], consisting of concentric circular annuli such that the interior and exterior circles $C_n^{(j)}$ and $\Gamma_n^{(j)}$ of $R_n^{(j)}$ have the center at the middle point of $S_{n+1}^{(j)}$, and $C_n^{(j)}$ has the radius $r_n=\frac{l_n}{4}\left(1+\frac{1}{p_{n+1}}\right)\left(1-\frac{1}{p_n}\right)$ and $\Gamma_n^{(j)}$ has the radius $\rho_n=\frac{l_n}{4}\left(1+\frac{1}{p_n}\right)$. Then, it can be verified by using the preceding relation (γ) that the segment $S_{n+1}^{(j)}$ lies inside the interior circle $C_n^{(j)}$ of $R_n^{(j)}$ and the exterior circle $\Gamma_{n+1}^{(j)}$ of $R_{n+1}^{(j)}$ lies inside some one of the interior circles of R_n. Further, we obtain as to the modulus of $R_n^{(j)}$ that

$$(1) \qquad \operatorname{mod} R_n^{(j)}=\log\frac{1+\frac{1}{p_n}}{\left(1+\frac{1}{p_{n+1}}\right)\left(1-\frac{1}{p_n}\right)}$$
$$\geqq\log\frac{1}{2\left(1-\frac{1}{p_n}\right)}=n-\log 2,$$

which is valid for $j=1, 2, \ldots, 2^n$.

2. Let $w=f(z)$ be any K-QC mapping stated in our theorem. If we define $w=f(z)$ in $\operatorname{Im} z<0$ by $\overline{f(\bar{z})}$, then it is well known that $w=f(z)$ can be extended to a K-QC mapping in the whole plane. In this case, we may assume without loss of generality that the point at ∞ corresponds to each other under $w=f(z)$. Let E be the Cantor set $E(p_1, p_2, \ldots)$, where $p_n = \frac{e^n}{e^n-1}$ $(n=1, 2, \ldots)$, lying on $\mathrm{Im}\ z=0$ and let $R_n = \{R_n^{(j)}\}_{j=1}^{2^n}$ $(n=1, 2, \ldots)$ be the systems constructed in **1.** Denote by $\tilde{E}$ and $\tilde{R}_n = \{\tilde{R}_n^{(j)}\}_{j=1}^{2^n}$ $(n=1, 2, \ldots)$ the images of E and $R_n = \{R_n^{(j)}\}_{j=1}^{2^n}$ $(n=1, 2, \ldots)$ under $w=f(z)$ respectively. Then it is evident that $\tilde{E}$ is also closed and that $\tilde{R}_n (n=1, 2, \ldots)$ are the systems of doubly connected domains separating $w=\infty$ from $\tilde{E}$ in the domain obtained by excluding $\tilde{E}$ from the whole w-plane. It is well known that

$$\frac{1}{K}\ \mathrm{mod}\ R_n^{(j)} \leqq \mathrm{mod}\ \tilde{R}_n^{(j)}\ (j=1, 2, \ldots, 2^n).$$

Next, denote by $\tilde{C}_n^{(j)}$ and $\tilde{\Gamma}_n^{(j)}$ the images of $C_n^{(j)}$ and $\Gamma_n^{(j)}$ under $w=f(z)$. Then, $\tilde{C}_n^{(j)}$ and $\tilde{\Gamma}_n^{(j)}$ are the interior contour and the exterior contour bounding $\tilde{R}_n^{(j)}$ and are both symmetric with respect to the real axis $\mathrm{Im}\ w=0$. Moreover, we denote by $\tilde{r}_n^{(j)}$ the largest distance of the image point of the center of $R_n^{(j)}$ from the contour $\tilde{C}_n^{(j)}$, and denote by $\tilde{\rho}_n^{(j)}$ the smallest distance from the contour $\tilde{\Gamma}_n^{(j)}$. Then, by Teichmüller's theorem [6], we have

$$\mathrm{mod}\ \tilde{R}_n^{(j)} \leqq \log \Psi\left(\frac{\tilde{\rho}_n^{(j)}}{\tilde{r}_n^{(j)}}\right) \qquad (j=1, 2, \ldots, 2^n),$$

where $\log \Psi(P)$ is the modulus of Teichmüller's extremal domain whose two complementary continua are $\{w;\ -1 \leqq Re\ w \leqq 0,\ \mathrm{Im}\ w=0\}$ and $\{w;\ P \leqq Re\ w \leqq +\infty,\ \mathrm{Im}\ w=0\}$.

From the two relations stated above, we have

$$(2) \qquad \frac{1}{K}\ \mathrm{mod}\ R_n^{(j)} \leqq \log \Psi\left(\frac{\tilde{\rho}_n^{(j)}}{\tilde{r}_n^{(j)}}\right) \qquad (j=1, 2, \ldots, 2^n).$$

3. First, note that if $P \geqq 8+6\sqrt{2}$, then $16P+8 \leqq P^2$, and hence $\Psi(P) < P^2$ from Teichmüller's inequality $\Psi(P) < 16P+8$.

Now, put $\log \Psi(8+6\sqrt{2}) = m_0$. For any $K (1 \leqq K < \infty)$, choose an integer n_K which is not less that $Km_0 + \log 2$. Then, for $n \geqq n_K$,

$$(3) \qquad m_0 \leqq \frac{1}{K}(n - \log 2).$$

If we combine (3) with (1) and (2), then we have for $n \geqq n_K$,

$$m_0 = \log \Psi(8+6\sqrt{2}) \leqq \log \Psi\left(\frac{\tilde{\rho}_n^{(j)}}{\tilde{r}_n^{(j)}}\right) \qquad (j=1, 2, \ldots, 2^n).$$

Since $\Psi(P)$ is an increasing function of P, it follows that $8+6\sqrt{2} \leqq \tilde{\rho}_n^{(j)}/\tilde{r}_n^{(j)}$. Therefore, we obtain for $n \geqq n_K$,

$$m_0 \leqq \log \Psi\left(\frac{\tilde{\rho}_n^{(j)}}{\tilde{r}^{(j)}}\right) < 2 \log \frac{\tilde{\rho}_n^{(j)}}{\tilde{r}_n^{(j)}},$$

or

$$\tilde{r}_n^{(j)} e^{m_0/2} \leqq \tilde{\rho}_n^{(j)} \qquad (j=1, 2, \ldots, 2^n).$$

Summing up these, we have, for $n \geqq n_K$,

$$e^{m_0/2} \leqq \tilde{\rho}_n/\tilde{r}_n,$$

where $\tilde{\rho}_n = \sum_{j=1}^{2^n} \tilde{\rho}_n^{(j)}$ and $\tilde{r}_n = \sum_{j=1}^{2^n} r_n^{(j)}$.

Further, by a geometric consideration it is not difficult to see that

$$\tilde{\rho}_n \leqq \tilde{r}_{n-1}.$$

Consequently, we have, for $n \geqq n_K$,

$$e^{m_0/2} \leqq \tilde{r}_{n-1}/\tilde{r}_n,$$

so that

$$\prod_{n=n_K}^{N} e^{m_0/2} \leqq \prod_{n=n_K}^{N} \frac{\tilde{r}_{n-1}}{\tilde{r}_n},$$

or

$$e^{m_0/2(N-n_K+1)} \leqq \frac{\tilde{r}_{n_K-1}}{\tilde{r}_N}.$$

Thus we have $\lim_{N\to\infty} \tilde{r}_N = 0$. This shows that the image set $\tilde{E}$ of E is of linear measure zero, and hence the Cantor set $E = E(p_1, p_2, \ldots)$, where $p_n = \frac{e^n}{e^n-1}$, is a required example.

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