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Resolving maps which commute with a power of the shift

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Abstract. In this paper, we prove an extension of a theorem of Marcus, which says that every subshift of finite type of entropy log n, n an integer, factors onto the full n-shift. Let $p(x)$ be a monic polynomial, irreducible over $\mathbb{Q}$, whose coefficients (except for the leading coefficient) are non-positive integers. Suppose $C(\lambda)$ is the companion matrix of $p(x)$, where λ is the largest real root of $p(x)$ (λ exists, by the Perron-Frobenius theorem). Then for any aperiodic, non-negative, integral matrix A, with Perron value λ, we give necessary and sufficient conditions for the existence of a positive integer n and a right-closing map $f:\Sigma_A\rightarrow\Sigma_{C(\lambda)}$ satisfying $f\sigma^n=\sigma^n f$ (where σ is the shift map).

Section 1. *Preliminaries*

We begin by briefly outlining a few of the basic facts of symbolic dynamics. A more complete reference is found in [**1**].

Given an m by m matrix A whose entries are non-negative intgers, let $G(A)$ denote the directed graph with m nodes and A_{ij} labelled edges from node i to node j, for each pair i, j. (Throughout this paper, all matrices will be integral, with the exception of the Jordan matrices in lemma 5.) Let $\mathcal{S}$ be the set of edges of $G(A)$. We say that edge f *follows* edge e if e joins node i to node j, and f joins node j to node k, for some i, j, k. We then define the *subshift of finite type* (SFT) Σ_A to be $\{x\in\mathcal{S}^{\mathbb{Z}}: x_{i+1}$ follows x_i for all $i\in\mathbb{Z}\}$.

A non-negative matrix A is *irreducible* if, for each pair i, j, there exists a positive integer n with $A^n_{ij}>0$. If n can be chosen independently of i and j, we say that A is *aperiodic*.

At times, it is convenient to describe an SFT by a transition matrix A whose entries are 0's and 1's, and where the symbols are the nodes of $G(A)$. A node j follows a node i if there is an edge from i to j in $G(A)$. Clearly, an SFT given by an arbitrary non-negative matrix A (where the symbols are the edges of $G(A)$) can also be described by a 0–1 matrix: the edges of $G(A)$ become the nodes of a new graph and one node follows another in the obvious sense.

We will need the following parts of the Perron-Frobenius theorem:

Let A be a non-negative, irreducible matrix. Then

(i) *A has a real eigenvalue* λ, *corresponding to a strictly positive eigenvector, such that, if* γ *is any other eigenvalue, then* $|\gamma|\leq|\lambda|$. λ *is called the* *__Perron__ __value__* *of A.*

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(ii) *The eigenspace of* λ *is one dimensional.*

It is well known that $h(\Sigma_A, \sigma) = \log \lambda$ in this case, where $h(\Sigma_A, \sigma)$ denotes the topological entropy of (Σ_A, σ).

In [**5**], Williams gives a useful algebraic condition for two SFT's to be topologically conjugate. Two non-negative matrices A and B are called strong shift equivalent in one step if there are non-negative (not necessarily square) matrices U and V such that $A = UV$ and $B = VU$. Extending this relation transitively, we say that A and B are *strong shift equivalent* if there is a finite sequence of one-step equivalences leading from A to B. Williams proved that Σ_A and Σ_B are topologically conjugate if and only if A and B are strong shift equivalent.

If Σ_A and Σ_B are SFT's, we define a *factor map* to be a continuous, surjective map $f:\Sigma_A \to \Sigma_B$ such that $f\sigma = \sigma f$. (In this paper, all maps will be assumed to be continuous and surjective.) If such an f exists, we say that Σ_A *factors onto* Σ_B. It is natural to ask under what conditions Σ_A factors onto Σ_B. For the lower entropy case, $h(\Sigma_A, \sigma) > h(\Sigma_B, \sigma)$, Boyle [**2**] has given necessary and sufficient conditions for Σ_A to factor onto Σ_B. In the equal entropy case, $h(\Sigma_A, \sigma) = h(\Sigma_B, \sigma)$, no such conditions are known.

The first positive result in the equal entropy case was found by Marcus [**4**].

THEOREM. *Let* Σ_A *be an SFT of entropy* $\log n$ *(*n *a positive integer). Then* Σ_A *factors onto the full n-shift.*

In this paper, we prove an analogous theorem, in which the role of the full shift is replaced by the SFT defined by the companion matrix of an irreducible monic polynomial whose coefficients are non-positive integers (except for the leading coefficient). We do not know under what circumstances we can obtain maps which commute with the first power of the shift. The example of Kitchens, which we present in § 3, shows that we cannot obtain such maps in general. However, we give necessary and sufficient conditions for the existence of a special type of map which commutes with some power of the shift. The main result (theorem 1) should be regarded as an attempt to extend Marcus' theorem to the non-integer entropy case.

With this in mind, we wish to represent the dynamical system (Σ_A, σ^n) in terms of the matrix A. It is a fact that A^n_{ij} counts the number of paths in $G(A)$ of length n from i to j. We may think of A^n as representing a graph whose edges are labelled by paths of length n in $G(A)$. Applying the shift to Σ_{A^n} corresponds to applying the nth power of the shift to Σ_A; i.e. (Σ_{A^n}, σ) is conjugate to (Σ_A, σ^n). A factor map $f:(\Sigma_{A^n}, \sigma) \to (\Sigma_{B^n}, \sigma)$ may be regarded as a continuous map $\Sigma_A \to \Sigma_B$ which commutes with σ^n.

The main technique used in this paper is that of state splitting. Given a non-negative matrix A and a node (or state) s of $G(A)$, let E be the set of edges leading out of s. Given a partition of E into E_1 and E_2, we define a new matrix A' by splitting s into two new nodes s_1 and s_2 and defining new transitions (edges) as follows:

(1) e is an edge of $G(A')$ joining s_1 to j iff e joined s to j in $G(A)$ and $e \in E_1$.

(2) e is an edge of $G(A')$ joining s_2 to j iff e joined s to j in $G(A)$ and $e \in E_2$.

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(3) For any edge e joining i to s in $G(A)$, there are two new edges e_1 and e_2, joining i to s_1 and i to s_2 respectively.

These rules hold if e is not a loop. If e is a loop and (for example) $e \in E_1$, then e gives rise to two new edges, a loop at s_1 and an edge from s_1 to s_2. A similar rule holds if $e \in E_2$. If A' is obtained from A by state splitting, then A' is strong shift equivalent to A.

Next, we describe the effect of state splitting on a right eigenvector for A. Suppose $r = (r_1, \ldots, r_m)$ is a right eigenvector for A. Then a right eigenvector $r' = (r'_1, \ldots, r'_{m+1})$ for A' is given by $r'_i = r_i$ for $i \neq s$, and the two new entries corresponding to s_1 and s_2 are given by

$$r'_{s_1} = \frac{1}{\lambda} \sum_{\substack{e \text{ is an edge joining } s \text{ to } j \\ e \in E_1}} r_j,$$

$$r'_{s_2} = \frac{1}{\lambda} \sum_{\substack{e \text{ is an edge joining } s \text{ to } j \\ e \in E_2}} r_j.$$

Finally, we mention the idea of right closing maps. A factor map $f : \Sigma_A \to \Sigma_B$ is called *right closing* if it never identifies two negatively asymptotic points; i.e. if $x, y \in \Sigma_A$ with $f(x) = f(y)$, and there exists an integer N such that $x_n = y_n$ for $n \leq N$, then $x = y$.

A special case are the right resolving maps. A one block map $f : \mathcal{S} \to \mathcal{T}$ (the symbol sets of Σ_A and Σ_B) is right resolving if $f(x_1) = y_1$ and y_2 follows y_1 implies that there is a unique x_2 following x_1 with $f(x_2) = y_2$. If $f : \mathcal{S} \to \mathcal{T}$ is onto, then $f : \Sigma_A \to \Sigma_B$ is onto (use compactness).

Right closing maps are related to right resolving maps by the following theorem, due to Kitchens [3]. A factor map $f : \Sigma_A \to \Sigma_B$ is right closing if and only if there exists $\bar{A}$ such that $\Sigma_{\bar{A}}$ is topologically conjugate to Σ_A, and a right resolving map $g : \Sigma_{\bar{A}} \to \Sigma_B$ which makes the following diagram commute:

$$\begin{tikzcd}
\Sigma_A \arrow[r, leftrightarrow] \arrow[d, "f"'] & \Sigma_{\bar{A}} \arrow[dl, "g"] \\
\Sigma_B &
\end{tikzcd}$$

Given a right resolving map $f : \Sigma_A \to \Sigma_B$, we define the *relation matrix* R by

$$R_{xy} = \begin{cases} 1 & \text{if } f(x) = y \\ 0 & \text{otherwise.} \end{cases}$$

Since f is right resolving, it is easy to check that $AR = RA$.

Section 2.

Let $p(X) = X^d - \sum_{i=0}^{d-1} a_i X^i$, $a_i \in \mathbb{Z}$, $a_i \geq 0$, with $p(X)$ irreducible over $\mathbb{Q}$. Let λ be the real root of $p(X)$ of maximum modulus, and let $C(\lambda)$ be the companion matrix of $p(X)$. Assume $C(\lambda)$ is aperiodic. Then we have the following:

THEOREM 1. *Let A be an $m \times m$ aperiodic matrix with Perron value λ. There exists a positive integer n and a right closing map $f : \Sigma_A \to \Sigma_{C(\lambda)}$ such that $f\sigma^n = \sigma^n f$ if and only*

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if A has a right eigenvector $r=(r_1,\ldots,r_m)$, $r_i\in\mathbb{Z}[\lambda]$, such that the ideal in $\mathbb{Z}[1/\lambda]$ generated by the r_i is principal.

We will denote this ideal by $\langle r\rangle$. It will follow from lemma 2 below that if $\langle r\rangle$ is principal in $\mathbb{Z}[1/\lambda]$, and $s=(s_1,\ldots,s_m)$ is any other right eigenvector ($s_i\in\mathbb{Z}[\lambda]$), then $\langle s\rangle$ is principal in $\mathbb{Z}[1/\lambda]$.

COROLLARY. *If $\mathbb{Z}[1/\lambda]$ is a principal ideal domain, then for every aperiodic matrix A with Perron value λ there exists an integer n and a right closing map $f:\Sigma_A\to\Sigma_{C(\lambda)}$ such that $f\sigma^n=\sigma^n f$.*

An example is λ, the largest root of $p(X)=X^2-X-1$ (the golden mean). Here $\mathbb{Z}[1/\lambda]=\mathbb{Z}[\lambda]$, which is well known to be a principal ideal domain.

By Kitchens' theorem, the existence of a right closing map $f:\Sigma_A\to\Sigma_{C(\lambda)}$ such that $f\sigma^n=\sigma^n f$ is equivalent to saying that for some n there exists a matrix B, such that Σ_B is topologically conjugate to Σ_{A^n} and a right resolving map $g:\Sigma_B\to\Sigma_{C^n(\lambda)}$. The condition is further clarified by the following:

LEMMA 1. *There exists a right resolving map $g:\Sigma_B\to\Sigma_{C^n(\lambda)}$ if and only if B has a right eigenvector r (corresponding to λ^n) with entries in $\{1,\lambda,\ldots,\lambda^{d-1}\}$.*

Proof. Suppose B has a right eigenvector r with entries in $\{1,\lambda,\ldots,\lambda^{d-1}\}$. Let x be a state for B with $r_x=\lambda^i$. Let $F(x)$ denote the set of edges beginning at x, and for any edge y, let r_y denote the eigenvector entry corresponding to the state at which y terminates. Since

$$\sum_{y\in F(x)} r_y=\lambda^n r_x=\lambda^{n+i}=\sum_{j=0}^{d-1} b_j\lambda^j,$$

(where the last expression is the unique representation of λ^{n+i} in terms of $1,\lambda,\ldots,\lambda^{d-1}$), it follows that for each $j=0,1,\ldots,d-1$, there are exactly b_j edges joining x to a state with eigenvector entry λ^j. The same is true for i, the unique state for $C^n(\lambda)$ whose eigenvector entry is λ^i, since the same equation holds. So, for each x we can assign a one-to-one correspondence from $\{y\in F(x)\colon r_y=\lambda^j\}$ to $\{y\in F(i)\colon r_y=\lambda^j\}$. This extends to a right resolving map $g:\Sigma_B\to\Sigma_{C^n(\lambda)}$. The map is well-defined, since for each j, $0\le j\le d-1$, $C(\lambda)$ has exactly one state whose eigenvector entry is λ^j. Conversely, suppose there exists a right resolving map $g:\Sigma_B\to\Sigma_{C^n(\lambda)}$. Let B' be the 0-1 matrix obtained from B by converting edges to nodes, and C' the corresponding 0-1 matrix for C. Then there is an equation $B'R=RC'$, where R is the relation matrix for g. Now C' has an eigenvector v whose entries lie in $\{1,\lambda,\ldots,\lambda^{d-1}\}$, since C does. Since $B'(Rv)=RC'v=\lambda(Rv)$, Rv is an eigenvector for B'. Since R has exactly one 1 in each row, the entries of Rv lie in $\{1,\lambda,\ldots,\lambda^{d-1}\}$. It then follows that B' (and hence B) must have an eigenvector whose entries lie in $\{1,\lambda,\ldots,\lambda^{d-1}\}$. □

Thus, the existence of a right closing map $f:\Sigma_A\to\Sigma_{C(\lambda)}$ such that $f\sigma^n=\sigma^n f$ is equivalent to the existence of an integer n and a non-negative matrix B, with Σ_B topologically conjugate to Σ_{A^n}, such that B has a right eigenvector whose entries lie in $\{1,\lambda,\ldots,\lambda^{d-1}\}$. To prove the 'if' direction of theorem 1, we will apply state

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splitting to Σ_{A^n}, for sufficiently large n, to produce Σ_B as above. Roughly, to split a given state s, we will partition most of the n-blocks leading out of s into exactly the right proportions to produce the desired new eigenvector entries. We then use the equations which follow from the condition on the ideal generated by an eigenvector to distribute the remaining blocks so that the correct proportions are maintained. Since the details of this procedure are rather technical, we will defer them for the moment, and turn our attention to some preliminary lemmas.

The condition that $\langle r\rangle$ is principal also has a simple reformulation:

LEMMA 2. *Suppose A has a right eigenvector $r=(r_1,\ldots,r_m)$, $r_i\in\mathbb{Z}[\lambda]$. Then $\langle r\rangle$ is principal in $\mathbb{Z}[1/\lambda]$ if and only if there exists $p\in\mathbb{Z}$, $p\geq 0$, and a linear combination $\sum_{i=1}^m c_is_i=\lambda^p$, s_i, $c_i\in\mathbb{Z}[\lambda]$, where $s=(s_1,\ldots,s_m)$ is some eigenvector for A.*

Proof. First, suppose there exists a linear combination $\sum_{i=1}^m c_is_i=\lambda^p$, for some eigenvector s. Then $\langle s\rangle=\mathbb{Z}[1/\lambda]$, since λ^p is a unit in $\mathbb{Z}[1/\lambda]$. We have $r=as$ for some $a\in\mathbb{R}$, since the eigenspace is one-dimensional. But then

$$\lambda^pa=\sum_{i=1}^m c_is_ia=\sum_{i=1}^m c_ir_i,$$

which implies $a\in\mathbb{Z}[1/\lambda]$. Thus

$$\langle r\rangle=\langle as\rangle=a\langle s\rangle=\langle a\rangle$$

(since $\langle s\rangle=\mathbb{Z}[1/\lambda]$) which says $\langle r\rangle$ is principal.

Conversely, suppose $\langle r\rangle$ is principal and let $\langle r\rangle=\langle a\rangle$, $a\in\mathbb{Z}[1/\lambda]$. Then there exists a linear combination $\sum_{i=1}^m d_ir_i=a$, $d_i\in\mathbb{Z}[1/\lambda]$. Also, we have $ak_i=r_i$, for some $k_i\in\mathbb{Z}[1/\lambda]$, $i=1$ to m. Substituting and dividing through by a yields $\sum_{i=1}^m d_ik_i=1$. Since any element of $\mathbb{Z}[1/\lambda]$ can be multiplied by a sufficiently high power of λ to obtain an element of $\mathbb{Z}[\lambda]$, we may multiply $\sum_{i=1}^m d_ik_i=1$ by a high power of λ to obtain $\sum_{i=1}^m c_is_i=\lambda^p$, c_i, $s_i\in\mathbb{Z}[\lambda]$, where $s=(s_1,\ldots,s_m)$ is an eigenvector. □

We can now easily prove the 'only if' direction of theorem 1. For suppose B is a non-negative integral matrix which has a right eigenvector $r=(r_1,\ldots,r_m)$, $r_i\in\mathbb{Z}[\lambda]$ and suppose there is a linear combination $\sum_{i=1}^m c_ir_i=\lambda^p$, $c_i\in\mathbb{Z}[\lambda]$. Let B' be strong shift equivalent to B in one step, so that $B=UV$, $B'=VU$, for non-negative integral matrices U and V. Then

$$B'(Vr)=VU(Vr)=VBr=V\lambda r=\lambda(Vr),$$

so Vr is a right eigenvector for B'. Clearly $(Vr)_i\in\mathbb{Z}[\lambda]$. Also,

$$\lambda^p=\sum_{i=1}^m c_ir_i=\sum_{i=1}^m c_i\frac{(UVr)_i}{\lambda}=\frac{1}{\lambda}\sum_{i=1}^m c_i\sum_{j=1}^m U_{ij}(Vr)_j,$$

so

$$\sum_{i=1}^m c_i\sum_{j=1}^m U_{ij}(V_r)_j=\lambda^{p+1}$$

and there is a linear combination of the $(Vr)_i$ equal to a power of λ. By induction, the same conclusion holds if B' is strong shift equivalent to B in k steps. Now, if there exists a right closing map $f:\Sigma_A\to\Sigma_{C(\lambda)}$ such that $f\sigma^n=\sigma^nf$, then by lemma 1 and Kitchen's theorem, A^n is strong shift equivalent to B, where B has a right

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eigenvector r whose entries lie in $\{1, \lambda, \ldots, \lambda^{d-1}\}$. Then clearly B has a linear combination $\sum_{i=1}^{m} c_i r_i = \lambda^p$, so by the above, A^n (and hence A) must have a right eigenvector with such a linear combination. As we have seen (lemma 2), this is equivalent to the condition that the eigenvector for A generates a principal ideal.

In what follows, we will be working in subrings of $\mathbb{Q}[\lambda]$. Note that every element of $\mathbb{Q}[\lambda]$ may be uniquely written $b=\sum_{i=0}^{d-1} b_i\lambda^i$, $b_i \in \mathbb{Q}$, where d is the degree of λ. If $b_i \geq 0$ for $0 \leq i \leq d-1$, we write $b \geq^* 0$.

Define a map $L: \mathbb{Q}[\lambda] \to \mathbb{Q}^d$ by $L(b) = (b_0, b_1, \ldots, b_{d-1})$. L is clearly a linear isomorphism of vector spaces.

LEMMA 3. *$L(\lambda b) = C^t(\lambda)(L(b))$ for any $b \in \mathbb{Q}[\lambda]$. ($C^t(\lambda)$ denotes the transpose of $C(\lambda)$.)*

Proof.

$$\lambda b = \sum_{i=0}^{d-1} b_i \lambda^{i+1} = \sum_{i=1}^{d-1} b_{i-1}\lambda^i + b_{d-1}\lambda^d$$

$$= \sum_{i=1}^{d-1} b_{i-1}\lambda^i + b_{d-1} \sum_{i=0}^{d-1} a_i\lambda^i = b_{d-1}a_0 + \sum_{i=1}^{d-1} (b_{i-1} + b_{d-1}a_i)\lambda^i.$$

Thus $L(\lambda b) = (b_{d-1}a_0,\ b_0 + b_{d-1}a_1, \ldots, b_{d-2} + b_{d-1}a_{d-1})$. A simple computation shows this equals $C^t(\lambda)(L(b))$. □

It is well known that any point of $\mathbb{R}^d$ which is not in the span of smaller eigenvalues of $C^t(\lambda)$ tends toward the eigenline associated with eigenvalue λ under iteration by $C^t(\lambda)$.

We now show that no non-zero point of $\mathbb{Q}^d$ lies in the span of smaller eigenvalues of $C(\lambda)$. In fact, no proper invariant subspace of $\mathbb{R}^d$ can contain a point of $\mathbb{Q}^d$. For let $L(q) \in \mathbb{Q}^d$ and consider $B = \{L(q), L(q\lambda), \ldots, L(q\lambda^{d-1})\}$. B is linearly independent over $\mathbb{Q}$, hence also over $\mathbb{R}$. Any invariant subspace containing $L(q)$ must also contain B, and so must be all of $\mathbb{R}^d$.

It follows from these remarks that $L(\lambda^n q)$ tends toward the eigenline for any $q \in \mathbb{Q}[\lambda]$, $q \neq 0$. $L(\lambda^n q)$ goes into the positive orthant if $q > 0$, and the negative orthant if $q < 0$.

Let $e = \sum_{i=0}^{d-1} e_i$, $\lambda^i \in \mathbb{Z}[\lambda]$, $e \geq^* 0$, and let $\Delta = \{L(e), L(e, \lambda), \ldots, L(e\lambda^{d-1})\}$. Let $\Gamma = \{\sum_{i=0}^{d-1} b_i L(e\lambda^i)\colon b_i \in \mathbb{Z},\ b_i \geq 0\}$ be the positive lattice generated by Δ, and $\Lambda = \{\sum_{i=0}^{d-1} \alpha_i L(e\lambda^i)\colon \alpha_i \in \mathbb{Q},\ \alpha_i \geq 0\}$ the positive cone generated by Δ. Let E be the positive eigenray corresponding to λ.

LEMMA 4. *$E \subset$ interior (Λ).*

Proof. Using lemma 3, it is easy to check that $C^t(\lambda)(\Lambda) \subseteq \Lambda$. But also elements of Λ tend toward E under iteration of $C^t(\lambda)$. Since E is a ray and Λ a cone, it follows that $E \subseteq \Lambda$. Now suppose that $E \not\subseteq$ interior (Λ). Then E must be contained in the positive cone generated by a proper subset of Δ. We show that this is impossible. Since $C^t(\lambda)$ is aperiodic, for sufficiently large n, $\lambda^n = \sum_{i=0}^{d-1} c_i\lambda^i$ with $c_i > 0$. Thus,

$$L(e\lambda^n) = L\left(\sum_{i=0}^{d-1} c_i e\lambda^i\right) = \sum_{i=0}^{d-1} c_i L(e\lambda^i),$$

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so that each generator of Λ gets pushed into the interior of Λ under iteration by $C^t(\lambda)$. It follows that the positive cone generated by any proper subset of Δ eventually get mapped completely outside itself, and hence cannot contain E. □

COROLLARY. *For all sufficiently large n we may write* $\lambda^n = q_n e + s_n$, q_n, $s_n \in \mathbb{Z}[\lambda]$, q_n, $s_n \geq^* 0$, *where* $s_n = \sum_{i=0}^{d-1} b_i\lambda^i$ *and* $0 \leq b_i \leq \|L(e\lambda^i)\|$. ($\|\ \|$ *denotes the norm in* $\mathbb{R}^d$.)

Proof. Since $L(\lambda^n)$ tends toward E, and $E \subset$ interior (Λ), $L(\lambda^n) \in \Lambda$ for sufficiently large n. Thus, $L(\lambda^n)$ lies in some parallelepiped P of the lattice Γ. Choose the corner $L(p)$ of P closest to the origin. Write

$$L(p) = \sum_{i=0}^{d-1} c_i L(e\lambda^i) = L\left(e \sum_{i=0}^{d-1} c_i\lambda^i\right), \qquad c_i \in \mathbb{Z},\ c_i \geq 0.$$

Set $q_n = \sum_{i=0}^{d-1} c_i\lambda^i$. If $L(s_n) = L(\lambda^n) - L(p)$, $s_n = \sum_{i=0}^{d-1} b_i\lambda^i$, then since $L(\lambda^n)$ lies in P, we must have $0 \leq b_i \leq \|L(e\lambda^i)\|$, the length of the ith generating edge of P. It follows that

$$L(\lambda^n) = L(p) + L(s_n) = L(q_n e + s_n),$$

so that $\lambda^n = q_n e + s_n$. □

We will also need the following

LEMMA 5. *If* $p(x) \in \mathbb{Z}[x]$ *with* $p(\lambda) = 0$, *then*

$$\lim_{k\to\infty} \frac{(p(A)A^k)_{ij}}{(A^k)_{ij}} = 0$$

for each pair i..j.

Proof. Write $A = RBR^{-1}$, with

$$B = \begin{pmatrix} \lambda & & & & \\ & J_2 & & 0 & \\ & & J_3 & & \\ & & & \ddots & \\ & 0 & & & J_r \end{pmatrix}$$

where the J_i are the Jordan blocks corresponding to eigenvalues of modulus smaller than λ. Then $p(A)A^k = Rp(B)B^kR^{-1}$, and

$$p(B)B^k = \begin{pmatrix} 0 & & & \\ & p(J_2)J_2^k & & 0 \\ & & \ddots & \\ & 0 & & p(J_r)J_r^k \end{pmatrix},$$

since $p(\lambda) = 0$. Since the J_i, $i \geq 2$, correspond to eigenvalues of modulus strictly smaller than λ, the entries of $p(B)B^k$ grow exponentially more slowly than those of A^k, which grow like λ^k, since A is aperiodic. The same is true of the entries of $Rp(B)B^kR^{-1}$, since this simply introduces a linear scaling. The result follows. □

Our task is to show that given a matrix A, as in theorem 1, there is an integer n and a matrix B, with Σ_B topologically conjugate to Σ_{A^n}, such that B has an eigenvector

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whose entries lie in $\{1, \lambda, \ldots, \lambda^{d-1}\}$. The matrix B will be obtained by applying the technique of state-splitting to A^n. To do this, we need to know that A has a right eigenvector whose entries are of the form $e = \sum_{i=0}^{d-1} e_i\lambda^i$, with $e \in \mathbb{Z}$ and $e \geq^* 0$.

For each such entry, we will then split the corresponding state s into $\sum_{i=0}^{d-1} e_i$ new states, with e_i of them having a new eigenvector entry of the form λ^i, for $0 \leq i \leq d-1$. This will be accomplished by partitioning the n-blocks beginning with s into $\sum_{i=0}^{d-1} e_i$ sets.

We may assume A has an eigenvector r whose entries are of the form $e = \sum_{i=0}^{d-1} e_i\lambda^i$, $e_i \in \mathbb{Z}$; this follows from linear algebra. To show that we may assume $e \geq^* 0$, observe that $L(e\lambda^n) = C^t(\lambda)(e)$ tends toward E, and so is eventually in the positive orthant of $\mathbb{Z}^d$.

If $\langle r\rangle$ is principal, then by lemma 2 we may assume, without loss of generality, that there is an equation $\sum_{i=1}^m c_ir_i = \lambda^p$, $c_i \in \mathbb{Z}[\lambda]$. We now show that we can further assume that the $c_i \in \mathbb{Z}$. For suppose we are given the equation $\sum_{i=1}^m c_ir_i = \lambda^p$, with $c_i \in \mathbb{Z}[\lambda]$. Then each c_i is of the form $c_i = \sum_{j=0}^{d-1} c_{ij}\lambda^j$, $c_{ij} \in \mathbb{Z}$, so that we have

$$\sum_{i=1}^{m} c_ir_i = \sum_{i=1}^{m}\sum_{j=0}^{d-1} c_{ij}\lambda^jr_i = \sum_{i=1}^{m}\sum_{j=0}^{d-1} c_{ij}(A^jr)_i,$$

since $\lambda^jr_i = (A^jr)_i$. Since $(A^jr)_i = \sum_{k=1}^m A^j_{ik}r_k$, and $A^j_{ik} \in \mathbb{Z}$, we may gather terms to get an equation with coefficients in $\mathbb{Z}$.

We fix some notation. Given an allowable n-block $x = x_1x_2 \ldots x_n$, call x an *i-end* if x_n terminates at state i. If x is an i-end, let $r(x) = r_i$.

In what follows, we deal with a fixed state s that we wish to split. For simplicity of notation, we let $r_s = e = \sum_{i=0}^{d-1} e_i\lambda^i$, with $e_i \in \mathbb{Z}$ and $e \geq^* 0$. Let $B_n =$ paths $x_1x_2 \cdots x_n$: x_1 begins at $s\}$. Given $R \subseteq B_n$, define the *weight* of R, $w(R)$, by $w(R) = \sum_{x\in R} r(x)$.

Definition. We say that R can be *divided* if it can be partitioned into $\sum_{i=0}^{d-1} e_i$ sets $U_{01}, \ldots, U_{0e_0}, U_{11}, \ldots, U_{1e_1}, \ldots, \quad, \ldots, U_{d-1,1} \ldots U_{d-1,e_{d-1}}$ such that the following holds:

(1) $w(U_{ij}) = w(U_{ik}) \quad 1 \leq j, k \leq e_i$;

(2) $\lambda w(U_{ij}) = w(U_{i+1,k}) \quad 0 \leq i \leq d-2$;

(3) λ^p divides $w(U_{ij})$ in $\mathbb{Z}[\lambda]$, for each U_{ij}.

Notice that if B_n itself can be divided, then S can be split in the desired manner. This is so because

$$w(B_n) = \sum_{x\in B_n} r(x) = \sum_{j=1}^{m} A^n_{sj}r_j = \lambda^nr_s = \lambda^ne = \sum_{i=0}^{d-1} e_i\lambda^{i+n}.$$

It follows that $w(U_{ij}) = \lambda^{i+n}$ for $0 \leq i \leq d-1$. Since we divide by λ^n to obtain the eigenvector entry corresponding to the new state defined by the set U_{ij}, we see that the new state will have an entry of λ^i.

We wish to show that, for sufficiently large n, B_n can be divided. Our strategy is as follows: divide almost all the blocks of B_n, so that what is left over has bounded weight. Use the equation $\sum_{k=1}^m c_kr_k = \lambda^p$ to move quantities from one U_{ij} to another. This is done as follows: Note that for any $q \in \mathbb{Z}[\lambda]$, there is an equation $\sum_{k=1}^m d_kr_k = \lambda^pq$, $d_k \in \mathbb{Z}$, which is found by multiplying both sides of the original equation by q and then expanding the left hand side as before. Then given any partition into U_{ij}'s,

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we can transfer $\lambda^p q$ from $U_{i_1 j_1}$ to $U_{i_2 j_2}$ by moving d_k k-ends from $U_{i_1 j_1}$ to $U_{i_2 j_2}$, for $d_k \geq 0$, and $-d_k$ k-ends from $U_{i_2 j_2}$ back to $U_{i_1 j_1}$ for $d_k < 0$. Since if x is a k-end, $r(x) = r_k$, the net effect of this procedure is to increase $w(U_{i_2 j_2})$ by $q\lambda^p$ and decrease $w(U_{i_1 j_1})$ by the same amount. There is one major obstacle to carrying out this procedure: $U_{i_1 j_1}$ must contain at least d_k k-ends, for $d_k \geq 0$, and $U_{i_2 j_2}$ must contain at least $-d_k$ k-ends, for $d_k < 0$. So our goal is to divide most of the blocks of B_n in such a way that each U_{ij} contains lots of k-ends. We now make these ideas more precise. First, for any $b = \sum_{i=0}^{d-1} b_i \lambda^i \in \mathbb{Z}[\lambda]$, there is an associated polynomial $b(x) \in \mathbb{Z}[x]$, namely $\sum_{i=0}^{d-1} b_i x^i$. For the matrix A, $b(A)$ denotes $\sum_{i=0}^{d-1} b_i A^i$.

Definition. A polynomial $p(A)$ *moves* a quantity $q\lambda^p \in \mathbb{Z}[\lambda]$ if there is an equation $\sum_{j=1}^{m} d_j r_j = q\lambda^p$ with $d_j \in \mathbb{Z}$ and $(p(A))_{sj} \geq |d_j|$ for $1 \leq j \leq m$.

Since for any $q\lambda^p$ there is always such an equation, any polynomial $p(A)$ with $(p(A))_{sj}$ sufficiently large moves $q\lambda^p$. If $p(A)$ moves $q\lambda^p$, and for some partition $U_{i_1 j_1}$ and $U_{i_2 j_2}$ each contain at least $(p(A))_{sk}$ k-ends for $1 \leq k \leq m$, then we can transfer the quantity $q\lambda^p$ from $U_{i_1 j_1}$ to $U_{i_2 j_2}$ as described earlier.

LEMMA 6. *If $p(A)$ moves $q\lambda^p$, then $p(A)A^l$ moves $q\lambda^{p+l}$ for $l = 1, 2, \ldots$.*

Proof. We have $\sum_{j=1}^{m} c_j r_j = q\lambda^p$ with $(p(A))_{sj} \geq c_j$. Hence,

$$\begin{aligned} q\lambda^{p+l} &= \sum_{j=1}^{m} c_j r_j \lambda^l \\ &= \sum_{j=1}^{m} c_j (A^l r)_j \\ &= \sum_{j=1}^{m} c_j \sum_{k=1}^{m} A^l_{jk} r_k \\ &= \sum_{k=1}^{m} \left(\sum_{j=1}^{m} c_j A^l_{jk} \right) r_k \end{aligned}$$

Since

$$(p(A)A^l)_{sk} = \sum_{j=1}^{m} (p(A))_{sj} A^l_{jk} \geq \left| \sum_{j-1}^{m} c_j A^l_{jk} \right|$$

(the last inequality since $(p(A))_{sj} \geq |c_j|$), we have the result. □

We will now prove that for sufficiently large L, B_L can be divided. The 'if' direction of theorem 1 then follows.

By the corollary to lemma 4, (using $e\lambda^p$ in place of e), for $n \geq 1$ we can write $\lambda^n = q_n e\lambda^p + s_n$, with $q_n, s_n \in \mathbb{Z}[\lambda]$, $q_n, s_n \geq^* 0$, $s_n = \sum_{i=0}^{d-1} s_i \lambda^i$ and $0 \leq s_i \leq \|L(e\lambda^{p+i})\|$. Note that for $n \geq p$, λ^p divides s_n. Also, for n sufficiently large, $q_n \neq 0$.

Let us dispose of a simple case. Suppose that for some N, $s_N = 0$. By lemma 5, we have a sequence of equations

$$A^{N+m} = q_N(A)e(A)A^{p+m} + E_m, \tag{1}$$

where

$$\frac{|(E_m)_{ij}|}{A^m_{ij}} \to 0 \quad \text{as } m \to \infty, \quad \text{for all } i, j. \tag{*}$$

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Observe that

$$\begin{aligned}(E_m r)_s &= (A^{N+m}r - q_N(A)e(A)A^{p+m}r)_s\\ &= (\lambda^{N+m} - q_N e\lambda^{p+m})r_s = 0.\end{aligned}$$

For fixed N, choose m large enough so that $q_N(A)A^{p+m} \geq |E_m|$. This is possible by $(*)$ above and the fact that $q_n(A) \geq 0$ and $q_n(A) \not\equiv 0$. We construct a partition of B_{N+m} as follows:

In U_{01} put $(q_N(A)A^{p+m} + E_m)_{sk}$ k-ends. This makes sense since $(q_N(A)A^{p+m} + E_m)_{sk} \geq 0$. Then

$$w(U_{01}) = [(q_N(A)A^{p+m} + E_m)r]_s = q_N\lambda^{p+m}e.$$

For $U_{ij} \neq U_{01}$, put $(q_N(A)A^{p+m+i})_{sk}$ k-ends in U_{ij}. By virtue of (1), this exhausts the blocks of B_{N+m}. Since $w(U_{ij}) = q_N\lambda^{p+m+i}e$, we have divided B_{N+m}.

Having treated this case, we may assume that $s_n \neq 0$ for all n. Since in the expression $\lambda^n = q_n e\lambda^p + s_n$, $L(s_n)$ is bounded, $q_n(A)A^p$ grows without bound as $n \to \infty$. We wish to choose N so that $q_n(A)A^p$ is very large. How large must it be? We require that

(a) $q_N(A)A^p \geq \sum_{i=0}^{d-1} e_i P_i(A)$, where $P_i(A)$ is a polynomial with non-negative coefficients in A, and $P_i(A)$ moves $s_N\lambda^i$.

(b) $q_N(A)A^{p+i}$ moves $s_N\lambda^i$, for $0 \leq i \leq d-1$.

Since $q_n(A)A^p$ grows without bound, we can meet both these requirements.

Again by lemma 5, we have a sequence of equations

$$A^{N+m} = q_N(A)e(A)A^{p+m} + s_N(A)A^m + E_m \tag{2}$$

where

$$\frac{|(E_m)_{ij}|}{A_{ij}^m} \to 0 \qquad \text{as } m \to \infty.$$

Choose m large enough so that $s_N(A)A^m \geq |E_m|$. Here we use the fact that $s_N(A) \geq 0$ and $s_n(A) \not\equiv 0$ We construct a partition of B_{N+m} as follows: Put $(q_N(A)A^{p+m+i})_{sk}$ k-ends in each U_{ij}, $0 \leq i \leq d-1$, $1 \leq j \leq e_i$. This makes sense, since what is left over, by (2), is $(s_N(A)A^m + E_m)_{sk}$ k-ends, which is a positive number for each k. Since $w(U_{ij}) = eq_N\lambda^{p+m+i}$, this collection of blocks has been divided. Also, if we denote the collection of leftover blocks by R, then

$$w(R) = [(s_N(A)A^m + E^m)r]_s = s_N\lambda^m e,$$

since $E_m r = 0$ as before. Now put all the leftover blocks in U_{01}. This increases $w(U_{01})$ by $s_N\lambda^m e$. We wish to redistribute this quantity by moving $s_N\lambda^{m+i}$ from U_{01} to each $U_{ij} \neq U_{01}$, $0 \leq i \leq d-1$. Conditions (a) and (b) guarantee that we can do this. For we have

$$q_N(A)A^{p+m} \geq \sum_{i=0}^{d-1} e_i P_i(A)A^m,$$

(since everything is non-negative), and by lemma 6, $P_i(A)A^m$ moves $s_N\lambda^{m+i}$. This ensures that we can move e_i of the quantity $s_N\lambda^{m+i}$ out of U_{01}, for $0 \leq i \leq d-1$, without exhausting the required number of k-ends in U_{01}. Since we added blocks to U_{01} when we put in the leftover R, we did not diminish the capacity to transfer quantities out of U_{01}. Also, by condition (b), and lemma 6, $q_N(A)A^{p+m+i}$ moves

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$s_N\lambda^{m+i}$, we can also move $s_N\lambda^{m+i}$ into U_{ij}. After redistributing the remaining $s_N\lambda^m e$, we have

$$\begin{aligned} w(U_{ij}) &= q_N\lambda^{p+m+i}e + s_N\lambda^{m+i} \\ &= (q_N\lambda^{p+m}e + s_N\lambda^m)\lambda^i, \end{aligned}$$

which shows that B_{N+m} has been divided. This completes the 'if' direction of theorem 1.

Section 3. *Examples*

We now give a few examples. The first, which appeared in [**3**], shows that in certain entropy classes it is necessary to go to higher powers. Let

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

C is the companion matrix for $X^3 - X - 1$ and A is the companion matrix for

$$X^5 - X^4 - 1 = (X^3 - X - 1)(X^2 - X + 1).$$

Σ_A and Σ_C have the same entropy, but there cannot be a map $f: \Sigma_A \to \Sigma_C$ with $f\sigma = \sigma f$. This is so because Σ_A has a fixed point while Σ_C does not. However, there is a continuous map $f: \Sigma_A \to \Sigma_C$ satisfying $f\sigma^2 = \sigma^2 f$. In figure 1 we give the sequence

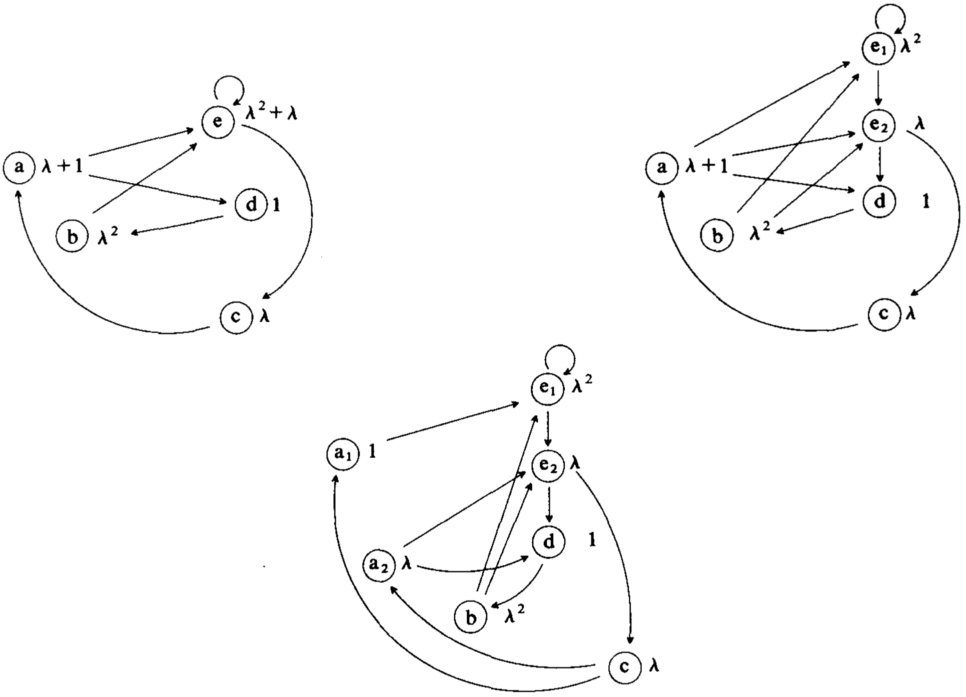

FIGURE 1

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of state splittings by which we obtain from A^2, a matrix B with a right eigenvector whose entries lie in $\{1, \lambda, \lambda^2\}$. The first graph represents A^2 (so that its edges correspond to paths of length 2 in $G(A)$). The states have been labelled a–e (going from top to bottom in the matrix A), and an eigenvector entry has been attached to each state. Note that we are using the equation $\lambda^3 = \lambda + 1$.

We first split state e into states e_1 and e_2, with new eigenvector entries λ^2 and λ, to obtain the second graph. We next split state a into states a_1 and a_2, with new entries 1 and λ, to obtain the third graph, which corresponds to a matrix B whose eigenvector entries lie in $\{1, \lambda, \lambda^2\}$.

It remains an open question in which entropy classes the integer n of theorem 1 can always be taken to be one. Marcus [**4**] showed that this is so in the integer entropy case. Perhaps it is also true for λ the golden mean, i.e. the root of $X^2 - X - 1$.

Here we mention that in certain entropy classes the condition that the eigenvector generates a principal ideal holds for every SFT of that entropy. In particular, it always holds if $\mathbb{Z}[1/\lambda]$ is a principal ideal domain. This is true, for example, if λ is an integer, or if λ is the golden mean.

There are other entropy classes in which the principal ideal condition does not always hold, as the next example shows: Let

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}.$$

The characteristic polynomial of A is $p(X) = X^2 - 4X - 1$. Let λ be the largest root of $p(X)$. A right eigenvector for A is $r = (2\lambda, 1 + \lambda)^t$. Claim: there is no linear combination $a(2\lambda) + b(1 + \lambda) = \lambda^p$ for any $p \in \mathbb{Z}$, $p \geq 0$, a, $b \in \mathbb{Z}[\lambda]$. For suppose there were. As usual, we may assume, without loss of generality, that a, $b \in \mathbb{Z}$. Observe that $\lambda^2 = 4\lambda + 1$, and if we reduce coefficients mod 2, we have $\lambda^2 = 1$. Thus $\lambda^p = 1$ or λ, mod 2, depending on whether p is even or odd. But $b + (2a + b)\lambda = b + b\lambda$, mod 2, which cannot equal 1 or λ. By the earlier observations, 2λ and $1 + \lambda$ do not generate a principal ideal in $\mathbb{Z}[1/\lambda]$. It follows from theorem 1 that there cannot exist a right closing $f: \Sigma_A \to \Sigma_C$ with $f\sigma^n = \sigma^n f$, for any n.

As a final example, we show that there are SFT's Σ_A and Σ_C and a map $f: \Sigma_A \to \Sigma_C$ which is right resolving and for which there cannot exist a left closing map $\Sigma_A \to \Sigma_C$. Let

$$B = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix}.$$

Let Σ_A be the almost conjugate extension of Σ_B and Σ_C (see [**1**]). Then there exists a right resolving map $f: \Sigma_A \to \Sigma_C$ and a left resolving map $g: \Sigma_A \to \Sigma_B$. As we have seen, A has a left eigenvector s with entries in $\{2\lambda, 1 + \lambda\}$ and $\langle s \rangle$ is not principal. Hence, no left eigenvector for Σ_A can generate a principal ideal. If there were a left closing map $h: \Sigma_A \to \Sigma_C$, then there would be a Σ_D conjugate to Σ_A and a left resolving map $i: \Sigma_D \to \Sigma_C$. Then D would have a left eigenvector with entries in $\{1, \lambda\}$, which geneates a principal ideal. It would then follow that A has a left eigenvector which generates a principal ideal, and this is a contradiction.

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