FAITHFUL REPRESENTATIONS OF FINITELY GENERATED METABELIAN GROUPS

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1. In [3] Remeslennikov proves that a finitely generated metabelian group $G$ has a faithful representation of finite degree over some field $F$ of characteristic zero (respectively, $p > 0$) if its derived group $G'$ is torsion-free (respectively, of exponent $p$). By the Lie-Kolchin-Mal'cev theorem any metabelian subgroup of $GL(n, F)$ has a subgroup of finite index whose derived group is torsion-free if char $F = 0$ and is a $p$-group of finite exponent if char $F = p > 0$. Moreover every finite extension of a group with a faithful representation (of finite degree) has a faithful representation over the same field. Thus Remeslennikov’s results have a gap which we propose here to fill.

1.1 Theorem. If the group $G$ is a finite extension of a finitely generated metabelian group $G_0$ whose derived group $G_0'$ is a $p$-group for some prime $p$, then $G$ has a faithful representation of finite degree over some field of characteristic $p$.

A quasi-linear group is a group of matrices over a direct sum of a finite number of fields, its characteristic being the set of the characteristics of the ground fields. (This is a slight modification of the definition in [4]). If $G$ is a metabelian group, by the characteristic of $G$ we mean the set of prime divisors of the orders of the elements of $G'$ of finite order, together with zero if $G'$ is not a torsion group. An immediate corollary of 1.1 above and Remeslennikov’s ‘characteristic zero’ case is the following.

1.2 Corollary. If the group $G$ is a finite extension, of a finitely generated metabelian group of characteristic $p$, then $G$ is isomorphic to a quasi-linear group of characteristic $p$.

There are no corresponding results without the finite generation. For example, for any non-trivial group $P$ the complete wreath product $P \wr \mathbb{Z}$ is not isomorphic to any group of automorphisms of any finitely generated module over any commutative Noetherian ring $R$. For $R$ a field this is a special case of [4, 10.22] and essentially the same proof works in general.

Given a field $F$ of characteristic $p > 0$ there exists one and, up to isomorphism, only one complete and unramified, discrete valuation ring with residue class field $F$ [1, Lemma 13 and Theorem 11, Corollary 2]. This ring we

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denote by $J(0, F)$. For each positive integer $m$ set
\[ J(m, F) = J(0, F)/(p^m). \]

$J(m, F)$ is a commutative local ring of characteristic $p^m$, with maximal ideal generated by $p$ and residue class field $F$. These properties uniquely determine $J(m, F)$ up to isomorphism in view of [1, Theorem 11, Corollary 3] and the ideal structure of $J(0, F)$.

To prove 1.1 we swiftly reduce to the split extension $X[M$ of a finitely generated $X$-module $M$ over the finitely generated abelian group $X$, by $X$. (Whenever $X$ is a group and $M$ is an $X$-group $X[M$ denotes the external semi-direct product of $M$ by $X$.) A series of module theoretic reductions leaves us with the case of the split extension of $J(m, F)$ by a finitely generated $p$-free subgroup of the group of units of $J(m, F)$.

Suppose that $R$ is a commutative local ring of characteristic $p^m$ and residue class field $F$. If $m$ is the maximal ideal of $R$ then there is a multiplicative exact sequence
\[ (1.3) \quad 1 \to 1 + m \to R\backslash m \to F^* \to 1. \]

Now $F^*$ is $p$-free since $\text{char } F = p$. If $m$ is nilpotent, for example if $m = pR$, then $1 + m$ is a $p$-group of finite exponent and the sequence (1.3) splits. When (1.3) splits a complement $U$ of $1 + m$ in $R\backslash m$ we call a unit complement of $R$. In general $U$ will not be unique though it will be of course if $F$ is a locally finite field. The final step of the proof of 1.1, which in fact we present first in § 2, is to construct representations of certain split extensions of the type $U[R$ with $U$ and $R$ as above. The proof of 1.1 is then completed in § 3.

In this note all rings have an identity, all modules are unital and ring homomorphisms are identity preserving.

2. Let $F$ be a field of characteristic $p > 0$, let $m$ be a positive integer and set $n = p^{m-1} + 1$. For each $\alpha \in F$ put
\[ t_\alpha = (\alpha_{ij}) \in \text{Tr}(n, F) \text{ where } \]
\[ \alpha_{ij} = 1 \text{ if } i = j \]
\[ = \alpha \text{ if } i = j + 1 \]
and is zero otherwise. Set $A = \langle t_\alpha : \alpha \in F \rangle \subseteq \text{Tr}(n, F)$. It is easy to check that $[t_\alpha] = p^n$ if $\alpha \neq 0$ and that $t_\alpha t_\beta = t_\theta t_\alpha$ for all $\alpha, \beta$ in $F$, [4, pp. 19–20]. In particular $A$ is an abelian group of exponent $p^n$.

We now define a new law of composition on $A$ to make $A$ into a ring. For $\beta \in F^*$ let
\[ d_\beta = \text{diag}(\beta^{n-1}, \beta^{n-2}, \ldots, \beta, 1) \in GL(n, F). \]

Then $D = \{d_\beta : \beta \in F^*\}$ is an abelian group, and since $d_\beta^{-1}t_\alpha d_\beta = t_{\alpha \theta}$ for all $\alpha, \beta \in F^*$ conjugation makes $A$ into a cyclic $D$-module generated by $t_1$. Thus $A$ is an image of the commutative ring $\mathbb{Z}D$ and hence $A$ can be made into a
commutative ring with identity \( t \). Note that the multiplication on \( A \), which we denote by circle, is determined by
\[
t_\alpha \circ t_\beta = t_{\alpha \beta} = d_\beta^{-1} t_\alpha d_\beta.
\]

Any field automorphism of \( F \) induces a ring automorphism of \( A \) by acting on the matrix entries. In particular if \( F \) is perfect the Frobenius automorphism \( a \mapsto a^p \) of \( F \) induces an automorphism \( \theta \) of \( A \) by acting on the matrix entries. In particular if \( F \) is perfect the Frobenius automorphism \( a \mapsto a^p \) of \( F \) induces an automorphism \( \theta \) of \( A \). Since \( A / A^p \) is a commutative ring of characteristic \( p \) the binomial theorem yields that modulo \( A^p \) the circle product of \( a \) with itself \( p \) times is
\[
o^p a = \prod_i (t_\alpha a^p)^{r_i} = a^{p^k}, \quad \text{where} \quad a = \prod_i t_\alpha^{r_i},
\]
for any positive integer \( k \). If \( a \) is a nilpotent element of \( A \) then for sufficiently large \( k \) we have \( o^p a = 1 \). In this situation \( a^{p^k} \in A^p \), whence \( a \in A^p \). We have now proved that \( A^p \) is the nilradical of \( A \).

For \( l \geq 0 \) put
\[
M_l = \{ a = (a_{ij}) \in A : a_{ij} = 0 \text{ whenever } 0 < i - j \leq l \}.
\]
Clearly \( M_l \) is a subgroup of \( A \) and \( d_\beta^{-1} M d_\beta \subseteq M_l \) for all \( \beta \in F^* \). Thus \( M_l \) is an ideal of \( A \) for \( l \geq 1 \) and \( M_0 = A \). Suppose \( a = (a_{ij}) \in M_{l-1} - M_l \). Then \( a_{i,i-1} \neq 0 \) and the \( (i, i-1) \) component of \( a \circ b \) is
\[
(a \circ b)_{i,i-1} = a_{i,i-1} \left( \sum_{j=1}^{i-1} f_j \beta_j \right).
\]
In particular for \( l \geq 1 \) with \( M_{l-1} \neq M_l \) the map \( \varphi_l : A \to F \) given by \( b \varphi_l = \sum f_j \beta_j \) is well defined. Also \( \varphi_l \) is a ring homomorphism—this can easily be checked directly but it is also an immediate consequence of
\[
(a \circ (bc))_{i,i-1} = ((a \circ b)(a \circ c))_{i,i-1} = (a \circ b)_{i,i-1} + (a \circ c)_{i,i-1}
\]
and
\[
(a \circ (b \circ c))_{i,i-1} = ((a \circ b) \circ c)_{i,i-1}
\]
Now assume that every polynomial \( X^l - \alpha \) has a root in \( F \) for every \( \alpha \) in \( F \) and every integer \( l \) satisfying \( 0 < l < n \) and \( M_{l-1} \neq M_l \). This will certainly be the situation if \( F \) is algebraically closed, which is the only case that we shall actually use. Then in particular \( A \varphi_l = F \) for each such \( l \) and for all \( l \geq 1 \) either \( M_{l-1} = M_l \) or \( M_{l-1} / M_l \) is an irreducible \( A \)-module such that modulo its annihilator, \( A \) is isomorphic to \( F \). Hence \( A \) satisfies the minimal condition and so is a direct sum of a finite number of local rings \( A_i \) whose maximal ideals are nilpotent (e.g. [5, p. 205]). Moreover the above implies that for each \( i \) the residue class field of \( A_i \) is isomorphic to \( F \) and that the maximal ideal of \( A_i \) is generated by \( p \). Thus \( A_i \) is isomorphic to \( J(m_i, F) \) for some integer \( m_i \leq m \) and since \( A \) has characteristic \( p \) we have \( m_i = m \) for at least one \( i \).

For \( M_{l-1} \neq M_l \), our assumption on \( F \) ensures that the ring homomorphism \( \varphi_l \) maps the subgroup \( W = \{ t_\alpha : \alpha \in F^* \} \) of the group of units of \( A \) isomorphically.
onto $F^\star$. Thus if $\pi_i$ denotes the projection of $A$ onto $A_i$ then $W\pi_i$ is a unit complement of $A_i$. For any $a \in A$ and $\beta \in F^\star$ we have

$$(a\pi_i)\beta = a\pi_i \circ \beta = a\pi_i \circ \beta \pi_i$$

since $A_i \circ A_j = \{1\}$ if $i \neq j$. Set $D = \langle d_\beta : \beta \in F^\star \rangle$. Then $\langle A_i, D \rangle \subseteq \text{Tr}(n, F)$ is isomorphic to the natural split extension of $A_i$ by $W\pi_i$. In particular we have now proved the following:

2.1 Theorem. If $F$ is any algebraically closed field of characteristic $p > 0$ and if $m$ is any positive integer there exists a unit complement $V$ of $J = J(m, F)$ such that the natural split extension $V[J$ is isomorphic to a subgroup of $\text{Tr}(p^m - 1 + 1, F)$.

An easy fact that we shall not need is that $J(m, F)$ has a unique unit complement whenever $F$ is perfect. I am indebted to Warren Dicks for pointing out to me that for $F$ a perfect field $J(m, F)$ is isomorphic to the ring of Witt vectors over $F$ of length $m$ and that the representations above of $U(m, F)[J$ can be given explicitly by means of the Artin-Hasse exponential.

3.

3.1 Lemma. If $E$ is a finitely generated subfield of the field $F$ of positive characteristic and if $m$ is a positive integer then $J(m, E)$ is isomorphic to a subring of $J(m, F)$.

A slight variant of the argument below yields the corresponding result for $m = 0$. The finite generation of $E$ is irrelevant.

**Proof.** $R = J(m, F)$ contains a finitely generated (and hence Noetherian) subring $S$ whose image in $R/pR$ generates a copy of $E$. The localization $T$ in $R$ of $S$ at $S \cap pR$ is a commutative Noetherian ring with residue class field $E$ and nilpotent maximal $T \cap pR$. Thus $T$ is also complete and Theorem 11, Corollary 1 of [1] yields a homomorphism $\varphi$ of $J = J(0, E)$ into $T$. Since $\varphi$ preserves the identity ker $\varphi = p^m J$ and hence $\varphi$ induces an embedding of $J(m, E) = J/p^m J$ into $R$.

3.2 Proof of Theorem 1.1. By hypothesis our group $G$ contains a finitely generated metabelian group $G_0$ of finite index such that $G_0$ is a $p$-group. Putting $H = G_0/G_0'$ consider the Kalmižin-Krasner embedding $\varphi$ of $G_0$ into $W = G_0' \cap H$ and denote the base group of $W$ by $B$. Then $G_1 = \langle G_0 e, H \rangle$ is a finitely generated metabelian group, $M = G_1 \cap B$ is an abelian normal $p$-subgroup of $G_1$ and $G_1$ is the split extension of $M$ by $H$. $M$ is a finitely generated $H$-module and in particular has finite exponent (e.g. [4], p. 189). By [4, 2.3] it suffices to construct a faithful representation of $G_1 = HM$.

If $\sigma : G_1 \rightarrow GL(r, F)$ and $\tau : G_1 \rightarrow GL(s, F)$ are homomorphisms with $\ker \sigma \cap \ker \tau = \langle 1 \rangle$ then

$$x \mapsto \text{diag}(x\sigma, x\tau)$$

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is a faithful representation of $G_1$ into $GL(r + s, F)$. By choosing a primary decomposition of $\{1\}$ in the finitely generated $\mathbb{Z}H$-module $M$ and applying (3.3) it follows that we may assume that $M$ is a primary $\mathbb{Z}H$-module. Clearly there exist fields of characteristic $p$ over which $H$ has a faithful representation of finite degree, e.g. 2.2 of [4]. Hence it suffices to construct a faithful representation of $HM/C_H(M) \cong (H/C_H(M))[M]$. That is, we may also assume that $H$ acts faithfully on $M$.

Let $R$ denote the subring of $\text{End}_\mathbb{Z}M$ generated by $H$ and set $r = \text{rad } M$. Then $r$ is a nilpotent prime ideal. We localize at $r$; whence $S = R_r$ is a commutative Noetherian local ring with nilpotent maximal ideal $m = r$, and, since $M$ is primary, $M$ embeds into $N = M \otimes_R S$. Regarding $R$ as a subring of $S$ we have that $G_1 = HM$ is isomorphic to a subgroup of $H[N]$. Also $M$ has finite exponent $p^m$ say, whence $S$ has characteristic $p^m$.

By [1, Theorem 11] there exists a subring $J$ of $S$ satisfying $S = J + m$ and $J \cap m = pJ$. If we put $F = S/m$ then clearly $J$ is isomorphic to $J(m, F)$. Since $S$ is Noetherian each $m^n/m^{n+1}$ has finite $F$-dimension. Thus the nilpotency of $m$ yields that $S$ is a finitely generated $J$-module and therefore $N$ is also a finitely generated $J$-module.

If $U$ is a unit complement of $J$ then $U$ is also a unit complement of $S$ and $H \subseteq U \times (1 + m)$. Now $1 + m$ is a $p$-group and $H$ is finitely generated, hence $H \subseteq H_1 \times P$ for some finitely generated subgroup $H_1$ of $U$ and some finite subgroup $P$ of $1 + m$. If $H_1[N]$ is isomorphic to a subgroup of $GL(n, E)$ for some $n$ and some field $E$ then $G_1$ is isomorphic to a subgroup of $GL(n|P|, E)$ by [4, 2.3] again. $J$ is an image of the principal ideal domain $J(0, F)$ so $N$ is a direct sum of cyclic $J$-modules. The only cyclic $J$-modules up to isomorphism are the $J/p^iJ$ for $i = 1, 2, \ldots, m$ and as rings $J/p^iJ \cong J(i, F)$. Applying the reduction 3.3 again this shows that it suffices to construct a faithful representation of the split extension $H_1[J]$ of characteristic $p$.

Let $\bar{F}$ denote the algebraic closure of $F$. Now $F$ is the quotient field of the finitely generated ring $R/r$ and hence by 3.1 there exists a copy $J$ of $J(m, \bar{F})$ containing $J$ as a subring. Now there exists by 2.3 a unit complement $V$ of $J$ such that $GL(p^{-1} + 1, \bar{F})$ contains an isomorphic copy of $V[J]$. Since $H_1 \subseteq V \times (1 + pJ)$ the finite generation of $H_1$ yields that $H_1 \subseteq V \times Q$ for some finite $p$-subgroup $Q$. Then

$$H_1[J] \subseteq H_1[J] \subseteq QV[J]$$

and by [4, 2.3] the latter group is isomorphic to a subgroup of $GL(|Q|(p^{-1} + 1), \bar{F})$. This completes the proof of 1.1.

References

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