INEQUALITIES FOR SOME MONOTONE MATRIX FUNCTIONS

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I. Introduction. Let $V$ denote a unitary vector space with inner product $(x, y)$. A self-adjoint linear map $T: V \rightarrow V$ is positive (positive definite) if $(Tx, x) \geq 0$ $(Tx, x) > 0$ for all $x \neq 0$ in $V$. We write $S \geq T$ $(S > T)$ if $S$ and $T$ are self-adjoint and $S - T \geq 0$ $(S - T > 0)$. If $U$ is a unitary vector space, a function $f: \text{Hom}(V, V) \rightarrow \text{Hom}(U, U)$ is monotone if $S \geq T$ implies that $f(S) \geq f(T)$. If both $U$ and $V$ are taken to be the $n$-dimensional unitary space $C^n$ of $n$-tuples of complex numbers with standard inner product, then $f$ is a monotone matrix function, a notion introduced for a more restrictive class of functions by Löwner (3) which has important applications in pure and applied mathematics. For orientation we refer the reader to (1), where several interesting examples of monotone and related functions are displayed in detail.

Let $H$ denote a permutation group of degree $n$ and let $M: H \rightarrow \text{Hom}(U, U)$ denote a representation of $H$ as unitary linear mappings on a unitary vector space $U$ of dimension $s$. Let $A = [a_{ij}]$ denote an $n$-square, positive semi-definite hermitian matrix and consider the generalized matrix function $M_A: U \rightarrow U$ of Schur (11) defined by $M_A = \sum_{\sigma \in H} M(\sigma) \Pi_{i=1}^n a_{\sigma(i)}$. Schur proved that $M_A$ is positive and is, in fact, positive definite when $A$ is positive definite. Moreover, he proved the beautiful inequality

$$\text{tr}(M_A) = \sum_{\sigma \in H} \chi(\sigma) \prod_{i=1}^n a_{\sigma(i)} \geq s \det A$$

for the trace $(\text{tr})$ of $M_A$, where $\chi$ is the character of $M$ and “det” is the familiar determinant function. A considerably more general result was announced in (5), where the positivity of $M_A$ and the inequality (1) are direct consequences of the Cauchy-Schwarz inequality in the tensor product of $U$ with the $n$-fold tensor product of $C^n$. These multilinear techniques continue to be a fruitful source of inequalities for the generalized matrix functions $d_\chi^H(A) \equiv \text{tr}(M_A)$.

**Theorem 1.** Regarded as a function from $\text{Hom}(C^n, C^n)$ to $\text{Hom}(U, U)$, $M_A$ is monotone on the set of positive semi-definite hermitian matrices.

**Corollary 1.** If $A \geq B \geq 0$, then $d_\chi^H(A) \geq d_\chi^H(B) \geq 0$. 

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One can employ the preceding result to generate inequalities for the matrix functions $d_x^H(A)$. Generally speaking, given a class of positive $A$ we seek a positive $B$ for which all $A$ satisfy either $A \geq B$ or $B \geq A$. It is then desirable that $d_x^H(B)$ be easy to bound or to compute. The following assertion is an example.

**Theorem 2.** Let $A = [a_{ij}]$ denote an $n$-square, positive semi-definite hermitian matrix whose row sums $r_1, \ldots, r_n$ satisfy $\sum_{i=1}^n r_i = r \neq 0$. Let $H$ denote a permutation group of degree $m \leq n$ and order $h$ and let $\chi$ be the identity character of degree 1 on $H$. Let $A_1, \ldots, A_N$ denote the $N = \binom{n}{m}$ principal $m$-square submatrices of $A$ and let $p_m(|r_1|, \ldots, |r_n|)$ be the $m$th weighted elementary symmetric function of the moduli of $r_1, \ldots, r_n$. Then

$$\sum_{i=1}^N d_x^H(A_i)/N \geq (h/r^m)p_m^2(|r_1|, \ldots, |r_n|).$$

The inequality (2) is strict unless $A$ has rank 1 or unless $A$ has fewer than $m$ non-zero rows. If $A$ has rank 1, then equality holds in (2) if and only if $A$ has exactly $m$ non-zero rows, $m = n$, or all non-zero row sums of $A$ have equal modulus.

The preceding inequality is proved for the case $m = n$ in (4; 7). A second generalization, in terms of the completely symmetric functions (2) of the moduli of the row sums of $A$, is available as follows.

**Theorem 3.** Let $A$ be as in Theorem 2 and let $B_1, \ldots, B_N$ denote the set of all $N = \binom{n+1}{m-1}$ $m$-square principal submatrices of $A$, where repeated occurrences of row and column subscripts are permitted. Then

$$\sum_{i=1}^N d_x^H(B_i)/N \geq (h/r^m)q_m^2(|r_1|, \ldots, |r_n|),$$

where $q_m$ is the $m$th weighted completely symmetric function and $\chi \equiv 1$. Equality holds in (3) if and only if $A$ has rank 1 and all row sums $r_1, \ldots, r_n$ of $A$ have equal modulus.

Proofs of these theorems as well as detailed statements of further results require a rather comprehensive discussion of multilinear methods. However, once the proper setting is established, the proofs follow readily from well-known properties of inner product spaces.

**II. Tensor products of unitary spaces.** Let $f: V_1 \times \ldots \times V_r \rightarrow P$ denote an $r$-linear mapping from the cartesian product of vector spaces $V_1, \ldots, V_r$ over a field $F$ to a vector space $P$ over $F$. If the pair $(P, f)$ satisfies:

(i) the linear closure of the range of $f$ is $P$, i.e., $\langle \text{rng } f \rangle = P$;

(ii) for any $r$-linear $\mu: V_1 \times \ldots \times V_r \rightarrow Q, Q$ any vector space over $F$, there is a linear $\lambda: P \rightarrow Q$ for which $\mu = \lambda f$ (universal factorization property),
then \((P,f)\) is a tensor product of vector spaces \(V_1, \ldots, V_r\). A diagram conveniently illustrates the mappings.

\[
\begin{array}{ccc}
V_1 \times \ldots \times V_r & \xrightarrow{f} & P \\
\mu & \downarrow & \\
& Q \end{array}
\]

It is easy to show that the tensor product of vector spaces is unique up to isomorphism; we adopt the familiar notation \(\bigotimes_{i=1}^r V_i = V_1 \otimes \ldots \otimes V_r\) for \(P\) and call its elements tensors. If an element in \(V_1 \otimes \ldots \otimes V_r\) is the image under \(f\) of an \(r\)-tuple \((v_1, \ldots, v_r)\), we call this element a decomposable or pure tensor and denote it by \(v_1 \otimes \ldots \otimes v_r\). By (i), \(V_1 \otimes \ldots \otimes V_r\) contains a basis of pure tensors.

Let the field \(F\) be the field \(C\) of complex numbers and suppose that each of \(V_1, \ldots, V_r\) is a unitary vector space. It is well-known (8) that the inner products in \(V_1, \ldots, V_r\) together induce an inner product in \(\bigotimes_{i=1}^r V_i\) which for decomposable tensors satisfies

\[
(u_1 \otimes \ldots \otimes u_r, v_1 \otimes \ldots \otimes v_r) = \prod_{i=1}^r \langle u_i, v_i \rangle,
\]

where the possibly different inner products are identified only by their operands. In fact, (5) can be taken as the definition of the induced inner product for, since \((\operatorname{rng} f) = \bigotimes_{i=1}^r V_i\), the inner product (5) can be extended to all of \(\bigotimes_{i=1}^r V_i\) by conjugate-bilinear extension.

Linear mappings on \(V_1, \ldots, V_r\) induce linear mappings on \(\bigotimes_{i=1}^r V_i\) in a natural way. Let \(T_i: V_i \to V_i, \ i = 1, \ldots, r\), be linear mappings. In the diagram (4) set \(\mu(v_1, \ldots, v_r) = f(T_1 v_1, \ldots, T_r v_r)\), clearly an \(r\)-linear mapping. Thus, for \(P = Q = \bigotimes_{i=1}^r V_i\), the linear \(\lambda: \bigotimes_{i=1}^r V_i \to \bigotimes_{i=1}^r V_i\) satisfying \(f(T_1 v_1, \ldots, T_r v_r) = \lambda f(v_1, \ldots, v_r)\) is called the tensor product of mappings \(T_1, \ldots, T_r\), and we write \(T_1 \otimes \ldots \otimes T_r\) for \(\lambda\). Let \(S_i: V_i \to V_i\) and \(T_i: V_i \to V_i, \ i = 1, \ldots, r\), be linear. Then the tensor product of mappings satisfies the rule of composition,

\[
(S_1 \otimes \ldots \otimes S_r)(T_1 \otimes \ldots \otimes T_r) = (S_1 T_1 \otimes \ldots \otimes S_r T_r).
\]

Thus, if each of \(T_1, \ldots, T_r\) is invertible, then \(T_1 \otimes \ldots \otimes T_r\) is invertible. Moreover, if each of \(T_1, \ldots, T_r\) is normal, unitary, self-adjoint, or positive relative to the respective inner products in \(V_1, \ldots, V_r\), then the mapping \(T_1 \otimes \ldots \otimes T_r\) inherits the corresponding property relative to the induced inner product (5). Let \(\dim V_i = n_i, \ i = 1, \ldots, r\). If \(\lambda_1, \ldots, \lambda_{\text{int}}\) are the eigenvalues of \(T_i\) then it is clear that the eigenvalues of \(T_1 \otimes \ldots \otimes T_r\) are the \(\prod_{i=1}^r n_i\) homogeneous products \(\lambda_{\omega_1} \ldots \lambda_{\omega_r}\), \(1 \leq \omega_i \leq n_i, \ i = 1, \ldots, r\).
Symmetry classes of tensors have proven useful in the study of the generalized matrix functions $d^H_x(A)$. If $H$ is any permutation group of degree $m$ and order $h$ and $x$ is any character of degree 1 of $H$ to a field $F$ of characteristic exceeding $h$, then an $m$-linear function $f: V \times \ldots \times V \to P$ from the $m$-fold cartesian product of a vector space $V$ over $F$ to a vector space $P$ over $F$ is symmetric with respect to $H$ and $x$ if for every $m$-tuple $(v_1, \ldots, v_m)$ and every $\sigma \in H$, we have that

\begin{equation}
    f(v_{\sigma(1)}, \ldots, v_{\sigma(m)}) = x(\sigma)f(v_1, \ldots, v_m).
\end{equation}

If the pair $(P, f)$ satisfies (i) and every $m$-linear $\mu$ symmetric with respect to $H$ and $x$ satisfies (ii) (universal factorization property with respect to $H$ and $x$), then $(P, f)$ is a symmetry class of tensors over $V$ associated with $H$ and $x$. Uniqueness of symmetry classes follows from that property for tensor product, and we write $V_x^m(H)$ in this case for $P$. In case $H$ is the identity group, then $V_x^m(H)$ coincides with $\otimes^m V$, the $m$-fold tensor product of $V$.

A most convenient method of representing symmetry classes is through the medium of symmetry operators on $\otimes^m V$. In the diagram (4) let $P = Q = \otimes^m V$ and set $\mu(v_1, \ldots, v_m) = f(v_{\phi(1)}, \ldots, v_{\phi(m)})$, where $\phi = \sigma^{-1}$ and $\sigma \in H$. We denote the linear map $\lambda$ by $P(\sigma)$, a permutation operator on $\otimes^m V$. $P(\sigma)$ is a unitary operator relative to the induced inner product (5) and, as is easily shown, satisfies $P(\sigma \tau) = P(\sigma)P(\tau)$. Thus, the set of $P(\sigma), \sigma \in H$, form a unitary representation of $H$ as linear operators on $\otimes^m V$. If $M(\sigma)$ is any other unitary representation of $H$ to a unitary space $U$, then the mappings $M(\sigma) \otimes P(\sigma)$ are a unitary representation of $H$ in the unitary space $U \otimes (\otimes^m V)$. Let the linear mapping $T_M: U \otimes (\otimes^m V) \to U \otimes (\otimes^m V)$ be defined by

\begin{equation}
    T_M = (1/h)\sum_{\sigma \in H} M(\sigma) \otimes P(\sigma),
\end{equation}

where $h$ is the order of $H$. Now $T_M$ is hermitian relative to the induced inner product for

\[ T_M^* = ((1/h)\sum_{\sigma \in H} M(\sigma) \otimes P(\sigma))^* = (1/h)\sum_{\sigma \in H} M^*(\sigma) \otimes P^*(\sigma) = (1/h)\sum_{\sigma \in H} M(\sigma^{-1}) \otimes P(\sigma^{-1}) = (1/h)\sum_{\sigma \in H} M(\sigma) \otimes P(\sigma) = T_M. \]

If $\tau \in H$, then

\[ (M(\tau) \otimes P(\tau))(1/h)\sum_{\sigma \in H} M(\sigma) \otimes P(\sigma) = (1/h)\sum_{\sigma \in H} M(\tau \sigma) \otimes P(\tau \sigma) = (1/h)\sum_{\sigma \in H} M(\sigma) \otimes P(\sigma). \]

Hence,

\[ T_M^2 = ((1/h)\sum_{\tau \in H} M(\tau) \otimes P(\tau))(1/h)\sum_{\sigma \in H} M(\sigma) \otimes P(\sigma) = (h/h^2)\sum_{\sigma \in H} M(\sigma) \otimes P(\sigma) = T_M \]

so that $T_M$ is idempotent. Thus, $T_M$ is a projection, a positive operator.

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For applications to symmetry classes we restrict ourselves to representations \( M \) of degree 1. We are thus led to the symmetry operator

\[
T_x = \frac{1}{h} \sum_{\sigma \in H} \chi(\sigma) P(\sigma)
\]

from \( \otimes^n V \) to itself. It is not difficult to show that \( \text{rng } T_x = V_x^m(H) \). For pure tensors \( T_x(v_1 \otimes \ldots \otimes v_m) \) we write \( v_1 \ast \ldots \ast v_m \).

It is customary to index the elements of a basis of \( \otimes^n V \) using the sequence set \( \Gamma_{m,n} \) consisting of all \( n^m \) sequences \( \omega = (\omega_1, \ldots, \omega_m), \ 1 \leq \omega_i \leq n, \ i = 1, \ldots, m, \) and to order the basis of \( \otimes^n V \) using the lexicographical ordering of the sequence set \( \Gamma_{m,n} \). If \( e_1, \ldots, e_n \) is a (orthonormal) basis of the \( n \)-dimensional (unitary) space \( V \), then \( e_1^\otimes = e_{\omega_1} \otimes \ldots \otimes e_{\omega_n}, \ \omega \in \Gamma_{m,n} \), is a (orthonormal) basis of \( \otimes^n V \) (8). We can use the fact that \( \text{rng } T_x = V_x^m(H) \) to derive certain combinatorial properties of sequence sets which determine which sequences of \( \Gamma_{m,n} \) index elements of a basis for \( V_x^m(H) \) (7). If \( \sigma \in S_m \) and \( \omega \in \Gamma_{m,n} \) let \( \omega^\sigma \) denote the sequence \( (\omega_{\sigma(1)}, \ldots, \omega_{\sigma(m)}) \). Define a binary relation \( \sim \) on \( \Gamma_{m,n}; \ \alpha \sim \beta \) if and only if there is a \( \sigma \in H \) for which \( \alpha^\sigma = \beta \). The relation \( \sim \) is in fact an equivalence relation. If we represent each of the resulting classes by its lexicographical minimum sequence we call this system \( \Delta_{m,n} \) the class of all sequences \( \omega \in \Delta_{m,n} \) for which the character \( \chi \) is identically 1 on \( H \). In the context permits, we denote the above sequence sets by \( \Gamma, \Delta, \) and \( \Delta \). We have the following result (7). Let \( e_1, \ldots, e_n \) be a (orthonormal) basis of the (unitary) space \( V \) of dimension \( n \). Then the (unitary) space \( V_x^m(H) \) has a (orthonormal) basis \( (h/\nu(\omega))^{1/2} e_{\omega} = (h/\nu(\omega))^{1/2} e_{\omega_1} \ast \ldots \ast e_{\omega_m}, \ \omega \in \Delta, \) of decomposable tensors. For any pure tensors in \( V_x^m(H) \) over the unitary space \( V \), the induced inner product satisfies

\[
(u_\omega \ast, v_\gamma \ast) = (u_{\omega_1} \ast \ldots \ast u_{\omega_m}, v_{\gamma_1} \ast \ldots \ast v_{\gamma_m}) = (1/h) d_x^H ([u_{\omega_1}, v_{\gamma_1}])
\]

a result basic to the proofs of many inequalities for generalized matrix functions (6).

A useful and important subset of \( \text{Hom}(V_x^m(H), V_x^m(H)) \) is the class of linear mappings induced by linear mappings in \( \text{Hom}(V, V) \). In the diagram (4) set \( r = m, \ V_i = V, \ i = 1, \ldots, m, \) and \( P = Q = V_x^m(H) \). Assume that \( f \) is symmetric with respect to \( H \) and \( \chi \) and set \( \mu(v_1, \ldots, v_m) = f(Tv_1, \ldots, Tv_m) \). We write \( K(T) \) for the linear \( \lambda: V_x^m(H) \rightarrow V_x^m(H) \) satisfying

\[
Tv_1 \ast \ldots \ast Tv_m = \lambda v_1 \ast \ldots \ast v_m
\]

and call \( K(T) \) the \( m \)th induced transformation associated with \( H \) and \( \chi \). The composition rule (6) provides the formula \( K(ST) = K(S)K(T) \). Indeed, \( K(T) \) is just the restriction of the \( m \)th Kronecker power \( T \otimes \ldots \otimes T \equiv \Pi^m(T) : \otimes^m V \rightarrow \otimes^m V \) to the symmetry class \( V_x^m(H) = \text{rng } T_x \). Accordingly,
properties of $T$ such as invertible, normal, unitary, self-adjoint, or positive each carry over to $K(T)$. In fact, we have the following result.

**Theorem 4.** Let $S: V \to V$ and $T: V \to V$ be self-adjoint mappings on the unitary space $V$ satisfying $S \triangleright T \triangleright 0$. Then $K(S): V^m \to V^m(H)$ and $K(T): V^m \to V^m(H)$ satisfy $K(S) \geq K(T) \geq 0$.

Hence, $K(T)$ considered as a function

$$K: \text{Hom}(V, V) \to \text{Hom}(V^m(H), V^m(H)),$$

in addition to being multiplicative, is a monotone function of $T$.

A companion of the $m$th induced transformation is its matrix representation. Let $A = [a_{ij}]$ denote the matrix representation of a linear map $T$ from an $n$-dimensional vector space $V$ to itself relative to the ordered basis $e_1, \ldots, e_n$. Then it is easy to show that the matrix representation of the $m$th induced map $K(T): V^m \to V^m(H)$ relative to the induced basis $(h/\nu(\omega))^{1/2}e_\omega^*$, $\omega \in \Delta$, ordered lexicographically, is the matrix whose entry in row $\alpha$ and column $\beta$ is $(1/\nu(\alpha)\nu(\beta))^{1/2}d_{\alpha\beta}(A^T\beta\alpha))$, where $\alpha = \langle \alpha_1, \ldots, \alpha_m \rangle$ and $\beta = \langle \beta_1, \ldots, \beta_m \rangle$ are in $\Delta$, and $A[\alpha\beta]$ denotes the $m$-square matrix whose $i, j$ entry is $a_{\alpha_i\beta_j}$, $i, j = 1, \ldots, m$. The superscript $T$ denotes matrix transpose whose presence in the above formula is made necessary by the well-known perversity of indices in matrix notation. We call this matrix $K(A)$ the $m$th induced matrix of $A$ associated with $H$ and $\chi$. Suppose, for example, that $A^T$ is the Gram matrix of a set of vectors $v_1, \ldots, v_n$ in an unitary space $V$. Then $K^T(A)$ is the Gram matrix of the decomposable tensors $(h/\nu(\omega))^{1/2}u_\omega^*$, $\omega \in \Delta$, in $V^m(H)$.

The most familiar example of a symmetry class of tensors is the $m$th Grassman space $\wedge^m V$ whose decomposable tensors are usually written $v_1 \wedge \ldots \wedge v_m$. Here $H = S_m$, the full symmetric group of degree $m$, and $\chi = \varepsilon$, the alternating character on $S_m$. If $V$ is unitary, the induced inner product (9) satisfies

$$(u_1 \wedge \ldots \wedge u_m, v_1 \wedge \ldots \wedge v_m) = (1/m!)|\text{det}([u_i, v_j])|.$$  

The sequence set $\Delta$ is the set $G_{m,n}$ of all $^{m+n-1}_{m-1}$ non-decreasing sequences of length $m$ chosen from 1, \ldots, $n$ while $\Delta$ is the set $Q_{m,n}$ of all $\binom{n}{m}$ strictly increasing sequences in $G_{m,n}$. We note in passing that for any group $H$ and character $\chi$, $G_{m,n}$ and $Q_{m,n}$ are subsets of $\Delta$ and $\Delta$ respectively. A linear map $T$ induces in $\wedge^m V$ a linear map $C_m(T)$ called the $m$th compound of $T$. When $\chi \equiv 1$ on $S_m$, $V^m(H)$ is called the $m$th symmetric space which we denote by $V^{(m)}$ with decomposable tensors written simply $v_1 \ldots v_m$. For unitary $V$, the induced inner product (9) satisfies

$$(u_1 \ldots u_m, v_1 \ldots v_m) = (1/m!)|\text{per}([u_i, v_j])|,$$

where “per” is the permanent function. For $V^{(m)}$ both $\Delta$ and $\Delta$ coincide with the sequence set $G_{m,n}$. The $m$th induced map here is often called the $m$th induced power of $T$ and is usually denoted by $P_m(T)$. The corresponding induced matrices $C_m(A)$ and $P_m(A)$ when $A$ is an incidence matrix of a certain combinatorial configuration were studied by Ryser (9; 10). Much of the current interest in inequalities for matrix functions can be traced to these efforts.
III. Proofs. Before proving Theorem 1 we need two results the first of which is immediate from definitions.

**Lemma 1.** Let $S: V \to V$ and $T: V \to V$ be self-adjoint linear mappings on a unitary space $V$ satisfying $S \geq T$. If $X: V \to V$ is any linear mapping, then $X^*SX \geq X^*TX$.

We now assert our second preliminary result.

**Lemma 2.** Let $V_1, \ldots, V_r$ be unitary vector spaces and let $S_i: V_i \to V_i$ and $T_i: V_i \to V_i$ denote self-adjoint linear mappings satisfying $S_i \geq T_i \geq 0$, $i = 1, \ldots, r$, relative to the respective inner products. Then, relative to the induced inner product,

$$S_1 \otimes \ldots \otimes S_r \geq T_1 \otimes \ldots \otimes T_r \geq 0.$$

**Proof.** The right-hand side of the identity

$$S_1 \otimes S_2 - T_1 \otimes T_2 = (S_1 - T_1) \otimes S_2 + T_1 \otimes (S_2 - T_2)$$

is clearly positive. The proof now follows by induction on $r$.

We proceed to the proof of Theorem 1. Let $S \geq T \geq 0$ for $S: V \to V$ and $T: V \to V$, where $V$ is an $n$-dimensional unitary vector space. By Lemma 2, the $m$th Kronecker powers of $S$ and $T$ satisfy $\Pi^m(S) \geq \Pi^m(T) \geq 0$. Set $T_M = (1/h)\sum_{\sigma \in H} M(\sigma) \otimes P(\sigma)$ and let $I_U$ denote the identity map on $U$. By Lemmas 1 and 2,

$$T_M^*(I_U \otimes \Pi^m(S))T_M \geq T_M^*(I_U \otimes \Pi^m(T))T_M \geq 0.$$ 

However, $T_M = T_M^* = T_M^2$. Hence, for decomposable tensors $x = x_1 \otimes \ldots \otimes x_m$ in $\otimes^m V$ and $u \in U$,

$$(T_M(I_U \otimes \Pi^m(S))u \otimes x, u \otimes x) \geq (T_M(I_U \otimes \Pi^m(T))u \otimes x, u \otimes x) \geq 0.$$ 

Alternatively, with $\phi = \sigma^{-1}$, we have

$$\left( \sum_{\sigma \in H} M(\sigma) u \otimes S_{\phi(1)} \otimes \ldots \otimes S_{\phi(m)}, u \otimes x_1 \otimes \ldots \otimes x_m \right) \geq 0.$$ 

Hence,

$$\left( \left( \sum_{\sigma \in H} M(\sigma) \prod_{i=1}^m (S_{\phi(i)}, x_{\sigma(i)}) \right) u, u \right) \geq 0.$$ 

(10)
We set \( m = n \), \( V = C^n \). Let \( A = [a_{ij}] \) and \( B = [b_{ij}] \) be positive semi-definite hermitian matrices and set \( S = A^T \) and \( T = B^T \). Let \( e_i = (\delta_{1i}, \ldots, \delta_{mi})^T \), \( i = 1, \ldots, n \), the standard basis of \( C^n \). Then \( S \geq T \geq 0 \) implies, by (10), that

\[
\left( \sum_{\sigma \in \mathcal{H}} M(\sigma) \prod_{i=1}^n a_{i\sigma(i)} \right) u, u \geq \left( \sum_{\sigma \in \mathcal{H}} M(\sigma) \prod_{i=1}^n b_{i\sigma(i)} \right) u, u \geq 0
\]

which is the assertion of Theorem 1.

The preceding argument also proves Theorem 4. Let \( s = \dim U = 1 \). \( K(S) \) is just the restriction of \( \Pi^m(S) \) to \( \text{rng} \ T_x = V_x \cap \mathcal{H} \). Thus, \( K(S) \) satisfies \( T_x \Pi^m(S)T_x = \Pi^m(S)T_x = K(S) \). Before proving Theorem 2 we prove the following lemma.

**Lemma 3.** Let \( A = [a_{ij}] \) denote an \( n \)-square positive semi-definite hermitian matrix whose row sums \( r_i \) satisfy \( \sum_{i=1}^n r_i = r \neq 0 \). Then the \( n \)-square matrix \( R = [[r_{ij}]/r] \) of rank 1 is positive semi-definite hermitian and \( A \geq R \).

*Proof.* \( R \) is a hermitian matrix when \( r \) is real and is positive if \( r \) is positive. Let \( e \) denote the column vector in \( C^n \) each of whose entries is 1. Then \( r \geq 0 \) implies that \( r = e^T A e = (A e, e) > 0 \). Now let the positive semi-definite hermitian matrix \( A = [x_i] = [(x_i, x_j)] \) represent the Gram matrix of the set of vectors \( x_1, \ldots, x_n \) in \( C^n \). Let \( P \) denote the \( n \)-square matrix whose \( i \), \( j \) entry is \( x_j^T x_i \), \( i = 1, \ldots, n \), and let \( Q \) be the \( n \)-square matrix each of whose column vectors is the vector \( u = x_1 + \ldots + x_n \). Denote by \( J_n \) the positive semi-definite hermitian matrix of rank 1 each of whose entries is \( 1/n \). We observe that \( PP^* = A \), that \( Q^* Q = rnJ_n = QQ^* \), and that \( PQ \) has \( r \) as its \( i, j \) entry. We claim that the \( n \)-square identity matrix \( I_n \) satisfies \( I_n \geq J_n \). Since \( J_n e = e \), a unitary matrix \( U \) exists for which \( U^* J_n U = E_n \), the \( n \)-square matrix with 1 in the 1,1 entry and zeros elsewhere. Clearly, \( I_n \geq E_n \).

Lemma 1 now proves the claim. Finally, we compute using Lemma 1

\[
rnA = rnP_i P_i^* \geq rnP_i P_i^* = PQ^* P^* = PQ(PQ^*)^* = n[r_{ij}].
\]

The inequality (2) of Theorem 2 is now easy. \( A \geq R \) implies that \( A [\omega | \omega] \geq R [\omega | \omega] \) for every choice of \( \omega = (\omega_1, \ldots, \omega_n) \in \Gamma_{m,n} \). However, \( d^H_x (R [\omega | \omega]) = (h/r^m) \prod_{i=1}^m |r_{\omega_i}|^2 \) when \( \chi = 1 \). Thus, by Corollary 1,

\[
d^H_x (A [\omega | \omega]) \geq (h/r^m) \prod_{i=1}^m |r_{\omega_i}|^2.
\]

Summing over all \( \omega \in Q_{m,n} \) yields

\[
(12) \quad \sum_{\omega \in Q_{m,n}} d^H_x (A [\omega | \omega]) \geq (h/r^m) \prod_{i=1}^m |r_{\omega_i}|^2.
\]

The function \( f(t) = t^2 \) is everywhere strictly convex so that

\[
\left( \frac{n}{m} \right) \sum_{\omega \in Q_{m,n}} \prod_{i=1}^m |r_{\omega_i}|^2 \geq \left( \sum_{\omega \in Q_{m,n}} \prod_{i=1}^m |r_{\omega_i}| \right)^2 = \left( \left( \frac{n}{m} \right) p_m(|r_1|, \ldots, |r_n|) \right)^2;
\]

(12) and (13) together imply (2). In order that equality hold in (12), equality must hold for every \( \omega \in Q_{m,n} \) in (11). The vector \( u = x_1 + \ldots + x_n \) is
non-zero since its length is \( r^{1/2} \). Hence, the tensor \( u \star \ldots \star u/\|u \star \ldots \star u\| \) is non-zero and of unit length (7, Lemma 2.4). An easy calculation using (9) yields
\[
\|x_u^\ast, u \star \ldots \star u/\|u \star \ldots \star u\|\|^2 = hd_x^H(R[\omega|\omega])
\]
for every \( \omega \in Q_{m,n} \) so that the inequality (11) is just the Cauchy-Schwarz inequality in \( V_x^m(H) \)
\[
(14) \quad \|x_u^\ast\|^2 \geq \| (x_u^\ast, u \star \ldots \star u/\|u \star \ldots \star u\|) \|^2.
\]
Equality holds in (14) if and only if \( x_u^\ast = 0 \) or \( x_u^\ast \) and \( u \star \ldots \star u \) are multiples. Again by (7, Lemma 2.4), \( x_u^\ast = 0 \) if and only if some \( x_u^i = 0 \), \( i = 1, \ldots, m \). In other words, equality holds in (11) if and only if \( A[\omega|\omega] \) has a zero row or \( A[\omega|\omega] \) has rank 1. If for each \( \omega \in Q_{m,n} \), \( A[\omega|\omega] \) has a zero row, then fewer than \( m \) of the vectors \( x_u^i \), hence, fewer than \( m \) rows of \( A \), are non-zero. Conversely, if fewer than \( m \) rows of \( A \) are non-zero, then every principal submatrix \( A[\omega|\omega] \) of \( A \) has a zero row. Clearly, if each \( A[\omega|\omega] \) has rank at most 1, then \( A \) has rank at most 1. On the other hand, suppose that \( A \) has rank 1. Then \( A = [c, \tilde{e}] \). An easy calculation shows that both sides of (12) equal \( h\sum_{\omega \in Q_{m,n}} \prod_{i=1}^m |r_{\omega}^i|^2 \). Finally, equality holds in (13) if and only if all the non-zero products \( \prod_{i=1}^m |r_{\omega}^i|^2 \), \( \omega \in Q_{m,n} \), are equal. This is the case if and only if \( A \) has exactly \( m \) non-zero rows, \( m = n \), or all the non-zero \( |r_{\omega}^i| \), \( i = 1, \ldots, m \), are equal.

The proof of the inequality (3) of Theorem 3 follows from the proof of (2). However, instead of summing over the \( \binom{n}{m} \) sequences \( \omega \in Q_{m,n} \) we sum over the \( \binom{n+m-1}{m-1} \) sequences of \( G_{m,n} \). The proof in the cases of equality in (3) can be argued in a manner similar to that for (2).

The proof of Corollary 1 for a character \( \chi \) of degree 1 and, hence, Theorems 2 and 3 can also be obtained from the formula (9) and the \( m \)th induced mapping \( K(S) : V_x^m(H) \to V_x^m(H) \). Set \( V = C^n \) and \( \chi = 1 \). Let \( S = A^T \) and \( T = B^T \). Then \( A \geq B \geq 0 \) implies that \( S \geq T \geq 0 \) whence, by Theorem 4, \( K(S) \geq K(T) \). Let \( \omega = (\omega_1, \ldots, \omega_m) \in G_{m,n} \) and let \( e_1, \ldots, e_n \) denote the standard basis of \( C^n \). Then using (9), we have that
\[
d_x^H(A[\omega|\omega]) = d_x^H((Se_{\omega^i}, e_{\omega^i})) = h(K(S)e_{\omega^i}, e_{\omega^i}) \geq h(K(T)e_{\omega^i}, e_{\omega^i}) = d_x^H(Te_{\omega^i}, e_{\omega^i}) = d_x^H(B[\omega|\omega]).
\]
Set \( m = n \) and \( \omega = (1, \ldots, n) \) to obtain Corollary 1. Of course, Theorem 1 provides this result immediately.

References


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