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p-adic confluence of q-difference equations

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Abstract

We develop the theory of p-adic confluence of q-difference equations. The main result is the fact that, in the p-adic framework, a function is a (Taylor) solution of a differential equation if and only if it is a solution of a q-difference equation. This fact implies an equivalence, called *confluence*, between the category of differential equations and those of q-difference equations. We develop this theory by introducing a category of *sheaves* on the disk $\mathrm{D}^{-}(1,1)$, for which the stalk at 1 is a differential equation, the stalk at q is a q-difference equation if q is not a root of unity, and the stalk at a root of unity ξ is a mixed object, formed by a differential equation and an action of σ_ξ.

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Introduction

The main aim of this paper is to provide a *theory of confluence* for q-difference equations in the p-adic framework.

A motivation: the rough idea of the confluence

Heuristically we say that a family of q-difference equations $\{\sigma_q(Y_q) = A(q,T) \cdot Y_q\}_{q \in \mathrm{D}^-(1,\epsilon)-\{1\}}$ (where σ_q is the automorphism $f(T) \mapsto f(qT)$) is confluent to the differential equation $\delta_1(Y_q) = G(1,T) \cdot Y_q$, with $\delta_1 := T\,d/dT$, if one has

$$\lim_{q \to 1} G(q,T) = G(1,T)$$

where $G(q,T) = (A(q,T) - I)/(q-1)$ is the matrix of the q-derivation $\Delta_q := (\sigma_q - 1)/(q-1)$ acting on M, and moreover if, in some suitable meaning,

$$\lim_{q \to 1} Y_q = Y_1. \tag{0.1}$$

Roughly speaking, in this paper we show that in the p-adic framework, if a differential equation is given, then, for ϵ sufficiently small, one may choose the family $\{G(q,T)\}_q$ in order to have $Y_q = Y_1$, for all $q \in \mathrm{D}^-(1,\epsilon)$. Conversely if q_0 is not a root of unity, and if a single equation $\sigma_{q_0}(Y_{q_0}) = A(q_0,T) \cdot Y_{q_0}$ is given, then, under some assumptions on the radius of convergence of its *generic* Taylor solution Y_{q_0}, one can find a differential equation and family as above with the property that $Y_q = Y_{q_0} = Y_1$, for all $q \in \mathrm{D}^+(1, |q_0 - 1|)$. In this sense, in the p-adic context, the solutions of q-difference equations are not simply a *discretization* of the solutions of differential equations, but they are actually equal. We want now to state these facts more precisely.

The work of André and Di Vizio

In [ADV04] André and Di Vizio initiated the study of the phenomena of confluence in a p-adic setting. For K a complete discrete valuation field of mixed characteristic, they found an equivalence between the category of q-difference equations with Frobenius structure over the Robba ring $\mathcal{R}_{K^{\mathrm{alg}}}$ (here called σ_q-Mod$(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$), and the category of differential equations with Frobenius structure over the Robba ring $\mathcal{R}_{K^{\mathrm{alg}}}$ (here called δ_1-Mod$(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$).

One of the restrictions of [ADV04] is that the number q is required to satisfy $|q-1| < |p|^{1/(p-1)}$. Indeed, in the annulus $|q-1| = |p|^{1/(p-1)}$ one encounters the pth root of unity and, if $\xi^p = 1$, then the category $\sigma_\xi\text{-Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$ is different in nature from the category of differential equations, since it is not K^{alg}-linear (that is, the ring of automorphisms of the unit object is strictly larger than K^{alg}).

The equivalence of [ADV04] is obtained as follows. In [And02] one proves that the Tannakian group of $\delta_1\text{-Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$ is $\mathcal{I}_{k^{\mathrm{alg}}((t))} \times \mathbb{G}_a$, where k is the perfect residue field of K, and $\mathcal{I}_{k^{\mathrm{alg}}((t))}$ is the absolute Galois group of $k^{\mathrm{alg}}((t))$. On the other hand in [ADV04] one shows that $\sigma_q\text{-Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$ has the same Tannakian group $\mathcal{I}_{k^{\mathrm{alg}}((t))} \times \mathbb{G}_a$. By composition with the respective Tannakian equivalences (T_q and T_1 below), one then obtains the so-called *confluence functor* Conf_q (in the notation of [ADV04] one has $T_1 = \mathrm{V}_d^{(\phi)}$ and $T_q = \mathrm{V}_{\sigma_q}^{(\phi)}$):

$$\begin{array}{ccc} \sigma_q\text{-Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)} & \xrightarrow[\cong]{\ \mathrm{Conf}_q\ } & \delta_1\text{-Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)} \\ & {\scriptstyle T_q,\ \cong}\searrow \qquad \swarrow{\scriptstyle \cong,\ T_1} & \\ & \underline{\mathrm{Rep}}_{K^{\mathrm{alg}}}(\mathcal{I}_{k^{\mathrm{alg}}((t))} \times \mathbb{G}_a) & \end{array} \tag{0.2}$$

The strategy of [ADV04] consists in showing that, as in the case of differential equations (cf. [And02]), every object M in $\sigma_q\text{-Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$ is quasi-unipotent, i.e. becomes unipotent after scalar extension to a special extension of $\mathcal{R}_K$ (cf. § 8.3). Once a basis of M is fixed, this means that M admits a complete basis of solutions $\widetilde{Y} \in GL_n(\widetilde{\mathcal{R}_K}[\log(T)])$, where $\widetilde{\mathcal{R}_K}$ is the union of all special extensions of $\mathcal{R}_K$ (it is a sort of lifting of $k((t))^{\mathrm{alg}}$). We will call *étale* solutions the solutions of M in $\widetilde{\mathcal{R}_K}[\log(T)]$. The proof of this relevant result needs a substantial effort, and is actually not less complicated than the classical p-adic local monodromy theorem for differential equations itself (i.e. the fact that T_1 is an equivalence). Thanks to the fact that this important, but also very peculiar, class of q-difference and differential equations are trivialized by $\widetilde{\mathcal{R}_K}[\log(T)]$, one can define T_1 (respectively T_q) as the functor associating to a differential (respectively q-difference) equation $(\mathrm{M}, \delta_1^{\mathrm{M}})$ (respectively $(\mathrm{M}, \sigma_q^{\mathrm{M}})$) the K^{alg}-vector space $T_1(\mathrm{M}, \delta_1^{\mathrm{M}})$ (respectively $T_q(\mathrm{M}, \sigma_q^{\mathrm{M}})$) of its 'étale' solutions in $\widetilde{\mathcal{R}_K}[\log(T)]$.[1] The action of $\mathcal{I}_{k^{\mathrm{alg}}((t))} \times \mathbb{G}_a$ on the space of the 'étale' solutions arises from its action on $\widetilde{\mathcal{R}_K}[\log(T)]$ by $\mathcal{R}_K$-linear automorphisms commuting with δ_1 and σ_q on $\widetilde{\mathcal{R}_K}[\log(T)]$.

Hence one sees for the first time in [ADV04] the fact that the 'étale' solutions of a q-difference equation with Frobenius structure are also the 'étale' solutions of a differential equation. Moreover the functor Conf_q is nothing but the functor sending a q-difference equation (with (strong) Frobenius structure) into the differential equation having the same solutions.

In the present paper we prove that this 'permanence' of the solutions holds also for *Taylor solutions* (see below). We then develop a p-adic theory of confluence using, as a unique tool, this fact, here called *propagation principle*. We prove indeed that this principle is sufficient to define the confluence and deformation equivalences, over almost all p-adic ring of functions, with very basic assumptions on the equations. This theory requires only the definition and the formal properties of the *generic Taylor solution* $Y(x, y)$. For this reason it is not a consequence of the previously developed theory (as presented in [ADV04] and [DV04]). Conversely we deduce, as a special case, the confluence of [ADV04] by comparing Taylor solutions and 'étale' solutions (cf. the end of the introduction).

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[1] Following the definition in § 3.2, $\mathrm{V}_d^{(\phi)}(\mathrm{M}) := (\mathrm{M} \otimes_{\mathcal{R}_K} \widetilde{\mathcal{R}_K}[\log(T)])^{\delta_1 = 0}$ is actually the dual of the space of solutions $\mathrm{Hom}^{\delta_1}_{\mathcal{R}_K}(\mathrm{M}, \widetilde{\mathcal{R}_K}[\log(T)])$ (respectively same remark for $\mathrm{V}_{\sigma_q}^{(\phi)}(\mathrm{M}) := (\mathrm{M} \otimes_{\mathcal{R}_K} \widetilde{\mathcal{R}_K}[\log(T)])^{\sigma_q = \mathrm{Id}}$ and $\mathrm{Hom}^{\sigma_q}_{\mathcal{R}_K}(\mathrm{M}, \widetilde{\mathcal{R}_K}[\log(T)])$).

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The generic q-Taylor solution

Let now K be an arbitrary ultrametric complete valued field of mixed characteristic $(0,p)$. Let $X = \mathrm{D}^+(c_0,R_0) - \bigcup_{i=1,\dots,n} \mathrm{D}^-(c_i,R_i)$ be an affinoid, where $\mathrm{D}^-(c,R)$ denotes the open disk centered at c of radius R. Let $\mathcal{H}_K(X)$ be the ring of analytic elements on X. Consider a q-difference equation

$$\sigma_q(Y) = A(q,T)\cdot Y, \quad A(q,T) \in GL_n(\mathcal{H}_K(X)) \tag{0.3}$$

on X. Denote by $(\mathrm{M},\sigma_q^{\mathrm{M}})$ the q-difference module over X defined by this equation.

A major difference between the complex and the p-adic settings is that in the latter there are disks (not centered at 0) which are q-invariant. A disk $\mathrm{D}^-(c,R) \subset X(K)$ is q-invariant (i.e. the map $x \mapsto qx$ is a bijection of $\mathrm{D}^-(c,R)$) if and only if $|q-1||c| < R$, and $|q| = 1$ (cf. Lemma 5.1). Starting from this consideration, in [DV04] Di Vizio defines, for q-difference equations, the q-analog of the generic Taylor solution of a differential equation (cf. Definition 5.11):

$$Y(x,y) := \sum_{n\geqslant 0} H_n(q,T)\frac{(x-y)_{q,n}}{[n]_q^!}, \tag{0.4}$$

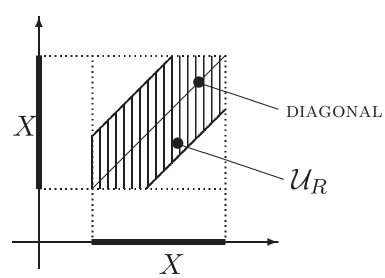
where $H_n(q,T)$ is obtained by iterating the equation (0.3): $d_q^n(Y) = H_n(q,T)\cdot Y$, where

$$d_q := \frac{\sigma_q - 1}{(q-1)T}.$$

For a large class of equations it happens that, for all $c \in X(K)$, the series $Y(x,c)$ represents a function which converges on a disk $\mathrm{D}^-(c,R)$, with $|q-1||c| < R$. More precisely $Y(x,y)$ converges in a neighborhood of the diagonal of the type $\mathcal{U}_R := \{(x,y) \in X\times X \mid |x-y| < R\}$, with

$$|q-1|\cdot \mathfrak{s}_X < R, \tag{0.5}$$

where $\mathfrak{s}_X := \sup_{c\in X}|c|$ as shown in the following picture (one easily sees that $\mathfrak{s}_X = \max(|c_0|,R_0)$).



We call such equations *Taylor admissible*. The matrix function $Y(x,y) : \mathcal{U}_R \to GL_n(K)$ is invertible and satisfies the cocycle conditions: $Y(x,y)\cdot Y(y,z) = Y(x,z)$ and $Y(x,y)^{-1} = Y(y,x)$, for all $(x,y),(y,z),(x,z) \in \mathcal{U}_R$. Moreover $Y(qx,y) = A(q,x)Y(x,y)$ and, for all $c \in X(K)$, the matrix $Y(x,c) \in GL_n(\mathcal{A}_K(c,R))$ is a fundamental basis of solutions of the equation (0.3). In particular the q-difference algebra $\mathcal{A}_K(c,R)$ of analytic functions over the disk $\mathrm{D}^-(c,R)$ trivializes $(\mathrm{M},\sigma_q^{\mathrm{M}})$.

The following fact is the main point of this paper (cf. Theorem 7.7). If now $q' \neq q$ belongs to the disk $\mathrm{D}^-(q,R/\mathfrak{s}_X) = \mathrm{D}^-(1,R/\mathfrak{s}_X)$, then the matrix

$$A(q',x) := Y(q'x,y)\cdot Y(x,y)^{-1} = Y(q'x,y)\cdot Y(y,x) = Y(q'x,x) \tag{0.6}$$

is an analytic function of x on all of X. Indeed $(q'x,x) \in \mathcal{U}_R$, for all $x \in X$, and hence the matrix $A(q',x)$ maps $x \mapsto (q'x,x) \mapsto Y(q'x,x) = A(q',x)$. One shows easily that $A(q',x) \in GL_n(\mathcal{H}_K(X))$, for all $q' \in \mathrm{D}^-(1,R/\mathfrak{s}_X)$, since $Y(x,y)$ is invertible. This fact implies that $Y(x,y)$ *is simultaneously the Taylor solution of every equation of the family* $\{\sigma_{q'}(Y) = A(q',T)Y\}_{q'}$, *for all* $q' \in \mathrm{D}^-(1,R/\mathfrak{s}_X)$. Equivalently, this means that the q-difference module $(\mathrm{M},\sigma_q^{\mathrm{M}})$ is canonically endowed with an action

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of $\sigma_{q'}$, for all $q' \in \mathrm{D}^-(1, R/\mathfrak{s}_X)$. This remarkable fact will be called the *propagation principle*. As one can see, this happens actually under the following weak assumptions on $(\mathrm{M}, \sigma_q^{\mathrm{M}})$:

(i) q is not a root of unity; (0.7)

(ii) $Y(x, y)$ converges on some $\mathcal{U}_R$ with $|q-1| \cdot \mathfrak{s}_X < R \leqslant r_X$; (0.8)

where $r_X = \min(R_0, R_1, \ldots, R_n)$ is a number depending on the geometry of X. The category of q-difference modules $(\mathrm{M}, \sigma_q^{\mathrm{M}})$ satisfying these two properties for a suitable unspecified R satisfying $|q-1|\mathfrak{s}_X < r \leqslant R \leqslant r_X$ will be denoted by $\sigma_q\text{-Mod}(\mathcal{H}_K(X))^{[r]}$.

The assumption $|q-1|\mathfrak{s}_X < R$ assures that the image of the map $x \mapsto (qx, x) : X \mapsto X \times X$ is contained in $\mathcal{U}_R$. The bound $R \leqslant r_X$ assures that the function $Y(x, y)$ does not converge outside X. Indeed the properties of $Y(x, y)$ outside X are not invariant under $\mathcal{H}_K(X)$-base changes in M. Finally condition (ii) also assures that the map $x \mapsto qx$ is a bijection of X globally fixing each individual hole of X (cf. § 5.2). Since $r_X \leqslant \mathfrak{s}_X$, we are assuming implicitly that $|q-1| < 1$. But no restrictive assumptions on X or on K are made.

Obviously this process works just as well if the initial function $Y(x, y)$ is the generic Taylor solution of a differential equation. The category of *differential* equations whose Taylor solution converges on $\mathcal{U}_R$, for an unspecified R satisfying $r \leqslant R \leqslant r_X$, will be denoted by $\delta_1\text{-Mod}(\mathcal{H}_K(X))^{[r]}$.

Discrete and analytic σ-modules

Let $\mathcal{Q}(X)$ be the set of $q \in K$ for which $x \mapsto qx$ is a bijection of X. Then $\mathcal{Q}(X)$ is a topological subgroup of $K^\times$, and the disk $\mathrm{D}^-(1, R/\mathfrak{s}_X)$, with $R \leqslant r_X$, is an open subgroup of $\mathcal{Q}(X)$. The group $\mathcal{Q}(X)$ acts continuously on $\mathcal{H}_K(X)$ via $q \mapsto \sigma_q$. The data of M, together with the simultaneous σ_q-semi-linear action of σ_q^{M}, for all $q \in \mathrm{D}^-(1, R/\mathfrak{s}_X)$, is then a *semi-linear representation of the subgroup* $\mathrm{D}^-(1, R/\mathfrak{s}_X) \subseteq \mathcal{Q}(X)$. This representation has the following three remarkable properties.

(a) The map $(q', x) \mapsto A(q', x)$ is analytic in (q', x). In particular, the representation is continuous.

(b) The group $\mathrm{D}^-(1, R/\mathfrak{s}_X)$ depends on R, and hence on M.

(c) The matrix $Y(x, y)$ is simultaneously the generic Taylor solution of the q-difference module $(\mathrm{M}, \sigma_q^{\mathrm{M}})$, for all $q \in \mathrm{D}^-(1, R/\mathfrak{s}_X)$.

Inspired by the first two properties we define a new class of objects called *discrete or analytic σ-modules* as follows. Consider a subset $S \subset \mathcal{Q}(X)$. A *discrete σ-module* on S is nothing but an $\mathcal{H}_K(X)$ semi-linear representation of the group $\langle S \rangle$ generated by S. If $S = U$ is an open subset of $\mathcal{Q}(X)$, we define *analytic σ-modules on U* to be a discrete σ-modules over U together with a certain condition of analyticity of σ_q^{M} with respect to q. These categories are denoted by $\sigma\text{-Mod}(\mathcal{H}_K(X))_S^{\mathrm{disc}}$ and $\sigma\text{-Mod}(\mathcal{H}_K(X))_U^{\mathrm{an}}$ respectively. In this paper the words *discrete* or *analytic* will refer to the discreteness or analyticity of σ_q^{M} with respect to q. We heuristically imagine the analytic σ-modules as *semi-linear representations of the (co-variant) sheaf of groups* $U \mapsto \langle U \rangle$.

Remark 0.1. It is important to notice that morphisms between analytic σ-modules over U are morphisms of representations. More precisely, once a basis of M (respectively N) is fixed, we have a family of operators $\{\sigma_q(Y) = A(q, T)Y\}_{q \in \langle U \rangle}$ (respectively $\{\sigma_q(Y) = \widetilde{A}(q, T)Y\}_{q \in \langle U \rangle}$) such that $A(q, T)$ (respectively $\widetilde{A}(q, T)$) depends analytically on (q, T).[2] A morphism $\alpha : \mathrm{M} \to \mathrm{N}$ then must simultaneously commute with σ_q^{M} and σ_q^{N}, for all $q \in \langle U \rangle$. In other words the matrix B of α must simultaneously verify $A(q, T)B = \sigma_q(B)\widetilde{A}(q, T)$, for all $q \in \langle U \rangle$. Actually there are *non-isomorphic* analytic σ-modules over U defining isomorphic q-difference equations at every $q \in \langle U \rangle$

[2]The data of an analytic σ-module is actually nothing but 'a family of q-difference equations depending analytically on q'.

(see Example 2.6). This is analogous to having *non-isomorphic* sheaves having isomorphic stalks at every point.

Taylor admissible σ-modules

We now want to analyze property (c): the constancy of the solutions. If $S \not\subseteq \boldsymbol{\mu}_{p^\infty}$ (where $\boldsymbol{\mu}_{p^\infty} := \{\xi \in K^{\mathrm{alg}} \mid \xi^{p^n} = 1, \exists n \geqslant 1\}$), we call *Taylor admissible σ-modules over S* those σ-modules for which the q-Taylor solution $Y(x,y)$ is the same for all $q \in \langle S\rangle$, and satisfy the condition (ii), for all $q \in S$ (cf. (0.8)). If $S = U$ is open, by the propagation principle, Taylor admissible σ-modules are *automatically* analytic on U (cf. Remark 7.8). This category is denoted by $\sigma\text{-Mod}(\mathcal{H}_K(X))_U^{\mathrm{adm}} \subseteq \sigma\text{-Mod}(\mathcal{H}_K(X))_U^{\mathrm{an}}$. We heuristically imagine Taylor admissible σ-modules as *semi-linear representations of the (co-variant) sheaf of groups $U \mapsto \langle U\rangle$*, which are *locally constant*.

Taylor admissibility is a particular case of a more classical notion. If $\mathrm{C}/\mathcal{H}_K(X)$ is an algebra admitting an action of the group $\langle S\rangle$ extending that on $\mathcal{H}_K(X)$, then a semi-linear representation of $\langle S\rangle$ over $\mathcal{H}_K(X)$ is called C-*admissible* if it is trivialized by C. For a discrete σ-module M over S to be trivialized by C means exactly that there exists $Y \in GL_n(\mathrm{C})$ which is a simultaneous solution of all operators defined by M. If M is trivialized by C we will say that M is C-*constant*. We observe that if $S = q^{\mathbb{Z}}$, then C is nothing but a q-difference algebra over $\mathcal{H}_K(X)$. So the constancy of the solutions does not depend on the analyticity of M; rather it is a *discrete* fact.

In § 3 we define *discrete σ-algebras*, and we develop a basic differential/difference Galois theory for discrete σ-algebras. The analog of the Picard–Vessiot theorem providing the existence of a discrete σ-algebra trivializing a given discrete σ-module *is missing*. We are thus obliged to work with the category of discrete σ-modules trivialized by a fixed discrete σ-algebra C. In § 4 we develop formally the theory of C-confluence and C-deformation, which will also depend on the chosen discrete σ-algebra C.

Remark 0.2. Notice that solutions will be defined formally as morphisms $\mathrm{M} \to \mathrm{C}$ commuting simultaneously with the actions of σ_q for all $q \in S$ (cf. § 3.2). This fact, together with Remark 0.1, explains why the notion of C-*constant* σ-module implies the constancy of the solutions (with respect to q).

The confluence functor

Let $(\mathrm{M}, \sigma^{\mathrm{M}})$ be an analytic σ-module over U. By analyticity we also have an action of the *Lie algebra* of $\langle U\rangle$ (here systematically identified with $K \cdot \delta_1$). In other words the following limit converges to a connection $\delta_1^{\mathrm{M}} : \mathrm{M} \to \mathrm{M}$ (cf. § 2.4):

$$\delta_1^{\mathrm{M}} := \lim_{q \in \langle U\rangle, q \to 1} \frac{\sigma_q^{\mathrm{M}} - 1}{q - 1} \in \mathrm{End}_K^{\mathrm{cont}}(\mathrm{M}), \tag{0.9}$$

where q runs over the (open) group $\langle U\rangle$ generated by U. In terms of matrices, the matrix $G(1,T)$ of δ_1^{M} is

$$G(1,T) = q\frac{\partial}{\partial q}(A(q,T))_{|_{q=1}}$$

(cf. Equation (2.4.5)). By continuity, morphisms of analytic σ-modules also commute with the connection (cf. Remark 2.5(1)). Hence we obtain a functor called $\mathrm{Conf}_U : \sigma\text{-Mod}(\mathcal{H}_K(X))_U^{\mathrm{an}} \longrightarrow \delta_1\text{-Mod}(\mathcal{H}_K(X))$, sending $(\mathrm{M}, \sigma^{\mathrm{M}})$ into $(\mathrm{M}, \delta_1^{\mathrm{M}})$ (cf. Remark 2.13). This functor is not an equivalence, but it does induce an equivalence:

$$\mathrm{Conf}_U^{\mathrm{Tay}} : \sigma\text{-Mod}(\mathcal{H}_K(X))_U^{[r]} \xrightarrow{\sim} \delta_1\text{-Mod}(\mathcal{H}_K(X))^{[r]}, \tag{0.10}$$

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where $\mathrm{Conf}_U^{\mathrm{Tay}}$ simply denotes the restriction of Conf_U to the category

$$\sigma\text{-Mod}(\mathcal{H}_K(X))_U^{[r]} \subseteq \sigma\text{-Mod}(\mathcal{H}_K(X))_U^{\mathrm{adm}}$$

of Taylor admissible σ-modules verifying condition (ii) with $r \leqslant R \leqslant r_X$ (cf. (0.8)), where $r > 0$ is large enough to have $U \subset \mathrm{D}^-(1, r/\mathfrak{s}_X)$ (cf. Corollary 7.9). The propagation principle gives a quasi-inverse functor (cf. Remark 2.13 for a formal presentation).

On the other hand let $q \in U - \boldsymbol{\mu}_{p^\infty}$. An analytic σ-module over U defines a q-difference module by forgetting the action of $\sigma_{q'}^{\mathrm{M}}$, for all $q' \neq q$. Again the propagation principle provides an equivalence

$$\mathrm{Res}_q^U : \sigma\text{-Mod}(\mathcal{H}_K(X))_U^{[r]} \xrightarrow{\sim} \sigma_q\text{-Mod}(\mathcal{H}_K(X))^{[r]}, \tag{0.11}$$

where $r \leqslant r_X$ is sufficiently large to have $U \subseteq \mathrm{D}^-(1, r/\mathfrak{s}_X)$ (cf. Corollary 7.9). We call the composite equivalence $\mathrm{Conf}_q^{\mathrm{Tay}}$. Thus we have

$$\mathrm{Conf}_q^{\mathrm{Tay}} := \mathrm{Conf}_U^{\mathrm{Tay}} \circ (\mathrm{Res}_q^U)^{-1} : \sigma_q\text{-Mod}(\mathcal{H}_K(X))^{[r]} \xrightarrow{\sim} \delta_1\text{-Mod}(\mathcal{H}_K(X))^{[r]}. \tag{0.12}$$

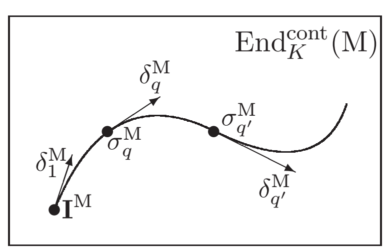
The equivalence $\mathrm{Conf}_q^{\mathrm{Tay}}$ sends a q-difference equation satisfying conditions (i) and (ii) (cf. (0.7) and (0.8)), into the differential equation having the same generic Taylor solution.

Roots of unity and q-tangent operators

In this last equivalence the number q must not belong to $\boldsymbol{\mu}_{p^\infty}$. If $q' = \xi$, with $\xi^{p^n} = 1$, the category of σ_ξ-difference equations is not K-linear and cannot be equivalent to the category of differential equations. Nevertheless, if, for $q \notin \boldsymbol{\mu}_{p^\infty}$, the radius R of the q-Taylor solution is large, the propagation principle gives an operator $\sigma_\xi^{\mathrm{M}} : \mathrm{M} \to \mathrm{M}$ acting on M. The idea is to replace the category σ_ξ-Mod$(\mathcal{H}_K(X))$ with another category. The expected object 'at ξ' should also be endowed with an action of the Lie algebra, *as we have just done in the case* $\xi = 1$. For all $q \in \langle U\rangle$ the action of the Lie algebra of $\langle U\rangle$ is given by the limit

$$\delta_q^{\mathrm{M}} := \lim_{q' \to q} \frac{\sigma_{q'}^{\mathrm{M}} - \sigma_q^{\mathrm{M}}}{q' - q} \in \mathrm{End}_K^{\mathrm{cont}}(\mathrm{M}),$$

for $q, q' \in \langle U\rangle$, as shown in the following diagram.



Clearly $\delta_q^{\mathrm{M}} = \sigma_q^{\mathrm{M}} \circ \delta_1^{\mathrm{M}}$, so to give δ_q^{M} is equivalent to give δ_1^{M}. In a root of unity the 'limit object' is a mixed data $(\mathrm{M}, \sigma_\xi^{\mathrm{M}}, \delta_1^{\mathrm{M}})$, i.e. a connection δ_1^{M} on M together with an action of σ_ξ^{M} on M. We call these new objects (σ_ξ, δ_ξ)*-modules*. In the sequel every terminology is given simultaneously for σ-modules and (σ, δ)-modules. The additional data of δ_ξ^{M} makes the category of (σ_ξ, δ_ξ)-modules K-linear. Moreover δ_ξ^{M} preserve the *information* in a neighborhood of ξ indeed we find equivalences

$$\mathrm{Conf}_\xi^{\mathrm{Tay}} := \mathrm{Conf}_U^{\mathrm{Tay}} \circ (\mathrm{Res}_\xi^U)^{-1} : (\sigma_\xi, \delta_\xi)\text{-Mod}(\mathcal{H}_K(X))^{[r]} \xrightarrow{\sim} \delta_1\text{-Mod}(\mathcal{H}_K(X))^{[r]}, \tag{0.13}$$

$$\mathrm{Def}_{\xi,q}^{\mathrm{Tay}} := \mathrm{Res}_q^U \circ (\mathrm{Res}_\xi^U)^{-1} : (\sigma_\xi, \delta_\xi)\text{-Mod}(\mathcal{H}_K(X))^{[r]} \xrightarrow{\sim} (\sigma_q, \delta_q)\text{-Mod}(\mathcal{H}_K(X))^{[r]}. \tag{0.14}$$

If q is not a root of unity, then the data of δ_1^{M} is superfluous; indeed if the module is Taylor admissible the propagation principle allows one to reconstruct δ_1^{M} from σ_q^{M}.

We notice that in the classical setting over the complex numbers $\mathbb{C}$, understanding of the case $q = \xi \in \boldsymbol{\mu}_{p^\infty}$ remains an open problem.

Quasi-unipotence and comparison with André–Di Vizio's confluence

Up to a correct definition for the notion of Taylor admissibility, the previous theory can be generalized to more general rings of functions. From § 7.4 on we obtain the theory over $\mathcal{R}_K$. We prove that every q-difference equation with Frobenius structure over $\mathcal{R}_K$ is quasi-unipotent (i.e. is trivialized by $\widetilde{\mathcal{R}_K}[\log(T)]$), for all $q \in \mathrm{D}^-(1,1) - \boldsymbol{\mu}_{p^\infty}$, generalizing the main result of [ADV04]. We actually prove this theorem in the more general context of σ-modules, and (σ,δ)-modules. We deduce it by the Quasi-unipotence of p-adic *differential* equations with Frobenius structure over $\mathcal{R}_K$, and by deformation. The idea is the following. As already mentioned, we are obliged to work with σ-modules trivialized by a fixed discrete σ-algebra C, and the C-confluence and C-deformation functors depend on C. In the 'quasi-unipotent' context this algebra is $\mathrm{C} := \widetilde{\mathcal{R}_K}[\log(T)]$, while in the context of the propagation theorem $\mathrm{C} := \mathcal{A}_K(c,R)$, for an arbitrary point $c \in X$, and suitable $R > 0$. To compare Taylor solutions to the 'étale solutions' in $GL_n(\widetilde{\mathcal{R}_K}[\log(T)])$, the idea is to find a discrete σ-algebra of *functions over a disk* containing $\widetilde{\mathcal{R}_K}[\log(T)]$. Actually such an algebra does not exist. Thus we use a theorem of Matsuda [Mat02] (cf. Theorem 8.13) providing an equivalence between $\delta_1\text{-Mod}(\mathcal{R}_K)^{(\phi)}$ with the sub-category of $\delta_1\text{-Mod}(\mathcal{H}_K^\dagger)^{(\phi)}$ formed by special objects. Special objects are trivialized by a special extension of $\mathcal{H}_K^\dagger$ (cf. § 8.3). The ring $\mathcal{A}_K(1,1)$ is a discrete σ-algebra over $\mathcal{H}_K^\dagger$. We then prove that the algebra $\widetilde{\mathrm{C}_K^{\text{ét}}}[\log(T)]$ generated over $\mathcal{H}_K^\dagger$ by all the 'étale solutions' of special objects admits an embedding $\widetilde{\mathrm{C}_K^{\text{ét}}}[\log(T)] \subset \mathcal{A}_{K^{\mathrm{alg}}}(1,1)$ commuting with δ_1, with the Frobenius, and with σ_q^{M}, for all $q \in \mathrm{D}^-(1,1) - \boldsymbol{\mu}_{p^\infty}$ (cf. Lemma 8.24). This will prove that the C-confluence and the C-deformation functors defined by using $\mathrm{C} = \mathcal{A}_K(1,1)$ or $\mathrm{C} = \widetilde{\mathcal{R}_K}[\log(T)]$ are actually the same (cf. Corollary 8.26). Moreover it proves also that the confluence of André–Di Vizio coincides with our $\mathrm{Conf}_q^{\mathrm{Tay}}$ (cf. § 8.5), and thus it is independent on the Frobenius.

Structure of the paper

Section 1 is devoted to notation. In § 2, we give definitions and basic facts on *discrete/analytic σ-modules and (σ,δ)-modules*. In § 3 we define *discrete σ-algebras and (σ,δ)-algebras*, and we give the abstract definition of *solutions*. In § 4 we give the formal notion of *confluence*. In § 5 we introduce *generic Taylor solutions* and in § 6 the *generic radius of convergence*. In § 7 we define *Taylor admissible objects* and obtain the main *propagation theorem* (Theorem 7.7). In the last (§ 8) we apply the previous theory to the Robba ring and to the *p-adic local monodromy theorem*.

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1. Notation

We refer to [DM82] for the definitions concerning Tannakian categories. In the sequel when we say that a given category $\mathcal{C}$ is (or is not) K-linear, we mean that the ring of endomorphisms of the unit object is (or is not) exactly *equal* to K. We set $\mathbb{R}_{\geqslant} := \{r \in \mathbb{R} \mid r \geqslant 0\}$, and $\delta_1 := T\,d/dT$.

1.1 Rings of functions

Let $R > 0$ and $c \in K$. The ring of analytic functions on the disk $\mathrm{D}^-(c, R)$ is

$$\mathcal{A}_K(c,R) := \left\{ \sum_{n\geqslant 0} a_n (T-c)^n \;\middle|\; a_n \in K, \liminf_n |a_n|^{-1/n} \geqslant R \right\}. \tag{1.1.1}$$

Its topology is given by the family of norms $|\sum a_i(T-c)^i|_{(c,\rho)} := \sup |a_i|\rho^i$, for all $\rho < R$. Let $\emptyset \neq I \subseteq \mathbb{R}_{\geqslant 0}$ be some interval. We denote the annulus relative to I by $\mathcal{C}_K(I) := \{x \in K \mid |x| \in I\}$. By $\mathcal{C}(I)$, without the index K, we mean the annulus itself and not its K-valued points. The ring of analytic functions on $\mathcal{C}(I)$ is

$$\mathcal{A}_K(I) := \left\{ \sum_{i\in\mathbb{Z}} a_i T^i \;\middle|\; a_i \in K, \lim_{i\to\pm\infty} |a_i|\rho^i = 0, \text{ for all } \rho \in I \right\}. \tag{1.1.2}$$

We set $|\sum_i a_i T^i|_\rho := \sup_i |a_i|\rho^i < +\infty$, for all $\rho \in I$. The ring $\mathcal{A}_K(I)$ is complete for the topology given by the family of norms $\{|\cdot|_\rho\}_{\rho\in I}$. Set $I_\varepsilon :=]1-\varepsilon, 1[$, $0 < \varepsilon < 1$. The Robba ring is defined as

$$\mathcal{R}_K := \bigcup_{\varepsilon>0} \mathcal{A}_K(I_\varepsilon), \tag{1.1.3}$$

and is complete with respect to the limit Frechet topology.

1.2 Affinoids

Definition 1.1. A K-*affinoid* is an analytic subset of $\mathbb{P}^1$ defined by

$$X := \mathrm{D}^+(c_0, R_0) - \bigcup_{i=1}^{n} \mathrm{D}^-(c_i, R_i), \tag{1.2.1}$$

for some $0 < R_1, \dots, R_n \leqslant R_0$, $c_0, \dots, c_n \in K$, $c_1, \dots, c_n \in \mathrm{D}_K^-(c_0, R_0)$. We denote by X the K-affinoid itself, and for all ultrametric valued K-algebras $(L, |\cdot|)$, we denote by $X(L)$ its L-rational points.

Let $H_K^{\mathrm{rat}}(X)$ be the ring of rational fractions $f(T)$ in $K(T)$, without poles in $X(K^{\mathrm{alg}})$, and let $\|\cdot\|_X$ be the norm on $H_K^{\mathrm{rat}}(X)$ given by $\|f(T)\|_X := \sup_{x\in X(K^{\mathrm{alg}})} |f(x)|$. We denote by

$$\mathcal{H}_K(X) \tag{1.2.2}$$

the completion of $(H_K^{\mathrm{rat}}(X), \|\cdot\|_X)$. It is known that if $\rho_1, \rho_2 \in |K^{\mathrm{alg}}|$, and if $X = \mathrm{D}^+(0,\rho_2) - \mathrm{D}^-(0,\rho_1)$, then $\mathcal{H}_K(X) = \mathcal{A}_K([\rho_1,\rho_2])$. Let now $\varepsilon > 0$. If $X = \mathrm{D}^+(c_0,R_0) - \bigcup_{i=1}^n \mathrm{D}^-(c_i,R_i)$, we set $X_\varepsilon := \mathrm{D}^+(c_0, R_0+\varepsilon) - \bigcup_{i=1}^n \mathrm{D}^-(c_i, R_i - \varepsilon)$. We then set

$$\mathcal{H}_K^\dagger(X) := \bigcup_{\varepsilon>0} \mathcal{H}_K(X_\varepsilon). \tag{1.2.3}$$

The ring $\mathcal{H}_K^\dagger(X)$ is complete with respect to the limit topology. Let $X_1 := \{x \mid |x| = 1\}$ we set

$$\mathcal{H}_K := \mathcal{H}_K(X_1), \quad \mathcal{H}_K^\dagger := \mathcal{H}_K^\dagger(X_1). \tag{1.2.4}$$

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1.3 Norms

Every semi-norm $|\cdot|_{\mathrm{B}}$ on a ring B will be extended to a semi-norm on $M_{n\times n}(\mathrm{B}) = M_n(\mathrm{B})$, by setting $|(b_{i,j})_{i,j}|_{\mathrm{B}} := \max_{i,j} |b_{i,j}|_{\mathrm{B}}$.

Definition 1.2. Let X be an affinoid. A *bounded multiplicative semi-norm* on $\mathcal{H}_K(X)$ is a function $|\cdot|_* : \mathcal{H}_K(X) \to \mathbb{R}_{\geqslant 0}$, such that $|0|_* = 0$, $|1|_* = 1$, $|f-g|_* \leqslant \max(|f|_*, |g|_*)$, $|fg|_* = |f|_*|g|_*$, and $|\cdot|_* \leqslant C\|\cdot\|_X$, for some constant $C > 0$.

1.3.1 Let $(L, |\cdot|)/(K, |\cdot|)$ be an extension of valued fields. Let $c \in X(L)$, then $|\cdot|_c : f \mapsto |f(c)|_L$ is a bounded multiplicative semi-norm on $\mathcal{H}_K(X)$. If $\mathrm{D}^+(c,R) \subseteq X$, then $|f|_{(c,R)} := \sup_{x\in \mathrm{D}^+_{L^{\mathrm{alg}}}(c,R)} |f(x)|$ is a bounded multiplicative semi-norm on $\mathcal{H}_K(X)$. Moreover if $f = \sum_{i\geqslant 0} a_i(T-c)^i$, $a_i \in L$, is the Taylor expansion of f at $c \in X(L)$, then $|f|_{(c,R)} = \sup_i |a_i|R^i$.

Definition 1.3. Let $f(T) = \sum_{i\in\mathrm{Z}} a_i(T-c)^i$, $a_i \in K$, be a formal power series. We set $|f|_{(c,\rho)} := \sup_i |a_i|\rho^i$; this number can be equal to $+\infty$.

Definition 1.4. Let $r \mapsto N(r) : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ be a function. The log-function attached to N is defined by $\widetilde{N}(t) := \log(N(\exp(t)))$, that is

$$\widetilde{N} : \mathbb{R}\cup\{-\infty\} \xrightarrow[\sim]{\exp} \mathbb{R}_{\geqslant 0} \xrightarrow{N} \mathbb{R}_{\geqslant 0} \xrightarrow[\sim]{\log} \mathbb{R}\cup\{-\infty\}.$$

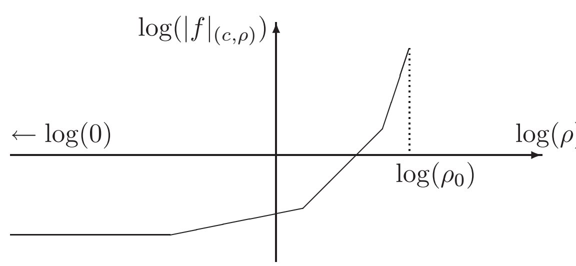
We will say that N has a given property logarithmically if $\widetilde{N}$ has that property.

Definition 1.5. Let $f(T) = \sum_{i\geqslant 0} a_i(T-c)^i$, $a_i \in K$, be a formal power series. The radius of convergence of $f(T)$ at c is $Ray(f(T), c) := \liminf_{i\geqslant 0} |a_i|^{-1/i}$. If $F(T) = (f_{h,k}(T))_{h,k}$ is a matrix, then we set $Ray(F(T), c) := \min_{h,k} Ray(f_{h,k}(T), c)$.

Lemma 1.6 [CR94, ch. II]. *Let $f(T) \in K[[T-c]]$. Suppose that $|f|_{(c,\rho_0)} < \infty$, for some $\rho_0 > 0$. Then one has the following:*

(i) *for all $\rho < \rho_0$ one has $Ray(f(T), c) \geqslant \rho$, and $|f|_{(c,\rho)} < \infty$;*

(ii) *the function $\rho \mapsto |f|_{(c,\rho)} : [0, \rho_0] \longrightarrow \mathbb{R}_{\geqslant 0}$ is log-convex, piecewise log-affine and log-increasing, as shown in the following picture:*



(iii) *one has $|f(T)|_{(c,\rho)} = \sup_{|x-c|\leqslant\rho, x\in K^{\mathrm{alg}}} |f(x)|_{K^{\mathrm{alg}}} = \lim_{r\to\rho^-} \sup_{|x-c|=r, x\in K^{\mathrm{alg}}} |f(x)|_{K^{\mathrm{alg}}}$;*

(iv) *all zeros of $f(T)$ are algebraic moreover $f(T)$ has a zero $\zeta \in K^{\mathrm{alg}}$, with $|\zeta - c| = \rho < \rho_0$, if and only if the previous graph has a break at $\log(\rho)$.*

1.4 Generic points

Let $(\Omega, |\cdot|)/(K, |\cdot|)$ be a complete field such that $|\Omega| = \mathbb{R}_{\geqslant 0}$, and that k_Ω/k is not algebraic.

Proposition 1.7 [CR94, 9.1.2]. *For every disk $\mathrm{D}^+(c,\rho)$, $c \in K$, there exists a point $t_{c,\rho} \in \Omega$, called* a generic point of $\mathrm{D}^+(c,\rho)$ *such that $|t_{c,\rho} - c|_\Omega = \rho$, and that $\mathrm{D}^-_\Omega(t_{c,\rho}, \rho) \cap K^{\mathrm{alg}} = \emptyset$.*

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1.4.1 A generic point defines a bounded multiplicative semi-norm on $\mathcal{H}_K(X)$, and hence defines a Berkovich point (cf. [Ber90]). The reader knowing the language of Berkovich will not find difficulties in translating the contents of this paper into the language of Berkovich.

For all $f(T) \in \mathcal{H}_K(\mathrm{D}^+(c,\rho))$, one has

$$|f(t_{c,\rho})|_\Omega = |f(T)|_{(c,\rho)} = \sup_{\substack{|x-c|\leqslant\rho \\ x\in K^{\mathrm{alg}}}} |f(x)| = \lim_{r\to\rho^-} \sup_{\substack{|x-c|=r \\ x\in K^{\mathrm{alg}}}} |f(x)|. \tag{1.4.1}$$

Hence, although the point $t_{c,\rho}$ is not uniquely determined by the fact that $\mathrm{D}^-_\Omega(t_{c,\rho},\rho)\cap K^{\mathrm{alg}} = \emptyset$, the norm $|\cdot|_{(c,\rho)}$ (i.e. the Berkovich point $|\cdot|_{(c,\rho)}$) does not depend on the choice of $t_{c,\rho}$.

By point (iii) of Lemma 1.6, if $\rho \in |K|$ (respectively $\rho \in |K^{\mathrm{alg}}|$; $\rho \notin |K^{\mathrm{alg}}|$), then one also has

$$|f(t_{c,\rho})| = \max_{\substack{|x|=\rho \\ x\in K}} |f(x)|$$

respectively

$$|f(t_{c,\rho})| = \max_{\substack{|x|=\rho \\ x\in K^{\mathrm{alg}}}} |f(x)|; \quad |f(t_{c,\rho})| = \lim_{r\to\rho^-} \max_{\substack{|x-c|=r\in|K^{\mathrm{alg}}| \\ x\in K^{\mathrm{alg}}}} |f(x)|.$$

PROPOSITION 1.8 [Ber90]. *Let $X = \mathrm{D}^+(c_0,R_0) - \bigcup_{i=1,\dots,n}\mathrm{D}^-(c_i,R_i)$ be an affinoid. Let $t_{c_i,R_i} \in X(\Omega)$ be the generic point of $\mathrm{D}^+(c_i,R_i)$. Then, for all $f \in \mathcal{H}_K(X)$, one has*

$$\|f(T)\|_X = \max(|f(t_{c_0,R_0})|_\Omega, \dots, |f(t_{c_n,R_n})|_\Omega). \tag{1.4.2}$$

LEMMA 1.9. *Let $X = \mathrm{D}^+(c_0,R_0) - \bigcup_{i=1,\dots,n}\mathrm{D}^-(c_i,R_i)$ be an affinoid. Let $r_X := \min(R_0,\dots,R_n)$. Then*

$$\left\|\frac{d}{dT}f(T)\right\|_X \leqslant r_X^{-1}\|f(T)\|_X.$$

Proof. This follows easily from the Mittag–Leffler decomposition of $f(T)$ together with the observations that $\|f(T)\|_X = \max_{i=0,\dots,n}(|f(t_{c_i,R_i})|)$ (cf. (1.4.2)), and $|f'(t_{c_i,R_i})| \leqslant R_i^{-1}|f(t_{c_i,R_i})|$, for all i. □

2. Discrete or analytic $\boldsymbol{\sigma}$-modules and $(\boldsymbol{\sigma},\boldsymbol{\delta})$-modules

DEFINITION 2.1. Let B be one of the rings of § 1.1. We denote by

$$\mathcal{Q}(\mathrm{B}) = \{q\in K \mid \sigma_q : f(T)\mapsto f(qT) \text{ is an automorphism of B}\}, \tag{2.0.1}$$

$$\mathcal{Q}_1(\mathrm{B}) = \mathcal{Q}(\mathrm{B})\cap \mathrm{D}^-(1,1). \tag{2.0.2}$$

We will write $\mathcal{Q}$ and $\mathcal{Q}_1$ when no confusion is possible.

Notice that $\mathcal{Q}(\mathrm{B}) \subset (K^\times,|\cdot|)$ is a topological group and always contains a disk $\mathrm{D}^-(1,\tau_0)$, for some $\tau_0 > 0$. One has $\mathcal{Q}(\mathcal{A}_K(I)) = \mathcal{Q}(\mathcal{R}_K) = \mathcal{Q}(\mathcal{H}_K^\dagger) = \{q\in K \mid |q| = 1\}$. One sees easily that $\mathcal{Q}(\mathcal{H}_K(X)) \subset \{q\in K \mid |q|=1\}$ (cf. § 5.2, and Lemma 5.1).

DEFINITION 2.2. Let $S \subseteq \mathcal{Q}$ be a subset. We denote by $\langle S\rangle$ the subgroup of $\mathcal{Q}$ generated by S. Let $\boldsymbol{\mu}(\mathcal{Q})$ be the set of all roots of unity belonging to $\mathcal{Q}$. Then we set

$$S^\circ := S - \boldsymbol{\mu}(Q). \tag{2.0.3}$$

2.1 Discrete $\boldsymbol{\sigma}$-modules

By assumption, every finite dimensional free B-module M has the product topology.

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Definition 2.3 (Discrete σ-modules). Let $S \subset \mathcal{Q}$ be an arbitrary subset. An object of

$$\sigma\text{-Mod}(\mathrm{B})_S^{\mathrm{disc}} \tag{2.1.1}$$

is a finite dimensional free B-module M, together with a group morphism

$$\sigma^{\mathrm{M}} : \langle S \rangle \longrightarrow \mathrm{Aut}_K^{\mathrm{cont}}(\mathrm{M}), \tag{2.1.2}$$

sending $q \mapsto \sigma_q^{\mathrm{M}}$, such that, for all $q \in S$, the operator σ_q^{M} is σ_q-semi-linear, that is

$$\sigma_q^{\mathrm{M}}(fm) = \sigma_q(f) \cdot \sigma_q^{\mathrm{M}}(m), \tag{2.1.3}$$

for all $f \in \mathrm{B}$, and all $m \in \mathrm{M}$. Objects (M, σ^{M}) in $\sigma\text{-Mod}(\mathrm{B})_S^{\mathrm{disc}}$ will be called *discrete σ-modules over S*. A morphism between $(\mathrm{M}, \sigma^{\mathrm{M}})$ and $(\mathrm{N}, \sigma^{\mathrm{N}})$ is a B-linear map $\alpha : \mathrm{M} \to \mathrm{N}$ such that

$$\alpha \circ \sigma_q^{\mathrm{M}} = \sigma_q^{\mathrm{N}} \circ \alpha, \tag{2.1.4}$$

for all $q \in S$. We will denote the K-vector space of morphisms by $\mathrm{Hom}_S^{\sigma}(\mathrm{M}, \mathrm{N})$.

Notation 2.4. If $S = \{q\}$ is reduced to a point, then the category of discrete σ-modules over $\{q\}$ is the usual category of q-difference modules. We will therefore use a simplified notation:

$$\sigma_q\text{-Mod}(\mathrm{B}) := \sigma\text{-Mod}(\mathrm{B})_{\{q\}}^{\mathrm{disc}}. \tag{2.1.5}$$

Remark 2.5. (1) Conditions (2.1.3) and (2.1.4) for $q \in S$ imply the same conditions for every $q \in \langle S \rangle$.

(2) If $\mathrm{M} \neq 0$, the map $\sigma^{\mathrm{M}} : \langle S \rangle \to \mathrm{Aut}_K^{\mathrm{cont}}(\mathrm{M})$ is injective. Indeed, since B is a domain and M is free, the equality $\sigma_q^{\mathrm{M}}(fm) = \sigma_{q'}^{\mathrm{M}}(fm)$, for all $f \in \mathrm{B}$, for all $m \in \mathrm{M}$, implies that $\sigma_q(f)\sigma_q^{\mathrm{M}}(m) = \sigma_{q'}(f)\sigma_{q'}^{\mathrm{M}}(m)$, and hence the contradiction: $\sigma_q(f) = \sigma_{q'}(f)$, for all $f \in \mathrm{B}$.

(3) The morphism σ^{M} on $\langle S \rangle$ is determined by its restriction to the set S. Conversely, if a map $S \to \mathrm{Aut}_K^{\mathrm{cont}}(\mathrm{M})$ is given, then this map extends to a group morphism $\langle S \rangle \to \mathrm{Aut}_K^{\mathrm{cont}}(\mathrm{M})$ if and only if the following conditions are verified:

(i) $\sigma_q^{\mathrm{M}} \circ \sigma_{q'}^{\mathrm{M}} = \sigma_{q'}^{\mathrm{M}} \circ \sigma_q^{\mathrm{M}}$, for all $q, q' \in S$;

(i) If $n, m \in \mathbb{Z}$, and $q_1, q_2 \in S$, such that $q_1^n = q_2^m$, then $(\sigma_{q_1}^{\mathrm{M}})^n = (\sigma_{q_2}^{\mathrm{M}})^m$;

(iii) If $1 \in S$, then $\sigma_1^{\mathrm{M}} = \mathrm{Id}$.

2.1.1 *Matrices of* σ^{M}. Let $\mathbf{e} = \{\mathrm{e}_1, \dots, \mathrm{e}_n\} \subset \mathrm{M}$ be a basis over B. If $\sigma_q^{\mathrm{M}}(\mathrm{e}_i) = \sum_j a_{i,j}(q,T) \cdot \mathrm{e}_j$, then in this basis σ_q^{M} acts as

$$\sigma_q^{\mathrm{M}}(f_1, \dots, f_n) = (\sigma_q(f_1), \dots, \sigma_q(f_n)) \cdot A(q,T), \tag{2.1.6}$$

where $A(q,T) := (a_{i,j}(q,T))_{i,j}$. By definition $A(1,T) = \mathrm{Id}$, and one has

$$A(qq', T) = A(q', qT) \cdot A(q,T). \tag{2.1.7}$$

In particular $A(q^n, T) = A(q, q^{n-1}T) \cdot A(q, q^{n-2}T) \cdots A(q,T)$.

2.1.2 *Internal* Hom *and* $\otimes$. Let $(\mathrm{M}, \sigma^{\mathrm{M}})$, $(\mathrm{N}, \sigma^{\mathrm{N}})$ be two discrete σ-modules over S. We define a structure of discrete σ-module on $\mathrm{Hom}_{\mathrm{B}}(\mathrm{M}, \mathrm{N})$ by setting $\sigma_q^{\mathrm{Hom(M,N)}}(\alpha) := \sigma_q^{\mathrm{N}} \circ \alpha \circ (\sigma_q^{\mathrm{M}})^{-1}$, for all $q \in S$, and all $\alpha \in \mathrm{Hom}_{\mathrm{B}}(\mathrm{M}, \mathrm{N})$. We define on $\mathrm{M} \otimes_{\mathrm{B}} \mathrm{N}$ a structure of discrete σ-module over S by setting $\sigma_q^{\mathrm{M}\otimes\mathrm{N}}(m \otimes n) := \sigma_q^{\mathrm{M}}(m) \otimes \sigma_q^{\mathrm{N}}(n)$, for all $q \in S$, and all $m \in \mathrm{M}$, $n \in \mathrm{N}$.

2.1.3 If $S^{\circ} \neq \emptyset$ (cf. (2.0.3)), then the category $\sigma\text{-Mod}(\mathrm{B})_S^{\mathrm{disc}}$ is K-linear. If B is a Bezout ring (i.e. every finitely generated ideal of B is principal), then $\sigma\text{-Mod}(\mathrm{B})_S^{\mathrm{disc}}$ is Tannakian (cf. [ADV04, 12.3]). The ring $\mathcal{H}_K(X)$ is always principal. If K is spherically closed, then $\mathcal{A}_K(I)$, $\mathcal{R}_K$, $\mathcal{H}_K^{\dagger}$ are Bezout rings.

2.1.4 As already mentioned in the introduction, the following is an example of two *non-isomorphic* analytic σ-modules over X (cf. Definition 2.9), having isomorphic 'stalks' at every $q \in U \subset \mathcal{Q}(X)$. This is analogous to having *non-isomorphic* sheaves having isomorphic stalks at every point.

Example 2.6. Let $X = \{|x| = 1\}$, then $\mathcal{Q}(X) = \{x \in K \mid |x| = 1\}$. Let $U := \mathrm{D}^-(1,1)$, and let $\pi \in K$ satisfy $|\pi| = |p|^{1/(p-1)}$. Put then $A(q,x) := \exp(\pi(q-1)x)$, and $\widetilde{A}(q,x) := \exp(\pi q(q-1)x)$. Let M (respectively N) be the discrete σ-module over U defined by the family $\{\sigma_q(Y) = A(q,x)\cdot Y\}_{q\in U}$ (respectively $\{\sigma_q(Y) = \widetilde{A}(q,x)\cdot Y\}_{q\in U}$). In this fixed basis of M and N, the matrices of every isomorphism between $(\mathrm{M},\sigma_q^{\mathrm{M}})$ and $(\mathrm{N},\sigma_q^{\mathrm{N}})$ are of the form $B(q,x) = \lambda\cdot\exp(\pi(1-q)x) \in \mathcal{H}_K(X)^\times$, with $\lambda \in K^\times$. Hence for all $q \in U$ the equation $\sigma_q(Y) = A(q,x)Y$ is isomorphic to $\sigma_q(Y) = \widetilde{A}(q,x)Y$. But since $B(q,x)$ depends on q, M *and* N *are not isomorphic as analytic σ-modules over U.*

2.2 Discrete (σ,δ)-modules

Let $S \subset \mathcal{Q}(\mathrm{B})$ be an arbitrary subset.

DEFINITION 2.7 (Discrete (σ,δ)-modules). An object of

$$(\sigma,\delta)\text{-Mod}(\mathrm{B})_S^{\mathrm{disc}} \tag{2.2.1}$$

is a discrete σ-module over S, together with a connection[3] $\delta_1^{\mathrm{M}} : \mathrm{M} \to \mathrm{M}$. Objects $(\mathrm{M},\sigma^{\mathrm{M}},\delta_1^{\mathrm{M}})$ of $(\sigma,\delta)\text{-Mod}(\mathrm{B})_S^{\mathrm{disc}}$ will be called *discrete* (σ,δ)*-modules over* S. A morphism between $(\mathrm{M},\sigma^{\mathrm{M}},\delta_1^{\mathrm{M}})$ and $(\mathrm{N},\sigma^{\mathrm{N}},\delta_1^{\mathrm{N}})$ is a morphism $\alpha : (\mathrm{M},\sigma^{\mathrm{M}}) \to (\mathrm{N},\sigma^{\mathrm{N}})$ of discrete σ-modules satisfying

$$\alpha \circ \delta_1^{\mathrm{M}} = \delta_1^{\mathrm{N}} \circ \alpha. \tag{2.2.2}$$

We will denote the K-vector space of morphisms by $\mathrm{Hom}_S^{(\sigma,\delta)}(\mathrm{M},\mathrm{N})$.

Notation 2.8. By analogy with (2.1.5), if $S = \{q\}$, then we set

$$(\sigma_q,\delta_q)\text{-Mod}(\mathrm{B}) := (\sigma,\delta)\text{-Mod}(\mathrm{B})_{\{q\}}^{\mathrm{disc}}. \tag{2.2.3}$$

If $q = 1$ we denote it by δ_1-Mod(B).

As already mentioned in the introduction, we introduce the operator

$$\delta_q^{\mathrm{M}} := \sigma_q^{\mathrm{M}} \circ \delta_1^{\mathrm{M}}. \tag{2.2.4}$$

For all $f \in \mathrm{B}$, all $m \in \mathrm{M}$, and all $q \in \langle S\rangle$, one has that

$$\delta_q^{\mathrm{M}}(f\cdot m) = \sigma_q(f)\cdot\delta_q^{\mathrm{M}}(m) + \delta_q(f)\cdot\sigma_q^{\mathrm{M}}(m). \tag{2.2.5}$$

Moreover, for all $\alpha \in \mathrm{Hom}^{(\sigma,\delta)}(\mathrm{M},\mathrm{N})$, and all $q \in \langle S\rangle$, one has $\alpha\circ\delta_q^{\mathrm{M}} = \delta_q^{\mathrm{N}}\circ\alpha$. Heuristically we imagine M as endowed with the map $q \mapsto \delta_q^{\mathrm{M}} : \langle S\rangle \to \mathrm{End}_K^{\mathrm{cont}}(\mathrm{M})$. This justifies notation (2.2.1) and (2.2.3).

2.2.1 *Matrices of* δ_q^{M}. Let $\mathbf{e} = \{\mathrm{e}_1,\dots,\mathrm{e}_n\} \subset \mathrm{M}$ be a basis over B. Let $A(q,T) \in GL_n(\mathrm{B})$ be the matrix of σ_q^{M} in the basis $\mathbf{e}$ (cf. (2.1.6)). If $\delta_q^{\mathrm{M}}(\mathrm{e}_i) = \sum_j g_{i,j}(q,T)\cdot\mathrm{e}_j$, and if $G(q,T) = (g_{i,j}(q,T))_{i,j}$, then δ_q^{M} acts in the basis $\mathbf{e}$ as

$$\delta_q^{\mathrm{M}}(f_1,\dots,f_n) = (\delta_q(f_1),\dots,\delta_q(f_n))\cdot A(q,T) + (\sigma_q(f_1),\dots,\sigma_q(f_n))\cdot G(q,T). \tag{2.2.6}$$

One has moreover the rule

$$G(q'\cdot q,T) = G(q',qT)\cdot A(q,T). \tag{2.2.7}$$

[3]That is, δ_1^{M} verifies $\delta_1^{\mathrm{M}}(fm) = \delta_1(f)\cdot m + f\cdot\delta_1^{\mathrm{M}}(m)$, for all $f \in \mathrm{B}$, for all $m \in \mathrm{M}$. Recall that $\delta_1 := T\,d/dT$.

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2.2.2 *Internal* Hom *and* $\otimes$. Let $(\mathrm{M},\sigma^{\mathrm{M}},\delta^{\mathrm{M}})$, $(\mathrm{N},\sigma^{\mathrm{N}},\delta^{\mathrm{N}})$ be two discrete (σ,δ)-modules over S. We define a structure of discrete (σ,δ)-module on $\mathrm{Hom}_{\mathrm{B}}(\mathrm{M},\mathrm{N})$ by setting

$$\delta_q^{\mathrm{Hom(M,N)}}(\alpha) := (\delta_q^{\mathrm{N}}\circ\alpha - \sigma_q^{\mathrm{Hom(M,N)}}(\alpha)\circ\delta_q^{\mathrm{M}})\circ(\sigma_q^{\mathrm{M}})^{-1}. \tag{2.2.8}$$

This definition gives the relation $\delta_q^{\mathrm{N}}(\alpha\circ m) = \sigma_q^{\mathrm{H}}(\alpha)\circ\delta_q^{\mathrm{M}}(m)+\delta_q^{\mathrm{H}}(\alpha)\circ\sigma_q^{\mathrm{M}}(m)$, for all $\alpha\in\mathrm{Hom}_{\mathrm{B}}(\mathrm{M},\mathrm{N})$, and all $m\in\mathrm{M}$, where $\mathrm{H}:=\mathrm{Hom}_{\mathrm{B}}(\mathrm{M},\mathrm{N})$. We define on $\mathrm{M}\otimes_{\mathrm{B}}\mathrm{N}$ a structure of discrete (σ,δ)-module over S by setting

$$\delta_q^{\mathrm{M}\otimes\mathrm{N}}(m\otimes n) := \delta_q^{\mathrm{M}}(m)\otimes\sigma_q^{\mathrm{N}}(n)+\sigma_q^{\mathrm{M}}(m)\otimes\delta_q^{\mathrm{N}}(n), \tag{2.2.9}$$

for all $q\in S$, and all $m\in\mathrm{M}$, $n\in\mathrm{N}$.

2.2.3 If B is Bezout, then $(\sigma,\delta)\text{-Mod}(\mathrm{B})_S^{\mathrm{disc}}$ is K-linear and Tannakian.

2.3 Analytic σ-modules

Analytic σ-modules are defined only if the ring B is equal to one of the following rings: $\mathcal{A}_K(I)$, $\mathcal{H}_K(X)$, $\mathcal{H}_K^\dagger(X)$, $\mathcal{H}_K$, $\mathcal{H}_K^\dagger$, $\mathcal{R}_K$. Notice that if $U\subset\mathcal{Q}(\mathrm{B})$ is an open subset, then the subgroup $\langle U\rangle\subseteq\mathcal{Q}(\mathrm{B})$ generated by U is open, i.e. $\langle U\rangle$ contains a disk $\mathrm{D}_K^-(1,\tau)$, for some $\tau>0$.

Definition 2.9. Let $\mathrm{B}:=\mathcal{H}_K(X)$. Let $(\mathrm{M},\sigma^{\mathrm{M}})$ be a discrete σ-module over U. Let $A(q,T)\in GL_n(\mathrm{B})$ be the matrix of σ_q^{M} in a fixed basis. We will say that $(\mathrm{M},\sigma^{\mathrm{M}})$ is an *analytic σ-module* if, for all $q\in U$, there exist a disk $\mathrm{D}^-(q,\tau_q)=\{q'\mid |q'-q|<\tau_q\}$, with $\tau_q>0$, and a matrix $A_q(Q,T)$ such that:

(i) $A_q(Q,T)$ is an analytic element on the domain $(Q,T)\in\mathrm{D}^-(q,\tau_q)\times X$;
(ii) for all $q'\in\mathrm{D}_K^-(q,\tau_q)$, one has $A_q(Q,T)_{|Q=q'}=A(q',T)$.

This definition does not depend on the choice of basis $\mathbf{e}$. We define

$$\sigma\text{-Mod}(\mathrm{B})_U^{\mathrm{an}} \tag{2.3.1}$$

as the full sub-category of $\sigma\text{-Mod}(\mathrm{B})_U^{\mathrm{disc}}$, whose objects are analytic σ-modules. Let $I\subset\mathbb{R}_{\geqslant 0}$ be an interval. We give the same definition over the ring $\mathrm{B}:=\mathcal{A}_K(I)$, namely, if $\mathcal{C}(I):=\{|T|\in I\}$, the point (i) is replaced by:

(i′) $A_q(Q,T)$ is an analytic function on the domain $(Q,T)\in\mathrm{D}^-(q,\tau_q)\times\mathcal{C}(I)$.

Example 2.10. The discrete σ-modules appearing in Example 2.6 are actually analytic.

2.3.1 *Analyticity of* $\mathrm{Hom}(\mathrm{M},\mathrm{N})$ *and* $\mathrm{M}\otimes\mathrm{N}$. If $(\mathrm{M},\sigma^{\mathrm{M}})$ and $(\mathrm{N},\sigma^{\mathrm{N}})$ are two analytic σ-modules over U, then $(\mathrm{Hom}(\mathrm{M},\mathrm{N}),\sigma^{\mathrm{Hom(M,N)}})$ and $(\mathrm{M}\otimes\mathrm{N},\sigma^{\mathrm{M}\otimes\mathrm{N}})$ are analytic. This follows from the explicit dependence of the matrices of $\sigma^{\mathrm{Hom(M,N)}}$ and $\sigma^{\mathrm{M}\otimes\mathrm{N}}$ on the matrices of σ^{M} and σ^{N}.

2.3.2 *Discrete and analytic σ-modules over $\mathcal{A}_K(I)$, $\mathcal{R}_K$ and $\mathcal{H}_K^\dagger(X)$.* If $I_1\subset I_2$, then the restriction functor $\sigma\text{-Mod}(\mathcal{A}_K(I_2))_U^{\mathrm{an}}\to\sigma\text{-Mod}(\mathcal{A}_K(I_1))_U^{\mathrm{an}}$ is faithful. Indeed the equality $f_{|_{I_1}}=g_{|_{I_1}}$ implies that $f=g$, for all $f,g\in\mathcal{A}_K(I_2)$ (analytic continuation [CR94, 5.5.8]).

Definition 2.11. Let $S\subseteq\mathcal{Q}$ be a subset, and let $U\subseteq\mathcal{Q}$ be an open subset. We set

$$\sigma\text{-Mod}(\mathcal{R}_K)_U^{\mathrm{an}} := \bigcup_{\varepsilon>0}\sigma\text{-Mod}(\mathcal{A}_K(]1-\varepsilon,1[))_U^{\mathrm{an}}, \tag{2.3.2}$$

$$\sigma\text{-Mod}(\mathcal{R}_K)_S^{\mathrm{disc}} := \bigcup_{\varepsilon>0}\sigma\text{-Mod}(\mathcal{A}_K(]1-\varepsilon,1[))_S^{\mathrm{disc}}. \tag{2.3.3}$$

Similarly, one can define $\sigma\text{-Mod}(\mathcal{H}_K^\dagger(X))_U^{\mathrm{an}}$ and $\sigma\text{-Mod}(\mathcal{H}_K^\dagger(X))_S^{\mathrm{disc}}$.

Remark 2.12. Since U is open, one has $U^\circ \neq \emptyset$ (cf. (2.0.3)). By § 2.1.3, if B is one of the previous rings (and if it is a Bezout ring), then $\sigma\text{-Mod}(\mathrm{B})^{\mathrm{an}}_U$ is K-linear and Tannakian.

2.4 Analytic (σ,δ)-modules

We maintain the previous notation. In § 2.4.1 below we define a *fully faithful* functor

$$(\text{Forget }\delta)^{-1}:\sigma\text{-Mod}(\mathrm{B})^{\mathrm{an}}_U \longrightarrow (\sigma,\delta)\text{-Mod}(\mathrm{B})^{\mathrm{disc}}_U, \tag{2.4.1}$$

which is a 'local' section of the functor Forget $\delta : (\sigma,\delta)\text{-Mod}(\mathrm{B})^{\mathrm{disc}}_U \longrightarrow \sigma\text{-Mod}(\mathrm{B})^{\mathrm{disc}}_U$. The essential image of the functor $(\text{Forget }\delta)^{-1}$ will be denoted by

$$(\sigma,\delta)\text{-Mod}(\mathrm{B})^{\mathrm{an}}_U. \tag{2.4.2}$$

By definition, the functor which 'forgets' the action of δ is therefore an equivalence

$$(\sigma,\delta)\text{-Mod}(\mathrm{B})^{\mathrm{an}}_U \xrightarrow[\sim]{\text{Forget }\delta} \sigma\text{-Mod}(\mathrm{B})^{\mathrm{an}}_U. \tag{2.4.3}$$

Notice that a morphism between analytic (σ,δ)-modules is, by definition, a morphism of *discrete* (σ,δ)-modules.

2.4.1 *Construction of* δ. Let $(\mathrm{M},\sigma^{\mathrm{M}})$ be an analytic σ-module. We shall define a (σ,δ)-module structure on M. It follows from Definitions 2.9 and 2.11 that the map $q\mapsto \sigma^{\mathrm{M}}_q : \langle U\rangle \to \mathrm{Aut}_K(\mathrm{M})$ is *derivable*, in the sense that, for all $q\in\langle U\rangle$, the limit

$$\delta^{\mathrm{M}}_q := q\cdot \lim_{q'\to q}\frac{\sigma^{\mathrm{M}}_{q'}-\sigma^{\mathrm{M}}_q}{q'-q} = \left(q\frac{d}{dq}\sigma^{\mathrm{M}}\right)(q) \tag{2.4.4}$$

exists in $\mathrm{End}^{\mathrm{cont}}_K(\mathrm{M})$, with respect to the simple convergence topology (cf. (2.4.5)). Moreover, for all $q\in\langle U\rangle$, the rule (2.2.5) holds, and $\delta^{\mathrm{M}}_q=\sigma^{\mathrm{M}}_q\circ\delta^{\mathrm{M}}_1$.

Let $\alpha:(\mathrm{M},\sigma^{\mathrm{M}})\to(\mathrm{N},\sigma^{\mathrm{N}})$ be a morphism of analytic σ-modules, that is $\alpha\circ\sigma^{\mathrm{M}}_q=\sigma^{\mathrm{N}}_q\circ\alpha$, for all $q\in U$. Passing to the limit in the definition (2.4.4), one shows that α commutes with δ^{M}_q, for all $q\in U$. Hence the inclusion $\mathrm{Hom}^{(\sigma,\delta)}_U(\mathrm{M},\mathrm{N})\subseteq\mathrm{Hom}^{\sigma}_U(\mathrm{M},\mathrm{N})$ is an equality. If $\mathbf{e}=\{\mathrm{e}_1,\dots,\mathrm{e}_n\}\subset\mathrm{M}$ is a basis in which the matrix of σ^{M}_q is $A(q,T)$, then the matrix of δ^{M}_q is (cf. (2.2.6), Definition 2.9 and 2.11)

$$G(q,T) := q\cdot\lim_{q'\to q}\frac{A(q',T)-A(q,T)}{q'-q} = (\partial_Q(A_q(Q,T)))_{|Q=q}, \tag{2.4.5}$$

where ∂_Q is the derivation $Q\,d/dQ$, and $A_q(Q,T)$ is the matrix of Definition 2.9.

Remark 2.13. By the above definitions, there is an obvious functor

$$\mathrm{Conf}_U:\sigma\text{-Mod}(\mathrm{B})^{\mathrm{an}}_U\to\delta_1\text{-Mod}(\mathrm{B}), \tag{2.4.6}$$

obtained by composing $(\text{Forget }\delta)^{-1}$ (cf. (2.4.3)) with Forget $\sigma:(\sigma,\delta)\text{-Mod}(\mathrm{B})^{\mathrm{an}}_U\longrightarrow\delta_1\text{-Mod}(\mathrm{B})$.

3. Solutions (formal definition)

3.1 Discrete σ-algebras and (σ,δ)-algebras

Let $S\subseteq\mathcal{Q}(\mathrm{B})$ be a subset.

Definition 3.1 (Discrete σ-algebra over S). A B-*discrete σ-algebra over S*, or simply a *discrete σ-algebra over S*, is a B-algebra C such that:

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(i) C is an *integral domain*, and the structural morphism $\mathrm{B} \to \mathrm{C}$ is injective;

(ii) there exists a group morphism $\sigma^{\mathrm{C}} : \langle S \rangle \to \mathrm{Aut}_K(\mathrm{C})$ such that σ_q^{C} is a ring automorphism extending σ_q^{B}, for all $q \in \langle S \rangle$;

(iii) one has $\mathrm{C}_S^{\sigma} = K$, where $\mathrm{C}_S^{\sigma} := \{c \in \mathrm{C} \mid \sigma_q(c) = c, \text{ for all } q \in S\}$.

We will call C_S^{σ} *the sub-ring of σ-constants of* C. We will write σ_q instead of σ_q^{C}, when no confusion is possible.

Observe that no topology is required on C. The word *discrete* is employed, here and later on, to emphasize that we do not ask for 'continuity' with respect to q. Notice also that if a discrete σ-algebra C is free and of finite rank as B-module, then it is a discrete σ-module.

3.1.1 If $S^{\circ} \neq \emptyset$ (cf. (2.0.3)), then $\mathrm{B}_S^{\sigma} = K$, and B itself is a discrete σ-algebra over S. On the other hand, if $S = \{\xi\}$ is reduced to a root of unity $\xi \in \boldsymbol{\mu}(\mathcal{Q})$, since $\mathrm{B}_S^{\sigma} = \mathrm{B}^{\sigma_\xi} \neq K$, it follows that B itself is not a discrete σ-algebra over S. Hence there is no discrete σ-algebra over $S = \{\xi\}$. To deal with this problem we introduce the following definition.

Definition 3.2 (Discrete (σ,δ)-algebra over S). A *discrete (σ,δ)-algebra* C *over* S is a B-algebra such that:

(i) C satisfies properties (i) and (ii) of Definition 3.1;

(ii) there exists a derivation δ_1^{C}, extending $\delta_1 = T\,d/dT$ on B, and commuting with σ_q^{C}, for all $q \in \langle S \rangle$;

(iii) one has $\mathrm{C}_S^{(\sigma,\delta)} = K$, where $\mathrm{C}_S^{(\sigma,\delta)} := \{f \in \mathrm{C} \mid f \in \mathrm{C}_S^{\sigma}, \text{ and } \delta_1(f) = 0\}$.

We will call $\mathrm{C}_S^{(\sigma,\delta)}$ the *sub-ring of (σ,δ)-constants of* C. We will write δ_1 instead of δ_1^{C}, if no confusion is possible.

The operator $\delta_q^{\mathrm{C}} := \sigma_q^{\mathrm{C}} \circ \delta_1^{\mathrm{C}}$ satisfies property (2.2.5). Since $\mathrm{B}_S^{(\sigma,\delta)} = K$, it follows that B is always a (σ,δ)-algebra over S, for an arbitrary subset $S \subseteq \mathcal{Q}(\mathrm{B})$, even for $S = \{\xi\}$, with $\xi \in \boldsymbol{\mu}(\mathcal{Q}(\mathrm{B}))$.

3.2 Constant solutions

Definition 3.3 (Constant solutions on S). Let $(\mathrm{M}, \sigma^{\mathrm{M}})$ (respectively $(\mathrm{M}, \sigma^{\mathrm{M}}, \delta^{\mathrm{M}})$) be a *discrete* σ-module (respectively (σ,δ)-module) over S, and let C be a discrete σ-algebra (respectively (σ,δ)-algebra) over S. A *constant solution* of M, with values in C, is a B-linear morphism

$$\alpha : \mathrm{M} \longrightarrow \mathrm{C}$$

such that $\alpha \circ \sigma_q^{\mathrm{M}} = \sigma_q^{\mathrm{C}} \circ \alpha$, for all $q \in S$ (respectively α simultaneously satisfies $\alpha \circ \delta_1^{\mathrm{M}} = \delta_1^{\mathrm{C}} \circ \alpha$, and $\alpha \circ \sigma_q^{\mathrm{M}} = \sigma_q^{\mathrm{C}} \circ \alpha$, for all $q \in S$). We denote by $\mathrm{Hom}_S^{\sigma}(\mathrm{M},\mathrm{C})$ (respectively $\mathrm{Hom}_S^{(\sigma,\delta)}(\mathrm{M},\mathrm{C})$) the K-vector space of the solutions of M in C.

3.2.1 *Matrices of solutions.* Let M be a discrete σ-module (respectively (σ,δ)-module). Let C be a discrete σ-algebra (respectively (σ,δ)-algebra) over S. Recall that, if $S = \{\xi\}$, with $\xi^n = 1$, then there is no discrete σ-algebra, over S (cf. § 3.1.1).

Let $\mathbf{e} = \{\mathrm{e}_1, \ldots, \mathrm{e}_n\}$ be a basis of M, and let $A(q,T)$ (respectively $G(q,T)$) be the matrix of σ_q^{M} (respectively δ_q^{M}) in this basis (cf. (2.2.6)). We identify a morphism $\alpha : \mathrm{M} \to \mathrm{C}$ with the vector $(y_i)_i \in \mathrm{C}^n$, given by $y_i := \alpha(\mathrm{e}_i)$. In this way constant solutions become solutions in the usual *vector* *form.* Indeed

$$\begin{pmatrix}\sigma_q(y_1)\\ \vdots \\ \sigma_q(y_n)\end{pmatrix} = A(q,T)\cdot\begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix},\qquad \text{for all } q\in S, \tag{3.2.1}$$

respectively

$$\begin{pmatrix}\delta_q(y_1)\\ \vdots \\ \delta_q(y_n)\end{pmatrix} = G(q,T)\cdot\begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix},\qquad \text{for all } q\in S. \tag{3.2.2}$$

DEFINITION 3.4. By a *fundamental matrix of solutions* of M (in the basis $\mathbf{e}$) we mean a matrix $Y \in GL_n(\mathrm{C})$ satisfying *simultaneously*

$$\sigma_q(Y) = A(q,T)\cdot Y,\qquad \text{for all } q\in S, \tag{3.2.3}$$

respectively satisfying *simultaneously*

$$\begin{aligned}\sigma_q(Y) &= A(q,T)\cdot Y,\qquad \text{for all } q\in S,\\ \delta_1(Y) &= G(1,T)\cdot Y.\end{aligned} \tag{3.2.4}$$

3.2.2 *Unit object and σ-constants.* Let $\mathbb{I} = \mathrm{B}$ be the unit object. By the description given above, every solution $\alpha \in \mathrm{Hom}^{\sigma}_S(\mathbb{I},\mathrm{C})$ (respectively $\alpha \in \mathrm{Hom}^{(\sigma,\delta)}_S(\mathbb{I},\mathrm{C})$) can be identified with $y := \alpha(1) \in \mathrm{C}^{\sigma}_S$ (respectively $y := \alpha(1) \in \mathrm{C}^{(\sigma,\delta)}_S$). We obtain $\mathrm{C}^{\sigma}_S \cong \mathrm{Hom}^{\sigma}_S(\mathbb{I},\mathrm{C})$ (respectively $\mathrm{C}^{(\sigma,\delta)}_S \cong \mathrm{Hom}^{(\sigma,\delta)}_S(\mathbb{I},\mathrm{C})$). In particular B^{σ}_S (respectively $\mathrm{B}^{(\sigma,\delta)}_S$) is identified with $\mathrm{End}^{\sigma}_S(\mathbb{I})$ (respectively $\mathrm{End}^{(\sigma,\delta)}_S(\mathbb{I})$), and the category is K-linear if and only if $\mathrm{B}^{\sigma}_S = K$ (respectively $\mathrm{B}^{(\sigma,\delta)}_S = K$).

3.2.3 *Dimension of the space of solutions.* Let $F := \mathrm{Frac}(\mathrm{C})$ be the fraction field of C, then both σ_q and δ_1 extend to F (cf. [vdPS03, Ex. 1.5]).

LEMMA 3.5 (Wronskian lemma). *Let* M *be a (σ,δ)-module (respectively σ-module) over S, and let* C *be a discrete (σ,δ)-algebra (respectively σ-algebra) over S. One has*

$$\dim_K \mathrm{Hom}^{(\sigma,\delta)}_S(\mathrm{M},\mathrm{C}) \leqslant \mathrm{rk}_{\mathrm{B}}(\mathrm{M}). \tag{3.2.5}$$

(respectively if $S^{\circ} \neq \emptyset$ (cf. (2.0.3)), then $\dim_K \mathrm{Hom}^{\sigma}_S(\mathrm{M},\mathrm{C}) \leqslant \mathrm{rk}_{\mathrm{B}}(\mathrm{M})$.*)*

Proof. One has $\dim_K \mathrm{Hom}^{(\sigma,\delta)}_S(\mathrm{M},\mathrm{C}) \leqslant \dim_K \mathrm{Hom}^{\delta_1}(\mathrm{M},\mathrm{C}) \leqslant \mathrm{rk}_{\mathrm{B}}(\mathrm{M})$. On the other hand, if $q \in S^{\circ}$, then $\mathrm{Hom}^{\sigma_q}(\mathrm{M},\mathrm{C}) \leqslant \mathrm{rk}_{\mathrm{B}}(\mathrm{M})$ (cf. [DV02, Lemma 1.1.11]). Hence

$$\dim_K \mathrm{Hom}^{\sigma}_S(\mathrm{M},\mathrm{C}) \leqslant \dim_K \mathrm{Hom}^{\sigma_q}(\mathrm{M},\mathrm{C}) \leqslant \mathrm{rk}_{\mathrm{B}}(\mathrm{M}). \qquad \square$$

4. C-constant confluence

In this section we state the formal results regarding confluence. We introduce the notion of C-constant modules. As explained in the introduction, this notion is an adaptation of the notion of C-admissibility in the sense of representation theory. On the other hand it can be interpreted as a generalization of the Galois theory for differential and q-difference equations. According to this point of view, in our context we have the problem that the analog of the Picard–Vessiot algebra trivializing a given object M does not exist for arbitrary objects M. Also the uniqueness of the Picard–Vessiot algebra remains an open problem. We avoid these problems by working with the category of modules trivialized by a given algebra C which is fixed once and for all. We hope that this problem will be overcome in the future; the recent work of C. Hardouin and M. Singer seems to be a first progress in this direction [HS08].

4.1 C-constant modules

Let B be one of the rings of §§ 1.1 and 1.2, let $S \subset \mathcal{Q}(\mathrm{B})$ be a subset, and let $U \subset \mathcal{Q}(\mathrm{B})$ be an open subset.

Definition 4.1 (C-constant modules). Let M be a discrete σ-module over S. We will say that M is C-*constant* on S, or equivalently that M *is trivialized by* C, if there exists a discrete σ-algebra C over S such that

$$\dim_K \mathrm{Hom}^{\sigma}_S(\mathrm{M}, \mathrm{C}) = \mathrm{rk}_{\mathrm{B}}\mathrm{M}. \tag{4.1.1}$$

We give the analogous definition for (σ, δ)-modules. The full sub-category of σ-$\mathrm{Mod}(\mathrm{B})^{\mathrm{disc}}_S$ (respectively (σ, δ)-$\mathrm{Mod}(\mathrm{B})^{\mathrm{disc}}_S$), whose objects are trivialized by C, will be denoted by

$$\sigma\text{-}\mathrm{Mod}(\mathrm{B}, \mathrm{C})^{\mathrm{const}}_S \qquad (\text{respectively } (\sigma, \delta)\text{-}\mathrm{Mod}(\mathrm{B}, \mathrm{C})^{\mathrm{const}}_S). \tag{4.1.2}$$

The full subcategory of σ-$\mathrm{Mod}(\mathrm{B}, \mathrm{C})^{\mathrm{const}}_U$ (respectively (σ, δ)-$\mathrm{Mod}(\mathrm{B}, \mathrm{C})^{\mathrm{const}}_U$) whose objects are analytic will be denoted by

$$\sigma\text{-}\mathrm{Mod}(\mathrm{B}, \mathrm{C})^{\mathrm{an,const}}_U \qquad (\text{respectively } (\sigma, \delta)\text{-}\mathrm{Mod}(\mathrm{B}, \mathrm{C})^{\mathrm{an,const}}_U). \tag{4.1.3}$$

Notice that M is trivialized by C if there exists $Y \in GL_n(\mathrm{C})$, $n := \mathrm{rk}_{\mathrm{B}}\mathrm{M}$, such that Y is *simultaneously* a solution, for all $q \in S$, of the family of equations (3.2.3) (respectively both the conditions of (3.2.4)). Roughly speaking, M is C-constant on S if it admits a basis of q-solutions in $GL_n(\mathrm{C})$ which 'does not depend on $q \in S$'.

Lemma 4.2. *Let* M, N *be two discrete σ-modules (respectively (σ, δ)-modules). If* M, N *are both trivialized by* C*, then* $\mathrm{M} \otimes \mathrm{N}$, $\mathrm{Hom}(\mathrm{M}, \mathrm{N})$, $M^{\vee}$, $N^{\vee}$ *are trivialized by* C.

Proof. The fundamental matrix solution of $\mathrm{M} \otimes \mathrm{N}$ (respectively $\mathrm{Hom}(\mathrm{M}, \mathrm{N})$) is obtained by taking products of entries of the two matrices of solutions of M and N respectively. Hence 'it does not depend on $q \in S$'. The assertion on $\mathrm{M}^{\vee}$, $\mathrm{N}^{\vee}$ is a particular case of the previous one. □

Lemma 4.3. *Let $S' \subseteq S$ be a non-empty subset. Let* C *be a discrete (σ, δ)-algebra over S. Then the restriction functor* $\mathrm{Res}^S_{S'}$*, sending* $(\mathrm{M}, \sigma^{\mathrm{M}}, \delta^{\mathrm{M}}_1)$ *into* $(\mathrm{M}, \sigma^{\mathrm{M}}_{|_{\langle S' \rangle}}, \delta^{\mathrm{M}}_1)$,

$$\mathrm{Res}^S_{S'} : (\sigma, \delta)\text{-}\mathrm{Mod}(\mathrm{B}, \mathrm{C})^{\mathrm{const}}_S \longrightarrow (\sigma, \delta)\text{-}\mathrm{Mod}(\mathrm{B})^{\mathrm{disc}}_{S'}, \tag{4.1.4}$$

is fully faithful and its image is contained in the category (σ, δ)-$\mathrm{Mod}(\mathrm{B}, \mathrm{C})^{\mathrm{const}}_{S'}$*. The same fact is true for discrete σ-modules under the assumption* $(S')^{\circ} \neq \emptyset$.

Proof. The proof is the same in both cases: here we give the proof in the case of (σ, δ)-modules. We must show that the inclusion $\mathrm{Hom}^{(\sigma,\delta)}_S(\mathrm{M}, \mathrm{N}) \to \mathrm{Hom}^{(\sigma,\delta)}_{S'}(\mathrm{M}, \mathrm{N})$ is an isomorphism, for all M, N in (σ, δ)-$\mathrm{Mod}(\mathrm{B}, \mathrm{C})^{\mathrm{const}}_S$. In other words, we have to show that if $\alpha : \mathrm{M} \to \mathrm{N}$ commutes with $\sigma_{q'}$, for all $q' \in S'$, then it commutes also with σ_q, for all $q \in S$. One has

$$\mathrm{Hom}^{(\sigma,\delta)}_S(\mathrm{M}, \mathrm{N}) = \mathrm{Hom}^{(\sigma,\delta)}_S(\mathrm{M} \otimes \mathrm{N}^{\vee}, \mathrm{B}) = \mathrm{Hom}^{(\sigma,\delta)}_S(\mathrm{M} \otimes \mathrm{N}^{\vee}, \mathrm{C}) \cap \mathrm{Hom}_{\mathrm{B}}(\mathrm{M} \otimes \mathrm{N}^{\vee}, \mathrm{B}), \tag{4.1.5}$$
$$\mathrm{Hom}^{(\sigma,\delta)}_{S'}(\mathrm{M}, \mathrm{N}) = \mathrm{Hom}^{(\sigma,\delta)}_{S'}(\mathrm{M} \otimes \mathrm{N}^{\vee}, \mathrm{B}) = \mathrm{Hom}^{(\sigma,\delta)}_{S'}(\mathrm{M} \otimes \mathrm{N}^{\vee}, \mathrm{C}) \cap \mathrm{Hom}_{\mathrm{B}}(\mathrm{M} \otimes \mathrm{N}^{\vee}, \mathrm{B}).$$

Observe that $\mathrm{M} \otimes \mathrm{N}^{\vee}$ is the dual of the 'internal hom' $\mathrm{Hom}(\mathrm{M}, \mathrm{N})$. By Lemma 4.2, $\mathrm{M} \otimes \mathrm{N}^{\vee}$ is trivialized by C. The restriction of $\mathrm{M} \otimes \mathrm{N}^{\vee}$ to S' is obviously C-constant on S', since it is trivialized by C. This implies that

$$\mathrm{C}^n = \mathrm{Hom}^{(\sigma,\delta)}_S(\mathrm{M} \otimes \mathrm{N}^{\vee}, \mathrm{C}) = \mathrm{Hom}^{(\sigma,\delta)}_{S'}(\mathrm{M} \otimes \mathrm{N}^{\vee}, \mathrm{C}). \tag{4.1.6}$$

This shows that a morphism with values in $\mathrm{B} \subseteq \mathrm{C}$ commutes with all σ_q and δ_q, for all $q \in S$, if and only if it commutes with all σ_q and δ_q, for all $q \in S'$. Hence

$$\mathrm{Hom}^{(\sigma,\delta)}_S(\mathrm{M} \otimes \mathrm{N}^{\vee}, \mathrm{B}) = \mathrm{Hom}^{(\sigma,\delta)}_{S'}(\mathrm{M} \otimes \mathrm{N}^{\vee}, \mathrm{B}). \tag{4.1.7}$$

□

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4.1.1 *Restriction to roots of unity.* By the previous lemma, if $\xi \in S \cap \boldsymbol{\mu}(\mathcal{Q})$, then

$$\mathrm{Res}^S_{\{\xi\}} : (\sigma,\delta)\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{const}}_S \longrightarrow (\sigma_\xi,\delta_\xi)\text{-}\mathrm{Mod}(\mathrm{B}) \tag{4.1.8}$$

is again fully faithful. On the other hand, if $S^\circ \neq \emptyset$, then the restriction

$$\mathrm{Res}^S_{\{\xi\}} : \sigma\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{const}}_S \longrightarrow \sigma_\xi\text{-}\mathrm{Mod}(\mathrm{B}) \tag{4.1.9}$$

is *not* fully faithful, since σ-$\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{const}}_S$ is K-linear, while σ_ξ-$\mathrm{Mod}(\mathrm{B})$ is not K-linear (i.e. $K \subset \mathrm{End}(\mathbb{I})$, but $K \subsetneq \mathrm{End}(\mathbb{I})$; cf. § 1).

4.1.2 *The case of an open subset.* We observe that if U is open, then the condition $U^\circ \neq \emptyset$ is automatically verified. Hence, by Lemma 4.3, if $S \subset U$ is a (non-empty) subset, the restriction

$$\mathrm{Res}^U_S : (\sigma,\delta)\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{an,const}}_U \longrightarrow (\sigma,\delta)\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{const}}_S \tag{4.1.10}$$

is fully faithful. The same is true for σ-modules, under the assumption $S^\circ \neq \emptyset$. In particular, if $U' \subset U$ is an open subset, then the restriction functor is fully faithful:

$$\begin{aligned}\mathrm{Res}^U_{U'} &: (\sigma,\delta)\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{an,const}}_U \longrightarrow (\sigma,\delta)\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{an,const}}_{U'},\\ \mathrm{Res}^U_{U'} &: \sigma\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{an,const}}_U \longrightarrow \sigma\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{an,const}}_{U'}.\end{aligned} \tag{4.1.11}$$

4.2 C-constant deformation and C-constant confluence

In this section we give the formal definition of the confluence and deformation functors. As usual $S \subseteq \mathcal{Q}(\mathrm{B})$ is an arbitrary subset, and $U \subseteq \mathcal{Q}(\mathrm{B})$ is an open subset.

Definition 4.4 (Extensible objects). Let $q \in S$. Let C be a discrete σ-algebra over S. A q-difference module M is said to be C-*extensible to* S if it belongs to the essential image of the restriction functor

$$\mathrm{Res}^S_{\{q\}} : \sigma\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{const}}_S \longrightarrow \sigma_q\text{-}\mathrm{Mod}(\mathrm{B}).$$

The *full sub-category* of σ_q-Mod(B) whose objects are C-extensible to S will be denoted by σ_q-$\mathrm{Mod}(\mathrm{B},\mathrm{C})_S$. If U is open, and if $q \in U$, we will denote by

$$\sigma_q\text{-}\mathrm{Mod}(B,\mathrm{C})^{\mathrm{an}}_U \tag{4.2.1}$$

the full sub-category of σ_q-$\mathrm{Mod}(\mathrm{B})_U$ whose objects belong to the essential image of σ-$\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{an,const}}_U$. We give analogous definitions for (σ,δ)-modules.

Lemma 4.3 and Definition 4.4 easily give the following formal statement.

Corollary 4.5. *With the notation of Lemma 4.3, one has an equivalence*

$$\mathrm{Res}^S_{\{q\}} : (\sigma,\delta)\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{const}}_S \xrightarrow{\sim} (\sigma_q,\delta_q)\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})_S. \tag{4.2.2}$$

The same fact is true for σ-modules, under the additional hypothesis that $q \in S^\circ$.

Definition 4.6. (1) Let $S \subseteq \mathcal{Q}(\mathrm{B})$ be a subset and let $q, q' \in \langle S \rangle$. We will call the C-*constant deformation functor*, denoted by

$$\mathrm{Def}^{\mathrm{C}}_{q,q'} : (\sigma_q,\delta_q)\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})_S \xrightarrow{\sim} (\sigma_{q'},\delta_{q'})\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})_S, \tag{4.2.3}$$

the equivalence obtained by composition of the restriction functor (4.2.2):

$$\mathrm{Def}^{\mathrm{C}}_{q,q'} := \mathrm{Res}^S_{\{q'\}} \circ (\mathrm{Res}^S_{\{q\}})^{-1}. \tag{4.2.4}$$

(2) We will call the C-*constant confluence functor*, the equivalence

$$\mathrm{Conf}^{\mathrm{C}}_q := \mathrm{Def}^{\mathrm{C}}_{q,1} : (\sigma_q,\delta_q)\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})_S \xrightarrow{\sim} \delta_1\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})_S. \tag{4.2.5}$$

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(3) Suppose that $q \in S^\circ$ and $q' \in S$, then we will again call the C-*constant deformation functor*, denoted again by

$$\mathrm{Def}^{\mathrm{C}}_{q,q'} : \sigma_q\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})_S \longrightarrow \sigma_{q'}\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})_S, \tag{4.2.6}$$

the functor obtained by composition of the restriction functor (4.2.2): $\mathrm{Def}^{\mathrm{C}}_{q,q'} := \mathrm{Res}^S_{\{q'\}} \circ (\mathrm{Res}^S_{\{q\}})^{-1}$. If $q' \in S^\circ$, then $\mathrm{Def}^{\mathrm{C}}_{q,q'}$ is an equivalence.

It follows from Corollary 4.5 that, if $q, q' \in U$, one has an *equivalence*, again called $\mathrm{Def}^{\mathrm{C}}_{q,q'}$,

$$\mathrm{Def}^{\mathrm{C}}_{q,q'} : (\sigma_q,\delta_q)\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{an}}_U \xrightarrow[\sim]{} (\sigma_{q'},\delta_{q'})\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{an}}_U. \tag{4.2.7}$$

The same fact is true for analytic σ-modules under the condition $q, q' \notin \boldsymbol{\mu}(\mathcal{Q})$.

4.2.1 Notice that the functor $\mathrm{Res}^S_{\{q\}}$ does not depend on C, but $(\mathrm{Res}^S_{\{q\}})^{-1}$ is a particular section of $\mathrm{Res}^S_{\{q\}}$ with values in the category of objects trivialized by C (cf. Corollary 4.5). Hence $(\mathrm{Res}^S_{\{q\}})^{-1}$, $\mathrm{Conf}^{\mathrm{C}}_q$ and $\mathrm{Def}^{\mathrm{C}}_{q,q'}$ actually depend on C.

4.2.2 According to Definition 4.4 (cf. Equations (2.1.5) and (2.2.3)), if $q \in U \subset U'$, then, by Lemma 4.3 (cf. § 4.1.2), the following restriction functors are fully faithful immersions:

$$\begin{aligned} \mathrm{Res}^{U'}_U &: \sigma\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})_{U'} \longrightarrow \sigma\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})_U, \\ \mathrm{Res}^{U'}_U &: \sigma\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{an,const}}_{U'} \longrightarrow \sigma\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{an,const}}_U; \end{aligned}$$

respectively

$$\begin{aligned} \mathrm{Res}^{U'}_U &: (\sigma,\delta)\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})_{U'} \longrightarrow (\sigma,\delta)\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})_U, \\ \mathrm{Res}^{U'}_U &: (\sigma,\delta)\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{an,const}}_{U'} \longrightarrow (\sigma,\delta)\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{an,const}}_U. \end{aligned} \tag{4.2.8}$$

We can then consider the following diagram in which we heuristically imagine categories appearing in the first two lines as the stalks at q of suitable corresponding stacks over $\mathcal{Q}(X)$.

$$\begin{array}{ccc} \bigcup_U \ \sigma\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{an,const}}_U & \overset{\text{Equation (2.4.3)}}{=\!=\!=} & \bigcup_U \ (\sigma,\delta)\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{an,const}}_U \\ \Big\downarrow {\scriptstyle \bigcup_U \mathrm{Res}^U_{\{q\}}} & \odot & \wr\Big\downarrow {\scriptstyle \bigcup_U \mathrm{Res}^U_{\{q\}}} \\ \bigcup_U \ \sigma_q\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})_U & \underset{\text{Forget } \delta_q}{\longleftarrow} & \bigcup_U \ (\sigma_q,\delta_q)\text{-}\mathrm{Mod}(\mathrm{B},\mathrm{C})_U \\ \Big\downarrow {\scriptstyle i_\sigma} & \odot & \Big\downarrow {\scriptstyle i_{(\sigma,\delta)}} \\ \sigma_q\text{-}\mathrm{Mod}(\mathrm{B}) & \underset{\text{Forget } \delta_q}{\longleftarrow} & (\sigma_q,\delta_q)\text{-}\mathrm{Mod}(\mathrm{B}) \end{array} \tag{4.2.9}$$

Here U runs over the set of open neighborhoods of q, and i_σ and $i_{(\sigma,\delta)}$ are the trivial inclusions of full sub-categories. In the sequel we will study the full subcategory of σ_q-Mod(B) (respectively (σ_q,δ_q)-Mod(B)) formed by *Taylor admissible objects* this category is contained in the essential image of i_σ (respectively $i_{(\sigma,\delta)}$) (see Theorem 7.6). In this case we will obtain an analogous diagram (see Corollary 7.9) in which $i_{(\sigma,\delta)}$ is an equivalence (for all $q \in U$), and i_σ is an equivalence only if q is not a root of unity.

If q is not a root of unity, then all the arrows of this diagram will be equivalences, hence giving δ_q is superfluous. If q is a root of unity, then the right-hand side vertical arrows will be equivalences, while the arrow on the left-hand side will not. In this last case the q-tangent operator δ_q is necessary to *preserve the information in the neighborhood of* q. In this case the good notion of *stalk at* q of an analytic σ-module is the notion of (σ_q,δ_q)-module and not simply that of σ_q-module.

One may have the feeling that the functor 'Forget δ_q' contains 'information' if q is a root of unity, but we will see (Proposition 8.6) that, if $\mathrm{B}=\mathcal{R}_K$ or if $\mathrm{B}=\mathcal{H}_K^\dagger$, then this functor sends every (σ,δ)-module with Frobenius structure into a direct sum of copies of the unit object.

4.2.3 *Dependence on* C. Let $\mathrm{C}_1\subseteq\mathrm{C}_2$ be two algebras as above. Then clearly $\mathrm{Def}_{q,q'}^{\mathrm{C}_2}$ extends $\mathrm{Def}_{q,q'}^{\mathrm{C}_1}$ to the larger category of modules trivialized by C_2. One of the main problems of the theory is that, if there are no inclusions between C_1 and C_2, then it is not clear whether there exists a discrete σ-algebra (respectively (σ,δ)-algebra) C_3 containing both C_1 and C_2. For this reason, if the same object is trivialized by C_1, and also by C_2, it is not clear whether its deformations with respect to C_1 and C_2 are equal. We will encounter this problem in § 8.4.

5. Taylor solutions

In this section $\mathrm{B}=\mathcal{H}_K(X)$, for some affinoid $X=\mathrm{D}^+(c_0,R_0)-\bigcup_{i=1}^n\mathrm{D}^-(c_i,R_i)$, and $S=\{q\}\in\mathcal{Q}(\mathcal{H}_K(X))\subseteq\{q\in K\mid|q|=1\}$ is reduced to a point. Let $(\Omega,|\cdot|)/(K,|\cdot|)$ be an arbitrary extension of complete valued fields. Let $c\in X(\Omega)$ and let $\rho_{c,X}>0$ be the largest real number such that $\mathrm{D}^-_{\Omega'}(c,\rho_{c,X})\subseteq X(\Omega')$, for all complete valued field extensions $(\Omega',|\cdot|)/(\Omega,|\cdot|)$. One has

$$\rho_{c,X}=\min(R_0,|c-c_1|,|c-c_2|,\dots,|c-c_n|). \tag{5.0.1}$$

Notice that c can be equal to a generic point (cf. Proposition 1.7). We want to find solutions of q-difference equations converging in a disk centered at c, i.e. matrix solutions in the form (3.2.3), with values in the σ_q-algebra $\mathrm{C}:=\mathcal{A}_K(c,R)$, for some $0<R\leqslant\rho_{c,X}$.

5.1 The q-algebras $\Omega\{T-c\}_{q,R}$ and $\Omega[[T-c]]_q$

Unless we explicitly state the contrary, we will not assume that $q\notin\boldsymbol{\mu}(\mathcal{Q})$. The following results generalize the analogous constructions of [DV04] to the case of a root of unity.

LEMMA 5.1. *Let $0<R\leqslant\rho_{c,X}$. The algebra $\mathcal{A}_\Omega(c,R)$ is an $\mathcal{H}_\Omega(X)$-discrete σ-algebra over $S=\{q\}$, if and only if both of the following conditions hold:*

$$|q-1||c|<R\quad\text{and}\quad|q|=1. \tag{5.1.1}$$

DEFINITION 5.2. Let $q\in K^\times$ be an arbitrary number. Following [DV04] and [ADV04] we set

$$(T-c)_{q,n}:=(T-c)(T-qc)(T-q^2c)\cdots(T-q^{n-1}c), \tag{5.1.2}$$

$$[n]_q:=1+q+q^2+\cdots+q^{n-1}, \tag{5.1.3}$$

$$[n]_q^!:=\frac{(q-1)(q^2-1)(q^3-1)\cdots(q^n-1)}{(q-1)^n}. \tag{5.1.4}$$

5.1.1 *The q-binomial.* For all $q\in K^\times$, we define the q-binomial $\binom{n}{i}_q$ by the relation

$$(1-T)(1-qT)\cdots(1-q^{n-1}T)=\sum_{i=0}^n(-1)^i\binom{n}{i}_q q^{i(i-1)/2}T^i, \tag{5.1.5}$$

where, if $i=0$, the symbol $q^{i(i-1)/2}$ is by definition equal to 1. This extends the definition given in [DV04] (cf. Equation (5.1.7) below) to the case of a root of unity. If $1\leqslant i\leqslant n-1$, by induction one has

$$\binom{n}{i}_q=\binom{n-1}{i-1}_q+q^i\binom{n-i}{i}_q=q^{n-i}\binom{n-1}{i-1}_q+\binom{n-i}{i}_q. \tag{5.1.6}$$

If q is not a root of unity, then one can write

$$\binom{n}{i}_q = \frac{[n]_q \cdot [n-1]_q \cdots [n-i+1]_q}{[i]_q^!}. \tag{5.1.7}$$

If q is an mth root of unity, then $[n]_q^! = 0$, for all $n \geqslant m$. The family $\{(T-c)_{q,n}\}_{n\geqslant 0}$ is adapted to the q-derivation

$$d_q := \frac{\sigma_q - 1}{(q-1)T} = \frac{\Delta_q}{T} \tag{5.1.8}$$

in the sense that for all $n \geqslant 1$ one has $d_q((T-c)_{q,n}) = [n]_q \cdot (T-c)_{q,n-1}$. One always has the relation $d_q(fg) = \sigma_q(f)d_q(g) + d_q(f)g$. More generally our definition of q-binomials allows us to generalize the proof of [DV04, Lemma 1.2, (1.2.2)] to the case of a root of unity. We obtain the formula

$$d_q^n(fg)(T) = \sum_{i=0}^{n} \binom{n}{i}_q d_q^{n-i}(f)(q^iT)d_q^i(g)(T). \tag{5.1.9}$$

5.1.2 The following lemma extends [DV04, § 1.3] to the case of a root of unity.

Lemma 5.3. *Let $(\Omega, |\cdot|)/(K, |\cdot|)$ be a complete extension of valued fields. Let $|q-1||c| < R$, $|q| = 1$, and let $f(T) = \sum_{n\geqslant 0} a_n(T-c)^n \in \mathcal{A}_\Omega(c, R)$. Then the following hold:*

(a) *$f(T)$ can be written uniquely as the series of functions*

$$f(T) = \sum_{n\geqslant 0} \widetilde{a}_n(T-c)_{q,n} \in \mathcal{A}_\Omega(c, R), \tag{5.1.10}$$

with $\widetilde{a}_n \in \Omega$ satisfying $\sup_n |\widetilde{a}_n|\rho^n < \infty$, for all $\rho < R$;

(b) *for all $|q-1||c| < \rho < R$ one has $|f(T)|_{(c,\rho)} = \sup_{n\geqslant 0} |a_n|\rho^n = \sup_{n\geqslant 0} |\widetilde{a}_n|\rho^n$;*

(c) *one has $Ray(f(T), c) = \liminf_n |a_n|^{-1/n} = \liminf_n |\widetilde{a}_n|^{-1/n}$;*

(d) *if moreover $q \notin \boldsymbol{\mu}(\mathcal{Q})$, then one has the so-called q-Taylor expansion (cf. [DV04])*

$$f(T) = \sum_{n\geqslant 0} d_q^n(f)(c)\frac{(T-c)_{q,n}}{[n]_q^!}. \tag{5.1.11}$$

Proof. Since $\mathcal{A}_\Omega(c, R) = \varprojlim_{r\to R^-} \mathcal{H}_\Omega(\mathrm{D}^+(c, r))$, we need only prove the proposition for $\mathcal{H}_\Omega(\mathrm{D}^+(c, r))$, with $|q-1||c| < r < R$. We recall that a series of functions $\sum_{n\geqslant 0} f_n$, $f_n \in \mathcal{H}_K(\mathrm{D}^+(c, r))$, converges to a function $f \in \mathcal{H}_K(\mathrm{D}^+(c, r))$ if and only if $\lim_n |f_n|_{(c,r)} = 0$. Writing $(T-q^ic) = (1-q^i)c + (T-c)$, one sees easily that $(T-c)_{q,n} = \sum_{i=0}^n \widetilde{b}_{n,i}(T-c)^i$, with $\widetilde{b}_{i,j}$ satisfying (i) $\widetilde{b}_{0,0} = 1$, (ii) $\widetilde{b}_{0,i} = 0$ for all $i \geqslant 1$, (iii) $\widetilde{b}_{n,n} = 1$ for all $n \geqslant 0$, (iv) $\widetilde{b}_{n,i} = 0$ for all $i > n$, and (v) for all $0 \leqslant i < n$:

$$\widetilde{b}_{n,i} = c^{n-i} \cdot \sum_{0\leqslant k_1<\cdots<k_{n-i}\leqslant n-1} (1-q^{k_1})(1-q^{k_2})\cdots(1-q^{k_{n-i}}). \tag{5.1.12}$$

In other words $[1, (T-c)_{q,1}, (T-c)_{q,2}, \ldots, (T-c)_{q,n}]^{\mathrm{t}} = \widetilde{B} \cdot [1, (T-c), (T-c)^2, \ldots, (T-c)^n]^{\mathrm{t}}$ where $\widetilde{B} = (\widetilde{b}_{n,i})_{n,i=0,\ldots,n}$ is an $(n+1) \times (n+1)$ lower triangular matrix satisfying (i)–(v). Since $|q^i - 1| \leqslant |q-1|$, one also has the property (vi) $|\widetilde{b}_{n,i}| \leqslant (|q-1||c|)^{n-i} < r^{n-i}$, for all $0 \leqslant i < n$. Hence for all $n \geqslant 0$, one has $(T-c)_{q,n} = (T-c)^n + g_n(T)$, with $|g_n(T)|_{(c,r)} < r^n$, so $|(T-c)_{q,n}|_{(c,r)} = |(T-c)^n|_{(c,r)} = r^n$. It is easy to prove that also the matrix $B := \widetilde{B}^{-1} = (b_{n,i})_{n,i=0,\ldots,n}$ satisfies the properties (i)–(vi). Consider now $f(T) = \sum_{n\geqslant 0} a_n(T-c)^n$. Writing $f_m(T) := \sum_{n=0}^m a_n(T-c)^n = \sum_{n=0}^m a_n \sum_{i=0}^n b_{n,i}(T-c)_{q,i}$ and rearranging terms one finds $f_m(T) = \sum_{n=0}^m \widetilde{a}_{n,m}(T-c)_{q,n}$, with $\widetilde{a}_{n,m} = \sum_{k=0}^{m-n} a_{n+k}b_{n+k,n}$. By property (vi) and by the assumption that $\lim_n |a_n|r^n = 0$ the sum

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$\widetilde{a}_n := \sum_{k\geqslant 0} a_{n+k}b_{n+k,n}$ converges in Ω. Moreover

$$|\widetilde{a}_n|r^n \leqslant \max_{k\geqslant 0} |a_{n+k}||b_{n+k,n}| \cdot r^n \leqslant \max_{k\geqslant 0} |a_{n+k}|r^{n+k}. \tag{5.1.13}$$

This proves that $\lim_n |\widetilde{a}_n|r^n = 0$, and hence that the series of functions $\sum_{n\geqslant 0} \widetilde{a}_n(T-c)_{q,n}$ is convergent in $\mathcal{H}_\Omega(\mathrm{D}^+(c,r))$. If $f_m^0(T) := \sum_{n=0}^m \widetilde{a}_n(T-c)_{q,n}$, one sees that $|f_m^0 - f_m|_{(c,r)} \leqslant \sup_{k\geqslant 0} |a_{m+k}|r^{m+k}$ which tends to 0, so $\lim_m f_m^0(T) = \lim_m f_m(T) = f(T)$ in $\mathcal{H}_\Omega(\mathrm{D}^+(c,r))$. Now the inequality (5.1.13) shows that $\max_{n\geqslant 0} |\widetilde{a}_n|r^n \leqslant \max_{n\geqslant 0} |a_n|r^n$, and a symmetric argument using the matrix $\widetilde{B}$ instead of B proves the opposite inequality so $\max_{n\geqslant 0} |\widetilde{a}_n|r^n = \max_{n\geqslant 0} |a_n|r^n = |f(T)|_{(c,r)}$. This last equality shows the uniqueness of the coefficients $\{\widetilde{a}_n\}_n$ since if $\sum_{n\geqslant 0} \widetilde{a}_n(T-c)_{q,n} = \sum_{n\geqslant 0} \widetilde{a}'_n(T-c)_{q,n}$, then $\sum_{n\geqslant 0}(\widetilde{a}_n - \widetilde{a}'_n)(T-c)_{q,n} = 0$, and hence $\sup_n(|\widetilde{a}_n - \widetilde{a}'_n|r^n) = 0$, so that $\widetilde{a}_n = \widetilde{a}'_n$, for all $n \geqslant 0$. Clearly the radius of convergence of $f(T)$ is equal to both $\sup_{n\geqslant 0}\{r \geqslant 0 \mid |a_n|r^n \text{ is bounded}\}$ and $\sup_{n\geqslant 0}\{r \geqslant 0 \mid |\widetilde{a}_n|r^n \text{ is bounded}\}$. Hence, by classical arguments on the radius of convergence, one has $Ray(f(T),c) = \liminf_n |a_n|^{-1/n} = \liminf_n |\widetilde{a}_n|^{-1/n}$. The assertion (d) is proved in [DV04]. □

Remark 5.4. If $f(T) = \sum_{n\geqslant 0} f_n(T-c)_{q,n}$, and if $g(T) = \sum_{n\geqslant 0} g_n(T-c)_{q,n}$, then $f(T)g(T) = \sum_{n\geqslant 0} h_n(T-c)_{q,n}$, where $h_n = h_n(q;c;f_0,\dots,f_n;g_0,\dots,g_n)$ is a polynomial in $\{q,c,f_0,\dots,f_n,g_0,\dots,g_n\}$. Indeed one has $(T-c)_{q,n}\cdot(T-c)_{q,m} = \sum_{k=\max(n,m)}^{n+m} \alpha_k^{(n,m)}(T-c)_{q,k}$, with $\alpha_k^{(n,m)} = \alpha_k^{(n,m)}(q,c) \in \Omega$. This also shows that if $v_{q,c}(f) := \min\{n \mid f_n \neq 0\}$, then one has

$$v_{q,c}(fg) \geqslant \max(v_{q,c}(f), v_{q,c}(g)). \tag{5.1.14}$$

If moreover $q \notin \boldsymbol{\mu}(\mathcal{Q})$, then, by using equations (5.1.9) and (5.1.11), one has

$$h_n = \sum_{j=0}^{n}\sum_{s=0}^{j} \frac{[n]_q^![j]_q^![s+n-j]_q^!}{([s]_q^!)^2[n-j]_q^!} \cdot q^{s(s-1)/2}(q-1)^s c^s f_{s+n-j}g_j. \tag{5.1.15}$$

5.1.3 *The algebras* $\Omega[\![T-c]\!]_q$ *and* $\Omega\{T-c\}_{q,R}$. We have the following definitions.

DEFINITION 5.5. For all $q \in \mathcal{Q}(X)$ we set

$$\Omega[[T-c]]_q := \left\{\sum_{n\geqslant 0} f_n(T-c)_{q,n} \;\middle|\; f_n \in \Omega\right\}, \tag{5.1.16}$$

$$\Omega\{T-c\}_{q,R} := \left\{\sum_{n\geqslant 0} f_n(T-c)_{q,n} \;\middle|\; f_n \in \Omega,\ \liminf_n |f_n|^{-1/n} \geqslant R\right\}. \tag{5.1.17}$$

We define a multiplication on $\Omega[\![T-c]\!]_q$ and $\Omega\{T-c\}_{q,R}$ by the rule given in Remark 5.4.

LEMMA 5.6. *The algebras* $\Omega[\![T-c]\!]_q$ *and* $\Omega\{T-c\}_{q,R}$ *are commutative* Ω*-algebras, for all* $q \in \mathcal{Q}$.

Proof. We prove only the associativity, the other verifications being similar. We have to prove that $(fg)h = f(gh)$. By Lemma 5.3 the assertion is proved if $f,g,h \in \Omega\{T-c\}_{q,R}$, with $|q-1||c| < R$, since in this case $\Omega\{T-c\}_{q,R} \cong \mathcal{A}_\Omega(c,R)$. On the other hand one can assume that f,g,h are polynomials since, by Remark 5.4, the nth coefficient of $(fg)h$ and of $f(gh)$ is a polynomial in q and in the first n coefficients of f,g,h. □

Remark 5.7. If there exists a (smallest) integer k_0 such that $|\overline{q}^{k_0}-1||c| < R$, then one shows that $\Omega\{T-c\}_{q,R} = \prod_{i=0}^{k_0-1} \mathcal{A}_\Omega(q^ic,\widetilde{R})$, where $\widetilde{R}$ depends explicitly on R, c and q (cf. [DV04, Proposition 15.3]). In this case $\Omega\{T-c\}_{q,R}$ is not a domain and hence is not a $\mathcal{H}_\Omega(X)$-discrete σ-algebra over $S = \{q\}$.

Remark 5.8. If x,y are variables, then $\Omega[[x-y]]_q$ is not an algebra, but merely a vector space. Indeed the multiplication law involves y in the coefficients 'h_n' of Remark 5.4. This minor mistake occurs

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occasionally in [DV04], but it is an irrelevant inaccuracy and does not jeopardize any proposition of [DV04]. Indeed the matrix $Y(x,y)$ always seems to be used there under the assumption (5.5.6) (cf. Lemma 5.16).

5.2 q-invariant affinoids

Let $|q|=1$, $q\in K$. Let $X:=\mathrm{D}^+(c_0,R_0)-\bigcup_{i=1}^n\mathrm{D}^-(c_i,R_i)$, $c_1,\dots,c_n\in\mathrm{D}^+_K(c_0,R_0)$, $c_0\in K$, be a K-affinoid. Then X is q-invariant if and only if $|q-1||c_0|<R_0$, and the map $x\mapsto qx$ permutes the family of disks $\{\mathrm{D}^-(c_i,R_i)\}_{i=1,\dots,n}$. This happens if and only if for all $i=1,\dots,n$ there exists (a smallest) $k_i\geqslant 1$, such that $|q^{k_i}-1||c_i|<R_i$, and moreover the family of disks $\{\mathrm{D}^-(q^kc_i,R_i)\}_{k=1,\dots,k_i}$ is finite and contained in $\{\mathrm{D}^-(c_i,R_i)\}_{i=1,\dots,n}$. If k_0 is the minimum common multiple of the k_i, then $x\mapsto q^{k_0}x$ leaves every disk globally fixed and, by Lemmas 5.1 and 5.3, one has

$$\|d_{q^{k_0}}(f)\|_X\leqslant r_X^{-1}\|f\|_X, \tag{5.2.1}$$

for all $f\in\mathcal{H}(X)$ (cf. Lemma 1.9). Indeed by the Mittag–Lefler decomposition [CR94], we reduce to showing that every series $f=\sum_{j\leqslant -1}a_j(T-c_i)^j$, such that $|a_j|R_i^j$ tends to zero, satisfies $|d_{q^{k_0}}(f)|_{(c_i,R_i)}\leqslant R_i^{-1}\cdot|f|_{(c_i,R_i)}$, and this is true by Lemma 5.3.

Such a bound does not exist for d_q itself. One can easily construct counterexamples via the Mittag–Leffler decomposition.

5.3 The generic Taylor solution

We recall the definition of the classical Taylor solution of a differential equation.

Definition 5.9. Let $\delta_1-G(1,T)$, be a differential equation. Let $G_{[n]}(T)$ be the matrix of $(d/dx)^n$. We set

$$Y_{G(1,T)}(x,y):=\sum_{n\geqslant 0}G_{[n]}(y)\frac{(x-y)^n}{n!}. \tag{5.3.1}$$

By induction on the rule $G_{[n+1]}=G'_{[n]}+G_{[n]}G_{[1]}$, one finds $\|G_{[n]}\|_X\leqslant\max(\|G_{[1]}\|_X,r_X^{-1})^n$, hence

$$Ray(Y_G(T,c),c)=\liminf_n\left(\frac{|G_{[n]}(c)|_\Omega}{|n!|}\right)^{-1/n}\geqslant\frac{|p|^{1/(p-1)}}{\max(r_X^{-1},\|G_{[1]}\|_X)}. \tag{5.3.2}$$

In other words $Y_G(x,y)$ is an analytic function over a neighborhood $\mathcal{U}_R$ of the diagonal of the type

$$\mathcal{U}_R:=\{(x,y)\in X\times X\mid |x-y|<R\}, \tag{5.3.3}$$

for some $R>0$.

Lemma 5.10. *One has* $Y_G(x,x)=\mathrm{Id}$, *and, for all* $(x,y)\in\mathcal{U}_R$

$$(d/dy)(Y_G(x,y))=-Y_G(x,y)\cdot G_{[1]}(y), \tag{5.3.4}$$
$$Y_G(x,y)^{-1}=Y_G(y,x), \tag{5.3.5}$$
$$Y_G(x,y)\cdot Y_G(y,z)=Y_G(x,z), \tag{5.3.6}$$
$$(d/dx)(Y_G(x,y))=G_{[1]}(x)\cdot Y_G(x,y). \tag{5.3.7}$$

Proof. See [CM02, p. 137] (cf. Lemma 5.16). The proof is analogous to that of Lemma 5.16. □

Definition 5.11. Let $q\in\mathcal{Q}-\boldsymbol{\mu}(\mathcal{Q})$. Consider the q-difference equation

$$\sigma_q(Y)=A(q,T)\cdot Y,\quad A(q,T)\in GL_n(\mathcal{H}_K(X)). \tag{5.3.8}$$

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Let H_n be defined by $d_q^n(Y) = H_n \cdot Y$. We formally set

$$Y_{A(q,T)}(x,y) = \sum_{n \geqslant 0} H_n(y) \frac{(x-y)_{q,n}}{[n]_q^!}. \tag{5.3.9}$$

We will omit the index $A(q,T)$ if no confusion is possible. Observe that $Y_{A(q,T)}(x,y)$ is a symbol and does not necessarily define a convergent function.

Example 5.12. With the notation of Example 2.6, the generic Taylor solution of the equations $\sigma_q(Y) = A(q,x)Y$ and $\sigma_q(Y) = \widetilde{A}(q,x)Y$ are $Y_{A(q,x)}(x,y) = \exp(\pi(x-y))$ and $Y_{\widetilde{A}(q,x)}(x,y) = \exp(\pi q(x-y))$ respectively. Notice that $Y_{A(q,x)}(x,y)$ is *constant* with q.

DEFINITION 5.13. For all (not necessarily bounded nor multiplicative) semi-norms $|\cdot|_*$ on $\mathcal{H}_K(X)$ extending the absolute value of K we set

$$Ray(Y_{A(q,T)}(x,y), |\cdot|_*) := \liminf_n (|H_n(y)|_*/|[n]_q^!|)^{-1/n}. \tag{5.3.10}$$

If $Y_{A(q,T)}(x,y)$ is a convergent function on some neighborhood of the diagonal of $X \times X$, then, for $|f(T)|_* := |f(c)|_\Omega$, $c \in X(\Omega)$, one finds Definition 1.5, namely $Ray(Y(x,y), |\cdot|_c) = Ray(Y(x,c),c)$. In this case we will write $Ray(Y(x,y),c) := Ray(Y(x,y),|\cdot|_c)$ (cf. § 1.3.1). If $X' \subseteq X$ is a sub-affinoid we simply write $Ray(Y(x,y),X') := Ray(Y(x,y), \|\cdot\|_{X'})$.

5.4 Transfer principle

As in the differential setting, if $X' := \mathrm{D}^+(c'_0,R'_0) - \bigcup_{i=1}^s \mathrm{D}^-(c'_i,R'_i) \subseteq X$ is a q-invariant sub-affinoid of X, such that every disk $\mathrm{D}^-(c'_i,R'_i)$ is also q-invariant, then the estimate (5.2.1) holds (cf. Remark 7.12). Then, by induction on the rule $H_{n+1} = d_q(H_n) + \sigma_q(H_n)H_1$, one shows that $\|H_n\|_{X'} \leqslant \max(\|H_1\|_{X'}, r_{X'}^{-1})^n$, hence

$$\begin{aligned} Ray(Y(x,y),X') &:= \liminf_n (\|H_n\|_{X'}/|[n]_q^!|)^{-1/n} = \inf_{c \in X'(\Omega)} Ray(Y(x,y),c) \\ &= \min_{c \in \{t_{c'_0,R'_0},\dots,t_{c'_s,R'_s}\}} Ray(Y(x,y),c) \geqslant \frac{\liminf_n |[n]_q^!|^{1/n}}{\max(r_{X'}^{-1}, \|H_1\|_{X'})}, \end{aligned} \tag{5.4.1}$$

where $(\Omega,|\cdot|)/(K,|\cdot|)$ is sufficiently large to contain $\{t_{c'_0,R'_0},\dots,t_{c'_s,R'_s}\}$ (cf. § 1.4). As suggested by the referee, one can prove the second and the third equalities using a q-analog of a theorem of Dwork and Robba (cf. [DV04]). One may also observe that, since $\|H_n(T)\|_{X'} \geqslant |H_n(c)|_\Omega$, for all $c \in X'(\Omega)$, then $Ray(Y(x,y),X') \leqslant \inf_{c \in X'(\Omega)} Ray(Y(x,y),c)$. The converse of this inequality is proved as follows. By the properties of the Shilow boundary one has $\|H_n(T)\|_{X'} = \max_{i=0,\dots,s'} |H_n(t_{c'_i,R'_i})|$. Hence $(\|H_n(T)\|_{X'}/|[n]_q^!|)^{-1/n} = \min_{i=0,\dots,s'} |H_n(t_{c'_i,R'_i})|/|[n]_q^!|^{-1/n}$, and since 'lim inf' commutes with the 'minimum over a finite set', then $Ray(Y(x,y),X') = \min_{i=0,\dots,s'} Ray(Y(x,y),t_{c'_i,R'_i})$. Now since we have chosen Ω such that $t_{c'_i,R'_i} \in X'(\Omega)$, then

$$\min_{i=0,\dots,s'} Ray(Y(x,y),t_{c'_i,R'_i}) \geqslant \inf_{c \in X'(\Omega)} Ray(Y(x,y),c).$$

This proves the required equalities.

In particular if $X' = \mathrm{D}^+(c,\rho) \subseteq X$, with $|q-1||c| < \rho \leqslant \rho_{c,X}$, is a q-invariant disk, then $Ray(Y(x,y),c)$ is greater than or equal to

$$\begin{aligned} Ray(Y(x,y),\mathrm{D}^+(c,\rho)) &= \min_{c' \in \mathrm{D}^+_\Omega(c,\rho)} Ray(Y(x,y),c') \\ &= Ray(Y(x,y),t_{c,\rho}) \geqslant \frac{\liminf_n ([n]_q^!)^{1/n}}{\max(\rho^{-1}, |H_1|_{(c,\rho)})}. \end{aligned} \tag{5.4.2}$$

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Notice that if $|q-1||c| < R_c := Ray(Y(x,y),c)$, then $Y(x,c) \in M_n(\mathcal{A}_\Omega(c,R_c))$, but $Y(x,c)$ is invertible only in $GL_n(\mathcal{A}_\Omega(c,\widetilde{R}))$, with $\widetilde{R} := \min(\rho_{c,X}, Ray(Y(x,y),c))$ (cf. Lemmas 5.15 and 5.16).

5.5 Properties of the generic Taylor solution

The formal matrix solution $Y_A(x,y)$ is not always a function in a neighborhood of type $\mathcal{U}_R$ of the diagonal of $X \times X$. But if for all $c \in X(K^{\mathrm{alg}})$ one has $|q-1||c| < R \leqslant \min(\rho_{c,X}, Ray(Y(x,y),c))$, then, by Lemma 5.3, and by the transfer principle (cf. Equation (5.4.2)), $Y_A(x,y)$ actually defines an *invertible* function on $\mathcal{U}_R$ (cf. Lemmas 5.15 and 5.16). If $X = \mathrm{D}^+(c_0,R_0) - \bigcup_{i=1}^n \mathrm{D}^-(c_i,R_i)$, the condition $|q-1||c| < R \leqslant \min(\rho_{c,X}, Ray(Y(x,y),c))$, for all $c \in X(K^{\mathrm{alg}})$, implies that

$$|q-1|\sup(R_0,|c_0|) = |q-1|\max_{c\in X}|c| < R \leqslant \min_{c\in X}\rho_{c,X} = \min(R_0,\dots,R_n) = r_X. \tag{5.5.1}$$

In particular, since $r_X = \min(R_0,\dots,R_n) \leqslant \sup(|c_0|,R_0)$, this is possible only if

$$|q-1| < 1, \quad \text{i.e. if } q \in \mathcal{Q}_1(X). \tag{5.5.2}$$

Hypothesis 5.14. From now on, without explicit mention to the contrary, we will assume that

$$q \in \mathcal{Q}_1(X). \tag{5.5.3}$$

Lemma 5.15. *Let $q \in \mathcal{Q}_1(X) - \boldsymbol{\mu}(\mathcal{Q}_1(X))$. Let $f(x,y)$ be an analytic function in a neighborhood of type $\mathcal{U}_R \subset X \times X$ of the diagonal of $X \times X$. Assume that*[4]

$$|q-1|\max(|c_0|,R_0) < R \leqslant r_X. \tag{5.5.4}$$

If moreover $f(x,y)$ satisfies $f(x,qy) = a(y)\cdot f(x,y)$, with $a(y) \in \mathcal{H}_K(X)^\times$, then $f(x,y)$ is invertible.

Proof. Since f is an analytic function, it is sufficient to prove that f has no zeros in $\mathcal{U}_R$. We need only show that, for all $c \in X(\Omega)$, the function $g_c(y) := f(c,y)$ has no zeros in $\mathrm{D}^-(c,R)$. One has $d_q(g_c(y)) = h(y)\cdot g_c(y)$, with $h(y) = (a(y)-1)/((q-1)y)$. Assume that $g_c(\widetilde{c}) = 0$, for some $\widetilde{c} \in \mathrm{D}^-(c,R) = \mathrm{D}^-(\widetilde{c},R)$, then, by Lemma 5.3, $g_c(y) = \sum_{n\geqslant 0} a_k(y-\widetilde{c})_{q,n}$, with $a_0 = 0$. Since $q \notin \boldsymbol{\mu}(\mathcal{Q})$, we have $[n]_q a_n = 0$ if and only if $a_n = 0$. Hence, by Remark 5.4 one has $v_{q,\widetilde{c}}(d_q(g_c)) = v_{q,\widetilde{c}}(g_c) - 1$. On the other hand, $v_{q,\widetilde{c}}(hg_c) \geqslant v_{q,\widetilde{c}}(g_c)$, which contradicts $d_q(g_c) = hg_c$. □

Lemma 5.16. *Let $q \in \mathcal{Q}_1(X) - \boldsymbol{\mu}(Q_1(X))$, and let*

$$\begin{aligned} &\sigma_q^x : f(x,y) \mapsto f(qx,y), \quad && \sigma_q^y : f(x,y) \mapsto f(x,qy),\\ &d_q^x := \frac{\sigma_q^x - 1}{(q-1)x}, && d_q^y := \frac{\sigma_q^y - 1}{(q-1)y}. \end{aligned} \tag{5.5.5}$$

Suppose that $Y_A(x,y)$ converges on $\mathcal{U}_R$, with (cf. § 5.5)

$$|q-1|\max(|c_0|,R_0) < R \leqslant r_X. \tag{5.5.6}$$

Then $Y_A(x,y)$ is invertible on $\mathcal{U}_R$ and satisfies $Y_A(x,x) = \mathrm{Id}$ and

$$d_q^y\, Y_A(x,y) = -\sigma_q^y(Y_A(x,y))\cdot H_1(y), \tag{5.5.7}$$

$$\sigma_q^y Y_A(x,y) = Y_A(x,y)\cdot A(q,y)^{-1}, \tag{5.5.8}$$

$$Y_A(x,y)\cdot Y_A(y,z) = Y_A(x,z), \tag{5.5.9}$$

$$Y_A(x,y)^{-1} = Y_A(y,x), \tag{5.5.10}$$

$$d_q^x\, Y_A(x,y) = H_1(x)\cdot Y_A(x,y), \tag{5.5.11}$$

$$\sigma_q^x\, Y_A(x,y) = A(q,x)\cdot Y_A(x,y). \tag{5.5.12}$$

[4]That is, assume that $|q-1|\max(|c_0|,R_0) < R \leqslant \rho_{c,X}$ for all $c \in X(K^{\mathrm{alg}})$.

Proof. The relation $Y(x,x) = \mathrm{Id}$ is evident, while (5.5.7) is easy to compute explicitly, and is equivalent to (5.5.8). Since $Y(x,y)$ converges on $\mathcal{U}_R$, (5.5.8) implies that the determinant $d(x,y)$ of $Y(x,y)$ satisfies $d(x,qy) = a(y)d(x,y)$, with $a(y) = \det(A(q,y)^{-1}) \in \mathcal{H}_K(X)^\times$. By Lemma 5.15, $d(x,y)$ is then invertible on $\mathcal{U}_R$, and hence also $Y(x,y)$ is invertible. By (5.1.9), and since $q \notin \boldsymbol{\mu}(\mathcal{Q})$, the relation $d_q^y(Y(x,y)Y(x,y)^{-1}) = 0$ gives

$$d_q^y(Y(x,y)^{-1}) = -\sigma_q^y(Y(x,y)^{-1}) \cdot d_q^y(Y(x,y)) \cdot Y(x,y)^{-1}. \tag{5.5.13}$$

Hence, for all x,y,z such that $|x-y|, |z-y| < R$, the relation (5.5.13) together with relation (5.5.7) give $d_q^y(Y(x,y) \cdot Y(z,y)^{-1}) = 0$. Since $q \notin \boldsymbol{\mu}(\mathcal{Q})$, this implies, by Lemma 5.3, that the function $Y(x,y)Y(z,y)^{-1}$ does not depend on y. Specializing for $y = x$ and $y = z$, one finds $Y(x,z) = Y(z,x)^{-1}$ and $Y(x,y) \cdot Y(y,z) = Y(x,z)$. Then, by the above expression for $d_q^x(Y(y,x)^{-1}) = d_q^x(Y(x,y))$, the relations (5.5.11) and (5.5.12) follow from (5.5.10) and (5.5.7). □

5.5.1 *The case $|q-1| = 1$, $|q| = 1$.* If for a $c \in X$ one has $|q-1||c| \geqslant Ray(Y_{A(q,T)}(x,y),c)$, then Lemma 5.16 does not apply (cf. [DV04, § 15]). It may happen (cf. Remark 7.12) that there exists a (smallest) $k_0 \geqslant 0$ such that condition (5.5.6) holds for q^{k_0} instead of q, and for $Y_{A(q^{k_0},T)}(x,y)$ instead of $Y_{A(q,T)}(x,y)$. There then exists a Taylor solution $Y_c \in M_n(\mathcal{A}_\Omega(c,R))$ of the iterated system $\sigma_{q^{k_0}}(Y_c) = A(q^{k_0},T)Y_c$. In this case, for all $c \in X(\Omega)$, we can recover a solution Y^{big} of the system $\sigma_q(Y^{\mathrm{big}}) = A(q,T)Y^{\mathrm{big}}$ itself in the algebra of analytic functions over the disjoint union of disks $\bigcup_{i=0}^{k_0-1} \mathrm{D}^-(q^ic,R)$. Indeed σ_q acts on the algebra $\prod_{i\in\mathbb{Z}/k_0\mathbb{Z}} M_n(\mathcal{A}_K(q^ic,R))$ by $\sigma_q((M_{q^ic}(T))_{i\in\mathbb{Z}/k_0\mathbb{Z}}) = (M_{q^{i+1}c}(qT))_{i\in\mathbb{Z}/k_0\mathbb{Z}}$, and so one has

$$Y^{\mathrm{big}}(T) = (Y^{\mathrm{big}}_{q^ic}(T))_i := (A(q^i,q^{-i}T)\cdot Y_c(q^{-i}T))_{i\in\mathbb{Z}/k_0\mathbb{Z}}. \tag{5.5.14}$$

In fact $A(q^{i+1},q^{-i}T) = A(q,T)A(q^i,q^{-i}T)$. This and related matters are very well explained in [DV04].

5.5.2 Notice that the relations of Lemma 5.16 hold for $Y_A(x,y)$ as a function on $\mathcal{U}_R$, and not for $Y^{\mathrm{big}}(T)$ (cf. (5.5.14)). In other words the expression $Y_A^{\mathrm{big}}(x,y)$ has no meaning if $|x-y| \geqslant R$. In particular the expression (5.5.9), which is the main tool of the propagation theorem (Theorem 7.7), holds only if $|x-y|, |z-y| < R$.

5.5.3 *The case of a root of unity.* If $q \in \boldsymbol{\mu}(\mathcal{Q})$ is a root of unity, then even when a solution $Y \in GL_n(\mathcal{A}_\Omega(c,R))$ exists, *the radius is not defined* since we may have another solution with different radius (cf. Example 5.17 below). For this reason, the radius of convergence of the system (5.3.8) will be not defined if $q \in \boldsymbol{\mu}(\mathcal{Q})$.

Example 5.17. Let $q = \xi$ be a pth root of unity, with $\xi \neq 1$. The solutions of the unit object at $t^p \in \Omega$ are the functions $y \in \mathcal{A}_\Omega(t^p,R)$ such that $y(\xi T) = y(T)$. Every function in T^p has this property. For example the family of functions $\{y_\alpha := \exp(\alpha(T^p - t^p))\}_{\alpha\in\Omega}$ is such that for different values of α one has different radii.

5.6 Taylor solutions of (σ_q, δ_q)-modules

In this subsection q may be a root of unity. We preserve the previous notation. We consider now a system (the notion of the solution of such a system has been defined in § 3.2):

$$\begin{aligned} \sigma_q(Y) &= A(q,T)\cdot Y, \quad A(q,T) \in GL_n(\mathcal{H}_K(X)),\\ \delta_q(Y) &= G(q,T)\cdot Y, \quad G(q,T) \in M_n(\mathcal{H}_K(X)). \end{aligned} \tag{5.6.1}$$

It can happen that a solution of σ_q^{M} is not a solution of δ_q^{M} as shown by the following example.

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Example 5.18. Suppose that $q \in \mathrm{D}^-(1,1)$ is not a root of unity. Let $X := \mathrm{D}^+(0, |p|^{1/(p-1)})$, $A(q,T) := \exp((q-1)T) \in \mathcal{H}_K(X)^\times$, $G(q,T) := 0$. Let $c = 0$, and $R < |p|^{1/(p-1)}$. Then every solution $y(T) \in \mathcal{A}_K(0,R)$ of the operator $\sigma_q - A(q,T)$ is of the form $y(T) = \lambda \cdot \exp(T)$, with $\lambda \in K$. If $\delta_q(y) = 0$, then $y = 0$. Hence, the (σ_q, δ_q)-module defined by $A(q,T)$ and $G(q,T)$ has no (non-trivial) solutions in $\mathcal{A}_K(0,R)$.

To guarantee the existence of solutions we need a *compatibility condition* between σ_q and δ_q, which should be written explicitly in terms of matrices of σ_q^n and δ_1^n. This obstruction will not appear in the sequel of the paper since this condition is automatically satisfied by analytic σ-modules (cf. Lemma 5.19). This will follow from the fact that a solution $\alpha : \mathrm{M} \to \mathcal{A}_\Omega(c,R)$ is continuous (see the proof of Lemma 5.19). Observe that Lemma 5.19 below is not a formal consequence of the previous theory. Indeed, by Definition 3.2, the general (σ,δ)-algebra C has the discrete topology, hence the morphism $\alpha : \mathrm{M} \to \mathrm{C}$ defining the solution is not continuous in general.

Lemma 5.19. *Let $U \subseteq \mathcal{Q}(\mathcal{H}_K(X))$ be an open subset, and let* M *be an analytic (σ,δ)-module on U, representing the family of equations $\{\sigma_q(Y) = A(q,T)\cdot Y\}_{q\in U}$, with $A(q,T) \in GL_n(\mathcal{H}_K(X))$, for all $q \in U$. Let $Y_c(T) \in GL_n(\mathcal{A}_\Omega(c,R))$, $|q-1||c| < R \leqslant \rho_{c,X}$, be a simultaneous* solution *of every equation of this family. Then $Y_c(T)$ is also solution of the equation*

$$\delta_q(Y) = G(q,T)\cdot Y, \tag{5.6.2}$$

where $G(q,T) := q(d/dq)(A(q,T))$ (cf. (2.4.5)). Hence $Y_c(T)$ is a solution of the differential equation defined in § 2.4.1,

$$\delta_1(Y_c(T)) = G(1,T)\cdot Y_c(T), \tag{5.6.3}$$

where $G(1,T) = G(q,q^{-1}T)\cdot A(q,q^{-1}T)^{-1} \in M_n(\mathcal{H}_K(X))$ (cf. (2.2.7)).

Proof. In terms of modules, the columns of the matrix $Y_c(T)$ correspond to $\mathcal{H}_K(X)$-linear maps $\alpha : \mathrm{M} \to \mathcal{A}_\Omega(c,R)$, verifying $\sigma_q \circ \alpha = \alpha \circ \sigma_q^{\mathrm{M}}$, for all $q \in U$ (cf. § 3.2.1). We must show that such an α also commutes with δ_q. This follows immediately by the continuity of α. Indeed, the inclusion $\mathcal{H}_K(X) \to \mathcal{A}_\Omega(c,R)$ is continuous, and hence every $\mathcal{H}_K(X)$-linear map $\mathcal{H}_K(X)^n \to \mathcal{A}_\Omega(c,R)$ is continuous. □

5.7 Twisted Taylor formula for (σ,δ)-modules, and rough estimate of radius

Let X be a q-invariant affinoid. Let

$$D_q := \sigma_q \circ \frac{d}{dT} = \lim_{q'\to q} \frac{\sigma_{q'} - \sigma_q}{T(q'-q)} = \frac{1}{qT}\cdot \delta_q.$$

For all $q \in \mathcal{Q}(X)$ and all $f(T) \in \mathcal{H}_K(X)$, one has

$$\mathrm{D}_q(f\cdot g) = \sigma_q(f)\cdot \mathrm{D}_q(g) + \mathrm{D}_q(f)\cdot \sigma_q(g), \tag{5.7.1}$$

$$(d/dT \circ \sigma_q) = q\cdot(\sigma_q \circ d/dT), \tag{5.7.2}$$

$$\mathrm{D}_q^n = q^{n(n-1)/2}\cdot \sigma_q^n \circ (d/dT)^n, \tag{5.7.3}$$

$$\|\mathrm{D}_q^n(f(T))\|_X \leqslant \frac{|n!|}{r_X^n}\cdot \|f(T)\|_X \text{ (cf. Lemma 1.9)}. \tag{5.7.4}$$

Hence, for all $c \in K$,

$$\mathrm{D}_q^n(T-c)^i = \frac{i!}{(i-n)!}\cdot q^{n(n-1)/2}\cdot (q^nT-c)^{i-n}$$

if $n \leqslant i$, and $\mathrm{D}_q^n(T-c)^i = 0$ if $n > i$. This shows that if

$$f(T) := \sum_{i\geqslant 0} a_i \cdot \frac{(T-c)^i}{(i!)\cdot q^{i(i-1)/2}} \in \mathcal{A}_\Omega(c,R)$$

is a formal series, with $|q-1||c| < R \leqslant \rho_{c,X}$, then $a_n = \mathrm{D}_q^n(f)(c/q^n)$, and the usual Taylor formula can be written as

$$f(T) = \sum_{n\geqslant 0} \mathrm{D}_q^n(f)(c/q^n) \cdot \frac{(T-c)^n}{(n!)\cdot q^{n(n-1)/2}}. \tag{5.7.5}$$

The following proposition gives the analog of the classical rough estimate for differential and q-difference equations (cf. [Chr83, 4.1.2] and [DV04, 4.3]).

PROPOSITION 5.20. *Let $c \in X(\Omega)$. Assume that the system (5.6.1) has a Taylor solution $Y_c \in M_n(\mathcal{A}_\Omega(c, R_c))$, with $|q-1||c| < R_c \leqslant \rho_{c,X}$. For all q-invariant sub-affinoid $X' \subseteq X$, containing $\mathrm{D}^+(c, |q-1||c|)$, one has*

$$R_c \geqslant \frac{|p|^{1/(p-1)}}{\max(r_{X'}^{-1}\|A(q,T)\|_{X'},\ \|G(q,T)/qT\|_{X'})}. \tag{5.7.6}$$

In particular if X' is a disk $\mathrm{D}^+(c,\rho)$, with $|q-1||c| \leqslant \rho \leqslant \rho_{c,X}$, then

$$R_c \geqslant \frac{|p|^{1/(p-1)}\cdot\rho}{\max(|A(q,T)|_{(c,\rho)}, |G(q,T)|_{(c,\rho)}/\max(1,|c|/\rho))}. \tag{5.7.7}$$

Proof. The matrix $Y_c(T)$ satisfies $\sigma_q^n(Y_c(T)) = A_{[n]}(q,T)\cdot Y_c(T)$, and $D_q^n(Y_c(T)) = F_{[n]}(q,T)\cdot Y_c(T)$, where $F_{[0]} = \mathrm{Id} = A_{[0]}$, $A_{[1]} := A(q,T)$, $F_{[1]} := (1/qT)G(q,T)$, and

$$A_{[n]} := \sigma_q^{n-1}(A_{[1]})\cdots\sigma_q(A_{[1]})\cdot A_{[1]}, \tag{5.7.8}$$

$$F_{[n+1]} := \sigma_q(F_{[n]})\cdot F_{[1]} + D_q(F_{[n]})\cdot A_{[1]}. \tag{5.7.9}$$

Hence one has

$$Y_c(T) := \sum_{i\geqslant 0} F_{[n]}(c/q^n)\frac{(T-c)^n}{(n!)\cdot q^{n(n-1)/2}}, \tag{5.7.10}$$

which is a hybrid between the usual Taylor formula and the Taylor formula for q-difference equations. Inequalities (5.7.6) then follow from the inequality

$$|F_{[n]}(c/q^n)|_\Omega \leqslant \|F_{[n]}\|_{X'} \leqslant \max\left(\|F_{[1]}\|_{X'},\ \frac{1}{r_{X'}}\cdot\|A_{[1]}\|_{X'}\right)^n. \tag{5.7.11}$$

If $X' = \mathrm{D}^+(c,\rho)$, then the last term is equal to

$$\frac{1}{\rho^n}\cdot\max\left(\frac{|G(q,T)|_{(c,\rho)}}{\max(1,|c|/\rho)},\ |A(q,T)|_{(c,\rho)}\right)^n.$$

Indeed $r_{\mathrm{D}^+(c,\rho)} = \rho$, $F_{[1]} = (1/qT)G(q,T)$, and $|T|_{(c,\rho)} = |(T-c)+c|_{(c,\rho)} = \max(\rho,|c|)$, hence

$$|F_{[1]}|_{(c,\rho)} = \frac{1}{|q|\max(|c|,\rho)}\cdot|G(q,T)|_{(c,\rho)}$$

and $|q| = 1$. $\square$

6. Generic radius of convergence and solvability

DEFINITION 6.1 (Generic radius of convergence). Let $q \in \mathcal{Q}(X)$ (respectively $q \in \mathcal{Q}(X) - \boldsymbol{\mu}(\mathcal{Q})$), let $c \in X(K^{\mathrm{alg}})$, and let $\mathrm{D}^+(c,\rho)$, $|q-1||c| < \rho \leqslant \rho_{c,X}$, be a q-invariant disk. Let M be the (σ_q,δ_q)-module (respectively σ_q-module) defined by the system (5.6.1) (respectively (5.3.8)). Let $R_{t_{c,\rho}} := Ray(Y(x,y), t_{c,\rho}) = Ray(Y(x,y), |\cdot|_{(c,\rho)})$ be the radius of convergence[5] of $Y_{A(q,T)}(T, t_{c,\rho})$.

[5] In the case of the q-difference equation (5.3.8), the radius $R_{t_{c,\rho}}$ is given by definition (5.3.10). In the case of the system (5.6.1), the radius $R_{t_{c,\rho}}$ is given indifferently by definition (5.3.2) or by definition (5.3.10). Indeed under our assumptions these two definitions are equal since $Y_{A(q,T)}(x,y) = Y_{G(1,T)}(x,y)$. However observe that the definition (5.3.10) exists only if $q \in \mathcal{Q} - \boldsymbol{\mu}(\mathcal{Q})$, while definition (5.3.2) preserves its meaning on the root of unity.

Assume that[6]

$$|q-1||t_{c,\rho}| < R_{t_{c,\rho}}. \tag{6.0.1}$$

We define the (c,ρ)-*generic radius of convergence of* M to be the real number

$$Ray(\mathrm{M}, |\cdot|_{c,\rho}) := \min(R_{t_{c,\rho}}, \rho_{c,X}) > |q-1||c|. \tag{6.0.2}$$

6.0.1 The assumption (6.0.1) ensures that the disk of convergence of $Y(x,y)$ at $y = t_{c,\rho}$ is q-invariant. The bound $Ray(\mathrm{M}, |\cdot|_{c,\rho}) \leqslant \rho_{c,X}$ ensures that $Y(x,y)$ is invertible in the disk $\mathrm{D}^-(t_{c,\rho}, R)$, for all $0 < R \leqslant Ray(\mathrm{M}, |\cdot|_{c,\rho})$ (cf. Lemma 5.15). We recall that $|t_{c,\rho}| = \min(|c|, \rho)$, and that $\|\cdot\|_{\mathrm{D}^+(c,\rho)} = \max_{y_0 \in \mathrm{D}^+_{K^{\mathrm{alg}}}(c,\rho)} |\cdot|_{y_0}$. Hence, by the transfer principle (cf. § 5.4), one has

$$R_{t_{c,\rho}} := Ray(Y(x,y), t_{c,\rho}) = Ray(Y(x,y), \mathrm{D}^+(c,\rho)) = \min_{y_0 \in \mathrm{D}^+_{K^{\mathrm{alg}}}(c,\rho)} Ray(Y(x,y), y_0). \tag{6.0.3}$$

The number $Ray(\mathrm{M}, |\cdot|_{(c,\rho)})$ is invariant under change of basis in M, while the number $R_{t_{c,\rho}} = Ray(Y(x,y), |\cdot|_{(c,\rho)})$ depends on the choice of basis. Observe that $Ray(\mathrm{M}, |\cdot|_{(c,\rho)})$ depends on the affinoid X, and on the semi-norm $|\cdot|_{(c,\rho)}$ defined by $t_{c,\rho}$, but not on the particular choice of $t_{c,\rho}$ (cf. § 1.4.1).

Definition 6.2 (Solvability). Let M be a σ_q-module (respectively a (σ_q, δ_q)-module) on $\mathcal{H}_K(X)$. We will say that M is *solvable* at $t_{c,\rho}$ if

$$Ray(\mathrm{M}, |\cdot|_{(c,\rho)}) = \rho_{c,X}. \tag{6.0.4}$$

6.0.2 *Continuity and log-concavity of the radius.* Notice that every point $|\cdot|_*$ in the Berkovich space associated to X is of the form $|\cdot|_{c,\rho}$, for a suitable $\rho \geqslant 0$, and for a point c in $X(L)$, where $(L, |\cdot|)/(K, |\cdot|)$ is a sufficiently large extension of complete valued fields. One may verify that $|\cdot|_{(c,\rho)} \mapsto Ray(\mathrm{M}, |\cdot|_{(c,\rho)})$ is a well defined function on the Berkovich space (i.e. the radius does not depend on the chosen c, but only on $|\cdot|_*$). In a recent preprint [BDV07] it has been proved that the function $|\cdot|_* \mapsto Ray(\mathrm{M}, |\cdot|_*)$ is continuous on the Berkovich space. We refer to [BDV07] for a very inspiring treatment to this subject.

We notice that this generalizes a previous statement [CD94] proving, for all $c \in X(L)$, the continuity of the function $\rho \mapsto Ray(\mathrm{M}, |\cdot|_{c,\rho})$.

Let now $(L, |\cdot|)/(K, |\cdot|)$ be any extension of complete valued fields. Let $c \in X(L)$. The function $\rho \mapsto Ray(\mathrm{M}, |\cdot|_{c,\rho})$ defined on $[0, \rho_{c,X}]$ is log-concave (cf. Definition 1.4), and it can be proved that it is piecewise log-affine. This follows essentially by the definition of the radius (cf. (5.3.10)), and by Lemma 1.6.

6.1 Solvability over an annulus and over the Robba ring

Let $\mathrm{B} := \mathcal{A}_K(I)$, with $I =]r_1, r_2[$, and let M be a σ_q-module (respectively a (σ_q, δ_q)-module) on $\mathcal{A}_K(I)$. For all $c \in K$, $|c| \in I$, one has $t_{c,|c|} = t_{0,|c|}$. For all affinoid $X \subseteq \mathcal{C}(I)$ containing the disk $\mathrm{D}^-(c, |c|)$ one has $\rho_{c,X} = |c|$. Then the norm $|\cdot|_{c,|c|} : \mathcal{A}_K(I) \longrightarrow \mathbb{R}_{\geqslant}$ and the generic radius $Ray(\mathrm{M}, |\cdot|_{(c,|c|)})$ do not depend on the choice of c or the affinoid X, but only on $|c|$. Hence, for all $\rho \in I$, we choose an arbitrary $c \in \Omega$, with $|c| = \rho \in I$, and we set

$$t_\rho := t_{c,\rho} \quad \text{and} \quad Ray(\mathrm{M}, \rho) := Ray(\mathrm{M}, |\cdot|_{(c,\rho)}). \tag{6.1.1}$$

To define the radius we need the assumption $|q-1||t_\rho| < \rho_{t_\rho,X} = \rho$ (cf. Definition 6.1).

[6] Observe that $\rho_{c,X} = \rho_{t_{c,\rho},X}$ indeed $\mathrm{D}^+(c,r) = \mathrm{D}^+(t_{c,\rho}, r)$, for all $r \geqslant \rho$.

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Since $|t_\rho| = \rho$, this assumption is equivalent to

$$|q-1| < 1. \tag{6.1.2}$$

DEFINITION 6.3 (Solvability at ρ). Let $q \in \mathcal{Q}_1 - \boldsymbol{\mu}(\mathcal{Q}_1)$ (cf. (2.0.2)). Let M be a σ_q-module on $\mathcal{A}_K(I)$. We will say that M is *solvable* at $\rho \in I$ if

$$Ray(\mathrm{M}, \rho) = \rho. \tag{6.1.3}$$

6.1.1 *Solvability over $\mathcal{R}_K$ or $\mathcal{H}_K^\dagger$.* Let $q \in \mathcal{Q}_1 - \boldsymbol{\mu}(\mathcal{Q}_1)$. Let M be a σ_q-module over $\mathcal{R}_K$. By definition M comes, by scalar extension, from a module $\mathrm{M}_{\varepsilon_1}$ defined on an annulus $\mathcal{C}(]1-\varepsilon_1, 1[)$. If $\varepsilon_2 > 0$, and if $\mathrm{M}_{\varepsilon_2}$ is another module on $\mathcal{C}(]1-\varepsilon_2, 1[)$ satisfying $\mathrm{M}_{\varepsilon_2} \otimes_{\mathcal{A}_K(]1-\varepsilon_2,1[)} \mathcal{R}_K \xrightarrow{\sim} \mathrm{M}$, then there exists an $\varepsilon_3 \leqslant \min(\varepsilon_1, \varepsilon_2)$ such that

$$\mathrm{M}_{\varepsilon_1} \otimes \mathcal{A}_K(]1-\varepsilon_3, 1[) \xrightarrow{\sim} \mathrm{M}_{\varepsilon_2} \otimes \mathcal{A}_K(]1-\varepsilon_3, 1[). \tag{6.1.4}$$

Hence the limit $\lim_{\rho \to 1^-} Ray(\mathrm{M}_\varepsilon, \rho)$ is independent of the choice of the module M_ε.

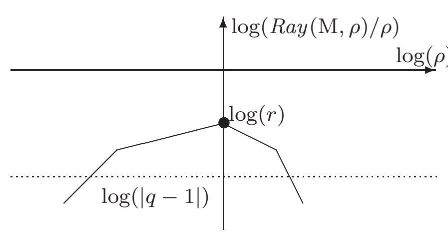
DEFINITION 6.4. Let $q \in \mathcal{Q}_1 - \boldsymbol{\mu}(\mathcal{Q}_1)$, and let $|q-1| < r \leqslant 1$. We define

$$\sigma_q\text{-Mod}(\mathcal{H}_K^\dagger)^{[r]}, \tag{6.1.5}$$

as the full subcategory of σ_q-Mod$(\mathcal{H}_K^\dagger)$ whose objects satisfy

$$Ray(\mathrm{M}, 1) \geqslant r \quad (r > |q-1|), \tag{6.1.6}$$

as illustrated below in the log-graphic of the function $\log(\rho) \mapsto \log(Ray(\mathrm{M}, \rho)/\rho)$ (cf. Definition 1.4).

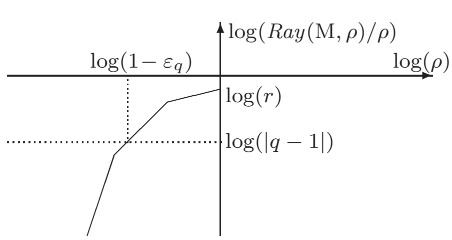


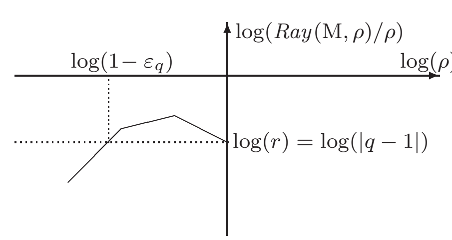
Objects in σ_q-Mod$(\mathcal{H}_K^\dagger)^{[1]}$ will be called *solvable*.

DEFINITION 6.5. Let $q \in \mathcal{Q}_1 - \boldsymbol{\mu}(\mathcal{Q}_1)$, and let $|q-1| \leqslant r \leqslant 1$. We define

$$\sigma_q\text{-Mod}(\mathcal{R}_K)^{[r]} \tag{6.1.7}$$

as the full subcategory of σ_q-Mod$(\mathcal{R}_K)$ formed by objects M satisfying $\lim_{\rho \to 1^-} Ray(\mathrm{M}, \rho) \geqslant r$, and there exists $\varepsilon_q > 0$ such that $Ray(\mathrm{M}, \rho) > |q-1|$, for all $\rho \in]1-\varepsilon_q, 1[$. There are two possible cases, $r > |q-1|$ and $r = |q-1|$, as illustrated in the following pictures.





Objects in σ_q-Mod$(\mathcal{R}_K)^{[1]}$ will be called *solvable*.

Remark 6.6. Notice that in Definition 6.4 the existence of $\varepsilon_q > 0$ such that $Ray(\mathrm{M}, \rho) > |q-1|$, for all $\rho \in]1-\varepsilon_q, 1+\varepsilon_q[$ is automatically verified since one assumes $r > |q-1|$.

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6.1.2 *Analogous definitions for (σ_q, δ_q)-modules.* In the case of (σ_q, δ_q)-modules, the generic radius of convergence is defined even if q is a root of unity. We then give analogous definitions of (σ_q, δ_q)-$\mathrm{Mod}(\mathrm{B})^{[r]}$, for $\mathrm{B} := \mathcal{R}_K$ or $\mathrm{B} := \mathcal{H}_K^{\dagger}$, without any restrictions on q.

6.2 Generic radius for discrete and analytic objects over $\mathcal{R}_K$ and $\mathcal{H}_K^{\dagger}$

In this section $\mathrm{B} = \mathcal{R}_K$ or $\mathrm{B} = \mathcal{H}_K^{\dagger}$.

Definition 6.7. For all $\varepsilon > 0$ let

$$I_\varepsilon := \begin{cases}]1-\varepsilon, 1[, & \text{if } \mathrm{B} = \mathcal{R}_K, \\ ]1-\varepsilon, 1+\varepsilon[, & \text{if } \mathrm{B} = \mathcal{H}_K^{\dagger}. \end{cases} \tag{6.2.1}$$

Definition 6.8. For all subsets $S \subseteq \mathrm{D}^-(1,1) = \mathcal{Q}_1$, for all $0 < \tau < 1$, we set

$$S_\tau := S \cap \mathrm{D}^-(1,\tau). \tag{6.2.2}$$

Definition 6.9. Let $0 < r \leqslant 1$. Let $S \subseteq \mathrm{D}^-(1,1)$, $S^\circ \neq \emptyset$. We denote by

$$\sigma\text{-}\mathrm{Mod}(\mathrm{B})_S^{[r]} \tag{6.2.3}$$

the full subcategory of σ-$\mathrm{Mod}(\mathrm{B})_S$ whose objects M have the following properties.

(i) The restriction of M to every $q \in S$ belongs to σ_q-$\mathrm{Mod}(\mathrm{B})^{[r]}$

(ii) For all τ such that $0 < \tau < r$, there exists $\varepsilon_\tau > 0$ such that the restriction $\mathrm{Res}^{\langle S\rangle}_{\langle S_\tau\rangle}(\mathrm{M})$ comes, by scalar extension, from an object

$$\mathrm{M}_{\varepsilon_\tau} \in \sigma\text{-}\mathrm{Mod}(\mathcal{A}_K(I_{\varepsilon_\tau}))^{\mathrm{disc}}_{S_\tau} \tag{6.2.4}$$

such that, for all $\rho \in I_{\varepsilon_\tau}$, and for all $q, q' \in S_\tau$, one has (cf. (5.3.9))

$$Y_{A(q,T)}(T, t_\rho) = Y_{A(q',T)}(T, t_\rho). \tag{6.2.5}$$

Objects in σ_q-$\mathrm{Mod}(\mathrm{B})_S^{[1]}$ will be called *solvable*.

Example 6.10. This example justifies the condition (i) given in the preceding definition. Let $r := \omega := |p|^{1/(p-1)}$, and let $S = \mathrm{D}^-(1,\omega)$. Let M be the discrete σ-module over the Robba ring defined by the family of equations $\{\sigma_q - A(q,T)\}_{q\in S}$, where $A(q,T) := \exp((q^{-1}-1)T^{-1})$. Then $Y(x,y) := \exp(x^{-1} - y^{-1})$ is the simultaneous solution of every equation of this family. Observe that $A(q,T) \in \mathcal{R}_K$ if and only if $|q^{-1} - 1| < \omega$, but if $|q-1|$ tends to ω^-, then *the matrices $A(q,T)$ do not all belong to the same annulus.* Indeed $A(q,T) \in \mathcal{A}_K(I_\varepsilon)$ if and only if $|q^{-1}-1| < \omega(1-\varepsilon)$.

Remark 6.11. Condition (i) implicitly implies that $S \subseteq \mathrm{D}^-(1,r)$ if $\mathrm{B} = \mathcal{H}_K^{\dagger}$ (cf. Definition 6.4), and $S \subseteq \mathrm{D}^+(1,r)$ if $\mathrm{B} = \mathcal{R}_K$ (cf Definition 6.5).

6.2.1 *Analogous definitions for (σ_q, δ_q)-modules.* One defines analogously (σ,δ)-$\mathrm{Mod}(\mathrm{B})_S^{[r]}$, *but without restrictions on $S \subseteq \mathrm{D}^-(1,r)$*, as the subcategory of (σ,δ)-$\mathrm{Mod}(\mathrm{B})_S$, whose objects verify conditions (i) and (ii), in which equation (6.2.5) is replaced by (cf. definitions (5.3.1) and (5.3.9))

$$Y_{G(1,T)}(T, t_\rho) = Y_{A(q,T)}(T, t_\rho), \tag{6.2.6}$$

for all $\rho \in I_{\varepsilon_\tau}$, and all $q \in S_\tau$.

7. The propagation theorem

7.1 Taylor admissible modules

Definition 7.1 (Taylor admissible discrete modules on S). Let $X := \mathrm{D}^+(c_0, R_0) - \bigcup_{i=1}^n \mathrm{D}^-(c_i, R_i)$ be an affinoid, and let $S \subseteq \mathcal{Q}_1(X)$ be a subset with $S^\circ \neq \emptyset$ (cf. (2.0.3)). Let $(\mathrm{M}, \sigma^{\mathrm{M}})$ be a discrete

σ-module defined by the family of equations

$$\{\sigma_q - A(q,T)\}_{q\in S}, \quad A(q,T) \in GL_n(\mathcal{H}_K(X)), \quad \forall q \in S. \tag{7.1.1}$$

We will say that $(\mathrm{M}, \sigma^{\mathrm{M}})$ is *Taylor admissible on X, with generic radius greater than r*, if:

(1) one has $S \subseteq \mathrm{D}^-(1, r/\max(|c_0|, R_0))$;

(2) there exists a matrix $Y(x,y)$, convergent in $\mathcal{U}_R$ (cf. (5.3.3)), with $R \geqslant r$ satisfying, for all $q \in S$, the condition (5.5.1), that is

$$r \leqslant R \leqslant r_X; \tag{7.1.2}$$

(3) $Y(x,y)$ is a simultaneous solution of every equation of the family (7.1.1).

The full subcategory of $\sigma\text{-Mod}(\mathcal{H}_K(X))_S^{\mathrm{disc}}$ whose objects are Taylor admissible, with generic radius greater than r, will be denoted by

$$\sigma\text{-Mod}(\mathcal{H}_K(X))_S^{[r]}. \tag{7.1.3}$$

Moreover we set

$$\sigma\text{-Mod}(\mathcal{H}_K(X))_S^{\mathrm{adm}} := \bigcup_r \sigma\text{-Mod}(\mathcal{H}_K(X))_S^{[r]}, \tag{7.1.4}$$

where $r \leqslant r_X$ runs in the set of real numbers such that $S \subseteq \mathrm{D}^-(1, r/\max(|c_0|, R_0))$. We define analogously the categories $(\sigma,\delta)\text{-Mod}(\mathcal{H}_K(X))_S^{[r]}$ and $(\sigma,\delta)\text{-Mod}(\mathcal{H}_K(X))_S^{\mathrm{adm}}$ of *admissible* (σ,δ)-*modules on* S. Namely the condition $S^\circ \neq \emptyset$ is suppressed, and if $(\mathrm{M}, \sigma^{\mathrm{M}}, \delta_1^{\mathrm{M}})$ is a discrete (σ,δ)-module on S defined by a system of equations (cf. (3.2.4)), then the Taylor solution $Y_{G(1,T)}(x,y)$ (cf. (5.3.1)) of the differential equation defined by δ_1^{M} satisfies (7.1.2), and moreover is simultaneously a solution of every equation defined by σ_q^{M}, for all $q \in S$.

7.1.1 *Taylor admissibility over $\mathcal{H}_K^\dagger(X)$.* We define

$$\sigma\text{-Mod}(\mathcal{H}_K^\dagger(X))_S^{[r]} \quad (\text{respectively } (\sigma,\delta)\text{-Mod}(\mathcal{H}_K^\dagger(X))_S^{[r]}) \tag{7.1.5}$$

as the full subcategory of $\sigma\text{-Mod}(\mathcal{H}_K^\dagger(X))_S$ (respectively $(\sigma,\delta)\text{-Mod}(\mathcal{H}_K^\dagger(X))_S$) formed by objects whose restriction belongs to $\sigma\text{-Mod}(\mathcal{H}_K(X))_S^{[r]}$ (respectively $(\sigma,\delta)\text{-Mod}(\mathcal{H}_K(X))_S^{[r]}$).

Remark 7.2. If $X = \{|T| = 1\}$, $\mathcal{H}_K^\dagger(X) = \mathcal{H}_K^\dagger$ (cf. (1.2.4)), this definition is equivalent to Definition 6.9.

7.1.2 *Taylor admissibility over $\mathcal{R}_K$.* We preserve the notation of § 6.2.

Definition 7.3. We will say that an object is *Taylor admissible* over an annulus $\mathcal{C}(I)$ if its restriction to every sub-annulus $\mathcal{C}(J)$, with J compact, $J \subseteq I$, is Taylor admissible (cf. Definition 7.1).

One defines Taylor admissibility over $\mathcal{R}_K$ by reducing to the case of modules over a single annulus $\mathcal{C}(I_\varepsilon)$, for some $\varepsilon > 0$ sufficiently close to 0. One finds in this way exactly Definition 6.9.

Definition 7.4. Let $S \subseteq \mathrm{D}^-(1,1)$, with $S^\circ \neq \emptyset$. Let $\tau_S := \sup_{q\in S} |q-1|$. We set

$$\sigma\text{-Mod}(\mathcal{R}_K)_S^{\mathrm{adm}} := \sigma\text{-Mod}(\mathcal{R}_K)_S^{[\tau_S]}. \tag{7.1.6}$$

We give the same definition for (σ,δ)-modules, without assuming that $S^\circ \neq \emptyset$: $(\sigma,\delta)\text{-Mod}(\mathcal{R}_K)_S^{\mathrm{adm}} := (\sigma,\delta)\text{-Mod}(\mathcal{R}_K)_S^{[\tau_S]}$.

7.2 Propagation theorem

Remark 7.5. We preserve the notation of Definition 7.1. If M is Taylor admissible on X, then, in particular, M is trivialized by $\mathcal{A}_K(c,R)$, for all $c \in X(K)$. Hence we can apply C-deformation and

C-confluence to M, with $\mathrm{C} = \mathcal{A}_K(c, R)$ (cf. § 4.2). It will follow from the proof of Theorem 7.7 that this confluence does not depend on the chosen point $c \in X(K)$.

Theorem 7.6 (Propagation theorem, first form). *Let X be an* affinoid. *Then, if $q \in \mathcal{Q}_1(X) - \boldsymbol{\mu}(\mathcal{Q}_1(X))$, the natural restriction functor*

$$\bigcup_U \mathrm{Res}_q^U : \bigcup_U \sigma\text{-Mod}(\mathcal{H}_K(X))_U^{\mathrm{adm}} \longrightarrow \sigma_q\text{-Mod}(\mathcal{H}_K(X))^{\mathrm{adm}} \tag{7.2.1}$$

is an equivalence, where U runs over the set of all open neighborhoods of q. The analogous fact is true for (σ, δ)-modules without supposing that $q \notin \boldsymbol{\mu}(\mathcal{Q})$.

Proof. By Lemma 4.3, $\bigcup_U \mathrm{Res}_{\{q\}}^U$ is fully faithful. Indeed for all modules M, N over U, by admissibility, there exists a number R, with $|q-1|\max(|c_0|, R_0) < R \leqslant r_X$, such that, for all $c \in X(K)$, the algebra $\mathrm{C} := \mathcal{A}_K(c, R)$ trivializes both M and N. The essential surjectivity of $\bigcup_U \mathrm{Res}_{\{q\}}^U$ will follow from Theorem 7.7 below. □

Theorem 7.7 (Propagation theorem, second form). *Let $X = \mathrm{D}^+(c_0, R_0) - \bigcup_{i=1}^n \mathrm{D}^-(c_i, R_i)$. Let $q \in \mathcal{Q}_1(X) - \boldsymbol{\mu}(\mathcal{Q}_1(X))$. Let*

$$Y(q \cdot T) = A(T) \cdot Y(T), \quad A(T) \in GL_n(\mathcal{H}_K(X)) \tag{7.2.2}$$

be a Taylor admissible *q-difference equation (cf. Definition 7.1). Then there exists a matrix $A(Q, T)$ uniquely determined by the following properties:*

(i) *$A(Q, T)$ is analytic and invertible in the domain*

$$\mathrm{D}^-\left(1, \frac{R}{\max(|c_0|, R_0)}\right) \times X \subset \mathbb{A}_K^2; \tag{7.2.3}$$

(ii) *the matrix $A(Q, T)$ specialized at (q, T) is equal to $A(T)$;*

(iii) *for all $q' \in \mathrm{D}^-(1, R/\max(|c_0|, R_0))$, the Taylor solution matrix $Y_A(x, y)$ of (7.2.2) (cf. (5.3.9)) simultaneously satisfies*

$$Y_A(q' \cdot T, y) = A(q', T) \cdot Y_A(T, y). \tag{7.2.4}$$

Moreover the matrix $A(Q, T)$ is independent of the choice of solution $Y_A(x, y)$.

Proof. By (7.2.4), the matrix $A(Q, T)$ must be equal to

$$A(Q, T) = Y_A(Q \cdot T, y) \cdot Y_A(T, y)^{-1} = Y_A(Q \cdot T, y) \cdot Y_A(y, T) = Y_A(Q \cdot T, T). \tag{7.2.5}$$

This makes sense since $Y_{A(q,T)}(x, y)$ is invertible in its domain of convergence (cf. Lemma 5.16). Hence $A(Q, T)$ converges in the domain of convergence of $Y_A(QT, T)$ and is invertible in that domain, since $Y_A(x, y)$ is. By admissibility, there exists $|q-1|\max(|c_0|, R_0) < R \leqslant r_X$ such that $Y_A(x, y)$ converges for all $(x, y) \in \mathcal{U}_R$, i.e. for all (x, y) such that $|x - y| < R$ (cf. (5.3.3)). Then $Y_A(QT, T)$ converges for $|Q-1||T| < R$. Since $|T| \leqslant \sup_{c \in A}|c| = \max(|c_0|, R_0)$, it follows that $Y(QT, T)$ converges for $|Q - 1| < R/\max(|c_0|, R_0)$. □

Remark 7.8. By the propagation theorem, every object of

$$\sigma\text{-Mod}(\mathcal{H}_K(X))_U^{\mathrm{adm}} \quad \text{and of} \quad (\sigma, \delta)\text{-Mod}(\mathcal{H}_K(X))_U^{\mathrm{adm}}$$

is automatically analytic.

Corollary 7.9. *Let $\max(|c_0|, R_0) < r \leqslant r_X$, and let $S \subseteq \mathrm{D}^-(1, r/\max(|c_0|, R_0))$, such that $S^\circ \neq \emptyset$. For all $q \in S^\circ$ one has the following diagram in which all functors are equivalences*

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by § 4.2.2

$$\begin{array}{ccc} \sigma\text{-Mod}(\mathcal{H}_K(X))^{[r]}_S & \overset{(2.4.3)}{=\!=\!=} & (\sigma,\delta)\text{-Mod}(\mathcal{H}_K(X))^{[r]}_S \\ {\scriptstyle \mathrm{Res}^S_{\{q\}}}\downarrow\wr & \odot & \wr\downarrow{\scriptstyle \mathrm{Res}^S_{\{q\}}} \\ \sigma_q\text{-Mod}(\mathcal{H}_K(X))^{[r]} & \underset{\text{Forget }\delta_q}{\overset{\sim}{\longleftarrow}} & (\sigma_q,\delta_q)\text{-Mod}(\mathcal{H}_K(X))^{[r]} \end{array} \tag{7.2.6}$$

By considering the union for all r *(cf.* (7.1.4)*) one has the following statement. If* $\tau_q := |q-1|\max(|c_0|, R_0)$*, one then has the equivalences given in the diagram below.*

$$\begin{array}{ccc} \bigcup_{r>\tau_q} \sigma\text{-Mod}(\mathcal{H}_K(X))^{\mathrm{adm}}_{\mathrm{D}^-(1,r)} & \overset{(2.4.3)}{=\!=\!=} & \bigcup_{r>\tau_q} (\sigma,\delta)\text{-Mod}(\mathcal{H}_K(X))^{\mathrm{adm}}_{\mathrm{D}^-(1,r)} \\ {\scriptstyle \bigcup_{r>\tau_q}\mathrm{Res}^{\mathrm{D}^-(1,r)}_{\{q\}}}\downarrow\wr & \odot & \wr\downarrow{\scriptstyle \bigcup_{r>\tau_q}\mathrm{Res}^{\mathrm{D}^-(1,r)}_{\{q\}}} \\ \sigma_q\text{-Mod}(\mathcal{H}_K(X))^{\mathrm{adm}} & \underset{\text{Forget }\delta_q}{\overset{\sim}{\longleftarrow}} & (\sigma_q,\delta_q)\text{-Mod}(\mathcal{H}_K(X))^{\mathrm{adm}} \end{array} \tag{7.2.7}$$

In particular, if $q, q' \in \mathrm{D}^-(1,1) - \boldsymbol{\mu}_{p^\infty}$ *verify* $\max(|q-1|, |q'-1|)\max(|c_0|, R_0) < r$*, then, by the formalism introduced in* § 4.2*, if* $\mathrm{D} := \mathrm{D}^-(1, r/\max(|c_0|, R_0))$*, one has an equivalence*

$$\mathrm{Res}^{\mathrm{D}}_{q'} \circ (\mathrm{Res}^{\mathrm{D}}_q)^{-1} : \sigma_q\text{-Mod}(\mathcal{H}_K(X))^{[r]} \xrightarrow{\ \sim\ } \sigma_{q'}\text{-Mod}(\mathcal{H}_K(X))^{[r]}. \tag{7.2.8}$$

The same statement holds for (σ,δ)*-modules without assuming that* $q, q' \notin \boldsymbol{\mu}_{p^\infty}$*.*

Definition 7.10. In the notation of Corollary 7.9 (cf. (7.2.8)), if $q, q' \notin \boldsymbol{\mu}_{p^\infty}$, we set

$$\mathrm{Def}^{\mathrm{Tay}}_{q,q'} := \mathrm{Res}^{\mathrm{D}}_{q'} \circ (\mathrm{Res}^{\mathrm{D}}_q)^{-1} : \sigma_q\text{-Mod}(\mathcal{H}_K(X))^{[r]} \xrightarrow{\sim} \sigma_{q'}\text{-Mod}(\mathcal{H}_K(X))^{[r]}. \tag{7.2.9}$$

We denote again by $\mathrm{Def}^{\mathrm{Tay}}_{q,q'}$, without assuming that $q, q' \in \boldsymbol{\mu}_{p^\infty}$, the analogous functor for (σ,δ)-modules. Moreover, if $q \notin \mathcal{Q}(X) - \boldsymbol{\mu}_{p^\infty}$, then we set

$$\mathrm{Conf}^{\mathrm{Tay}}_q := \mathrm{Def}^{\mathrm{Tay}}_{q,1} \circ (\text{Forget } \delta_q)^{-1} : \sigma_q\text{-Mod}(\mathcal{H}_K(X))^{[r]} \xrightarrow{\sim} \delta_1\text{-Mod}(\mathcal{H}_K(X))^{[r]}. \tag{7.2.10}$$

By Remark 7.5, the functor $\mathrm{Conf}^{\mathrm{Tay}}_q : (\sigma_q,\delta_q)\text{-Mod}(\mathcal{H}_K(X))^{[r]} \xrightarrow{\sim} \sigma_q\text{-Mod}(\mathcal{H}_K(X))^{[r]}$ of diagram (7.2.6) coincides with $\mathrm{Conf}^{\mathrm{C}}_q$ (cf. Definition 4.6), where C is equal to $\mathcal{A}_K(c,r)$, where r is as in Corollary 7.9, and where $c \in X(K)$ is arbitrarily chosen.

7.2.1 *Root of unity.* If $q \in \boldsymbol{\mu}_{p^\infty}$, then the categories $\sigma_q\text{-Mod}(\mathcal{H}_K(X))^{[r]}_S$ and $\sigma_q\text{-Mod}(\mathcal{H}_K(X))^{\mathrm{adm}}_S$ are not defined. In this case we cannot expect any equivalence between $(\sigma_q,\delta_q)\text{-Mod}(\mathcal{H}_K(X))^{\mathrm{adm}}$ with a full subcategory of $\sigma_q\text{-Mod}(\mathcal{H}_K(X))$ because the first category is K-linear and the second is not. In this case we will see in Proposition 8.6 that the functor 'Forget δ_q' is not very interesting since it sends every (σ_q,δ_q)-module with Frobenius structure into the trivial σ_q-module (i.e. a direct sum of the copies of the unit object).

7.2.2 Starting from a Taylor *admissible* σ_q-module M over B, one can *compute* the differential equation $\mathrm{Conf}^{\mathrm{Tay}}_q(\mathrm{M}) \in \delta_1\text{-Mod}(\mathrm{B})$ by the relation

$$G(1,T) = \lim_{q\to 1} \frac{A(q,T) - \mathrm{Id}}{q-1} = \lim_{n\to+\infty} \frac{A(q^{p^n},T) - \mathrm{Id}}{q^{p^n}-1}, \tag{7.2.11}$$

where $A(q^{p^n},T) = A(q, q^{p^n-1}T)A(q, q^{p^n-2}T)\cdots A(q,T)$. The propagation theorem provides the convergence of this limit in $M_n(\mathrm{B})$. The reader may have the feeling that this limit should be easy to compute, but (without introducing the Taylor solution) the convergence of this limit and its explicit computation are *highly non-trivial facts*. It is surprising to see that the admissibility condition, which is not a strong assumption, actually implies such a deep fact.

Remark 7.11. It should be possible to generalize the main theorem to other kinds of operators, different from σ_q. In other words it should be possible to 'deform' differential equations into 'σ-difference equations', where σ in an automorphism different from σ_q, but sufficiently close to the identity. We will describe this phenomenon in a forthcoming work [Pul08].

7.3 Extending the confluence functor to the case $|q-1| = |q| = 1$

Let $q \in \mathcal{Q}(X) - \boldsymbol{\mu}(\mathcal{Q}(X))$ be such that $q^{k_0} \in \mathcal{Q}_1(X)$, for some $k_0 \geqslant 1$.[7] By composing with the evident functor

$$\sigma_q\text{-Mod}(\mathcal{H}_K(X)) \longrightarrow \sigma_{q^{k_0}}\text{-Mod}(\mathcal{H}_K(X)), \tag{7.3.1}$$

one defines k_0-*Taylor admissible objects* of σ_q-Mod$(\mathcal{H}_K(X))$ as objects whose image is Taylor admissible in $\sigma_{q^{k_0}}$-Mod$(\mathcal{H}_K(X))$. Since the sequence $\{q^{k_0 p^n}\}_{n \geqslant 0}$ tends to 1, then, for k_0 sufficiently large, q^{k_0} satisfies the condition of § 5.2, in order that $d_{q^{k^0}}$ verifies equality (5.2.1). We obtain then a confluence functor:

$$\sigma_q\text{-Mod}(\mathcal{H}_K(X))^{k_0\text{-adm}} \longrightarrow \delta_1\text{-Mod}(\mathcal{H}_K(X))^{\text{adm}}. \tag{7.3.2}$$

The converse of this fact (i.e. the deformation of a differential equation into a q-difference equation with $|q| = 1$ and $|q-1|$ large) remains an open problem.

Remark 7.12. Notice that there exist equations in σ_q-Mod$(\mathcal{H}_K(X))$ which are not k_0-Taylor admissible, for all $k_0 \geqslant 1$. For example consider the rank one equation $\sigma_q - a$, with $a \in K$, $|a| > 1$. Suppose also that $|q-1| < |p|^{1/(p-1)}$, in order that $\liminf_n |[n]_q^!|^{1/n} = |p|^{1/(p-1)}$. Then the radius is small and one can compute it explicitly by applying [DV04, Proposition 4.6]. One has

$$Ray((\mathrm{M}, \sigma_q^{\mathrm{M}}), \rho) = |a|^{-1}|p|^{1/(p-1)}|q-1|\rho < |q-1|\rho$$

and

$$Ray((\mathrm{M}, \sigma_{q^{k_0}}^{\mathrm{M}}), \rho) = |a|^{-k_0}|p|^{1/(p-1)}|q^{k_0}-1|\rho < |q^{k_0}-1|\rho.$$

7.4 Propagation theorem over $\mathcal{H}_K^\dagger$ and $\mathcal{R}_K$

The propagation theorem is true over every base ring B appearing in this paper, up to a correct definition for the notion of 'Taylor admissible'. We state here the results for $\mathcal{H}_K^\dagger$ and $\mathcal{R}_K$.

Proposition 7.13. *Let again* $\mathrm{B} := \mathcal{H}_K^\dagger$, *or* $\mathrm{B} := \mathcal{R}_K$, *let* $0 < r \leqslant 1$, *and let* $S \subseteq \mathrm{D}^-(1,r)$ *be a subset, with* $S^\circ \neq \emptyset$. *Let* $\mathrm{M} \in \sigma\text{-Mod}(\mathrm{B})_S^{[r]}$ *(i.e. in particular* M *is admissible). Then* M *is the restriction to* S *of an* analytically C-constant *module over all the disk* $\mathrm{D}^-(1,r)$. *Moreover, the restriction functor is an equivalence:*

$$\sigma\text{-Mod}(\mathrm{B})_{\mathrm{D}^-(1,r)}^{[r]} \xrightarrow[\sim]{\mathrm{Res}_S^{\mathrm{D}^-(1,r)}} \sigma\text{-Mod}(\mathrm{B})_S^{[r]}. \tag{7.4.1}$$

In particular solvable modules extend to the whole disk $\mathrm{D}^-(1,1)$. *The analogous assertion holds for* (σ,δ)*-modules, without supposing that* $S^\circ \neq \emptyset$:

$$(\sigma,\delta)\text{-Mod}(\mathrm{B})_{\mathrm{D}^-(1,r)}^{[r]} \xrightarrow[\sim]{\mathrm{Res}_S^{\mathrm{D}^-(1,r)}} (\sigma,\delta)\text{-Mod}(\mathrm{B})_S^{[r]}. \tag{7.4.2}$$

[7] For an annulus centered at 0, the condition $q^{k_0} \in \mathcal{Q}_1(A) = \mathrm{D}^-(1,1)$ is equivalent to $\bar{q} \in \mathbb{F}_p^{\text{alg}}$.

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Proof. By Lemma 4.3, it suffices to prove the essential surjectivity of $\mathrm{Res}_S^{\mathrm{D}^-(1,r)}$. The proof is straightforward and essentially the same as the proof of the propagation theorem (Theorem 7.6). ☐

COROLLARY 7.14. *Let $q, q' \in \mathrm{D}^-(1,1) - \boldsymbol{\mu}_{p^\infty}$. Let $r \in \mathbb{R}$ satisfy*

$$\max(|q-1|, |q'-1|) < r \leqslant 1. \tag{7.4.3}$$

Then one has an equivalence

$$\sigma_q\text{-Mod}(\mathcal{R}_K)^{[r]} \xrightarrow[\sim]{\mathrm{Def}^{\mathrm{Tay}}_{q,q'}} \sigma_{q'}\text{-Mod}(\mathcal{R}_K)^{[r]}. \tag{7.4.4}$$

The same equivalence holds between (σ_q, δ_q)-Mod$(\mathcal{R}_K)^{[r]}$ and $(\sigma_{q'}, \delta_{q'})$-Mod$(\mathcal{R}_K)^{[r]}$, without assuming that $q \notin \boldsymbol{\mu}_{p^\infty}$. Moreover, if $q \notin \boldsymbol{\mu}_{p^\infty}$, and if $|q-1| < r$, then we have an equivalence

$$(\sigma_q, \delta_q)\text{-Mod}(\mathcal{R}_K)^{[r]} \xrightarrow[\sim]{\mathrm{Forget}\delta_q} \sigma_q\text{-Mod}(\mathcal{R}_K)^{[r]}. \tag{7.4.5}$$

As usual we set $\mathrm{Conf}^{\mathrm{Tay}}_q := \mathrm{Def}^{\mathrm{Tay}} \circ (\mathit{Forget}\ \delta_q)^{-1}$. The analogous statement holds for $\mathcal{H}^\dagger_K$.

7.4.1 *Unipotent equations.* We shall compute the deformation $\mathrm{Def}^{\mathrm{Tay}}_{1,q}$ of the differential module U_m defined by the equation

$$\delta_1(Y_{U_m}) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \cdot Y_{U_m}, \quad Y_{U_m}(x,y) = \begin{pmatrix} 1 & \ell_1 & \cdots & \ell_{m-2} & \ell_{m-1} \\ 0 & 1 & \ell_1 & \cdots & \ell_{m-2} \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & 1 & \ell_1 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}, \tag{7.4.6}$$

where $\ell_n := [\log(x) - \log(y)]^n/n!$. One has

$$\sigma_q^x(\ell_n(x,y)) = [\log(qx) - \log(y)]^n/n! = (\log(q) + \log(x) - \log(y))^n/n! = \sum_{i=0}^{n} \frac{\log(q)^{n-k}}{(n-k)!} \cdot \ell_k.$$

The matrix of $\sigma_q^{U_m}$ is then

$$A(q,T) = \begin{pmatrix} 1 & \log(q) & \log(q)^2/2 & \cdots & \log(q)^{m-1}/(m-1)! \\ 0 & 1 & \log(q) & \cdots & \log(q)^{m-2}/(m-2)! \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & \log(q) \\ 0 & 0 & \cdots & \cdots & 1 \end{pmatrix}. \tag{7.4.7}$$

7.5 Classification of solvable rank one q-difference equations over $\mathcal{R}_{K_\infty}$

In this section we classify rank one solvable q-difference equations over $\mathcal{R}_{K_\infty}$ by applying the deformation $\mathrm{Def}^{\mathrm{Tay}}_{1,q}$ to the classification of the differential equations obtained in [Pul07]. We recall the classification of the rank one solvable differential equations over $\mathcal{R}_{K_\infty} := \bigcup_{s \geqslant 0} \mathcal{R}_{K_s}$ (see below).

We fix a Lubin–Tate group $\mathfrak{G}_P$ isomorphic to $\widehat{\mathbb{G}}_m$ over $\mathbb{Z}_p$. We recall that $\mathfrak{G}_P$ is defined by a uniformizer w of $\mathbb{Z}_p$, and by a series $P(X) \in X\mathbb{Z}_p[[X]]$, satisfying $P(X) \equiv \mathrm{w} \cdot X \pmod{X^2\mathbb{Z}_p[[X]]}$ and $P(X) \equiv X^p \pmod{p\mathbb{Z}_p[[X]]}$. By simplicity we assume $p = \mathrm{w}$, in order that $\mathfrak{G}_P \cong \hat{\mathbb{G}}_m$. Such a formal series is called a *Lubin–Tate series*. We fix now a sequence $\boldsymbol{\pi} := (\pi_m)_{m \geqslant 0}$, $\pi_m \in \mathbb{Q}_p^{\mathrm{alg}}$, such that $P(\pi_0) = 0$, $\pi_0 \neq 0$ and $P(\pi_{m+1}) = \pi_m$, for all $m \geqslant 0$. The element $(\pi_m)_{m \geqslant 0}$ is a generator of the Tate module of $\mathfrak{G}_P$ which is a free rank one $\mathbb{Z}_p$-module. We set $K_s := K(\pi_s)$ and $K_\infty := \bigcup_{s \geqslant 0} K_s$. We denote by k_s and k_∞ the respective residual fields. The tower $K \subseteq K_0 \subseteq K_1 \subseteq \cdots$ does not depend on the choice of $\boldsymbol{\pi}$, nor on $\mathfrak{G}_P \cong \widehat{\mathbb{G}}_m$. One has $K_s = K(\xi_s)$, where ξ_s is a primitive p^{s+1}th

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root of unity. For example, one can choose $\mathfrak{G}_P = \mathbb{G}_m$, hence $P(X) = (X+1)^p - 1$, and $\pi_m = \xi_m - 1$, where ξ_m is a compatible sequence of primitive p^{m+1}th root of 1, i.e. $\xi_0^p = 1$ and $\xi_m^p = \xi_{m-1}$, for all $m \geqslant 0$. One has the following facts.

(i) Every rank one *solvable* differential module over $\mathcal{R}_K$ has a basis in which the associated operator is

$$\mathrm{L}(a_0, \boldsymbol{f}^-(T)) := \delta_1 - \left(a_0 - \sum_{j=0}^{s} \pi_{s-j} \sum_{i=0}^{j} f_i^-(T)^{p^{j-i}} \partial_{T,\log}(f_i^-(T))\right), \tag{7.5.1}$$

where $a_0 \in \mathbb{Z}_p$, and $\boldsymbol{f}^-(T) := (f_0^-(T), \ldots, f_s^-(T))$ is a Witt vector in $\mathbf{W}_s(T^{-1}\mathcal{O}_{K_s}[T^{-1}])$, with $K_s := K(\pi_s)$. Notice that even if π_j does not belong to K, the resulting polynomial $\sum_{j=0}^{s} \pi_{s-j} \sum_{i=0}^{j} f_i^-(T)^{p^{j-i}} \partial_{T,\log}(f_i^-(T))$ has, by assumption, coefficients in K.

(ii) The Taylor solution at ∞ of the differential module in this basis is given by the so-called $\boldsymbol{\pi}$-exponential attached to $\boldsymbol{f}^-(T)$:

$$T^{a_0} \cdot \mathrm{e}_{p^m}(\boldsymbol{f}^-(T), 1) := T^{a_0} \cdot \exp\left(\sum_{j=0}^{s} \pi_{s-j} \frac{\phi_j^-(T)}{p^j}\right), \tag{7.5.2}$$

where $\langle \phi_0^-(T), \ldots, \phi_s^-(T)\rangle \in (T^{-1}\mathcal{O}_{K_s}[T^{-1}])^{s+1}$ is the phantom vector of $\boldsymbol{f}^-(T)$, namely one has $\phi_j^-(T) = \sum_{i=0}^{j} p^i f_i^-(T)^{p^{j-i}}$.

(iii) The correspondence $\boldsymbol{f}^-(T) \mapsto \mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1)$ is a group morphism

$$\mathbf{W}_s(T^{-1}\mathcal{O}_{K_s}[T^{-1}]) \xrightarrow{\mathrm{e}_{p^s}(-,1)} 1 + \pi_s T^{-1}\mathcal{O}_{K_s}[[T^{-1}]]. \tag{7.5.3}$$

Notice that if $L(0, \boldsymbol{f}^-(T))$ has its coefficients in $\mathcal{R}_K$ $(\subset \mathcal{R}_{K_s})$ then also $\mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1)$ lies in $1 + T^{-1}\mathcal{O}_K[[T^{-1}]]$ (because it is its Taylor solution at ∞).

(iv) Conversely, $\mathrm{L}(a_0, \boldsymbol{f}^-(T))$ is solvable for all pairs $(a_0, \boldsymbol{f}^-(T)) \in \mathbb{Z}_p \times \mathbf{W}_s(T^{-1}\mathcal{O}_{K_s}[T^{-1}])$.

(v) The operator $\mathrm{L}(a_0, \boldsymbol{f}^-(T))$ has a (strong) Frobenius structure (cf. Definition 8.5) if and only if $a_0 \in \mathbb{Z}_{(p)} := \mathbb{Z}_p \cap \mathbb{Q}$.

(vi) The operators $\mathrm{L}(a_0, \boldsymbol{f}_1^-(T))$ and $\mathrm{L}(b_0, \boldsymbol{f}_2^-(T))$ (with coefficients in $\mathcal{R}_K(\subset \mathcal{R}_{K_s})$) define isomorphic differential modules (over $\mathcal{R}_K$) if and only if $a_0 - b_0 \in \mathbb{Z}$ and the Artin–Schreier equation

$$\bar{\mathrm{F}}(\overline{\boldsymbol{g}^-(T)}) - \overline{\boldsymbol{g}^-(T)} = \overline{\boldsymbol{f}_1^-(T) - \boldsymbol{f}_2^-(T)} \tag{7.5.4}$$

has a solution $\overline{\boldsymbol{g}^-(T)}$ in $\mathbf{W}_s(k^{\mathrm{alg}}(\!(t)\!))$, where t is the reduction of T, and $\bar{\mathrm{F}}$ is the Frobenius of $\mathbf{W}_s(k^{\mathrm{alg}}(\!(t)\!))$ (sending $(\bar{g}_0, \ldots, \bar{g}_s)$ into $(\bar{g}_0^p, \ldots, \bar{g}_s^p)$). This happens if and only if the equation $L(0, \boldsymbol{f}_1^-(T) - \boldsymbol{f}_2^-(T))$ is trivial over $\mathcal{R}_K$, and also if and only if $\mathrm{e}_{p^s}(\boldsymbol{f}_1^-(T) - \boldsymbol{f}_2^-(T), 1)$ is overconvergent.[8]

By deformation, every solvable q-difference equation, with $|q-1| < 1$, has a solution at ∞ of the form $T^{a_0} \cdot \mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1)$. Its matrix in this basis is then

$$A(q, T) = \mathrm{e}_{p^s}(\boldsymbol{f}^-(qT), 1)/\mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1) = \mathrm{e}_{p^s}(\boldsymbol{f}^-(qT) - \boldsymbol{f}^-(T), 1).$$

The deformation guarantees that $A(q, T) \in \mathcal{R}_K$. This is confirmed by the fact that $\boldsymbol{f}^-(qT)$ and $\boldsymbol{f}^-(T)$ have the same reduction in $\mathbf{W}_s(k^{\mathrm{alg}}(\!(t)\!))$, and hence $\mathrm{e}_{p^s}(\boldsymbol{f}^-(qT) - \boldsymbol{f}^-(T), 1) \in \mathcal{R}_K$ by point vi) of the previous classification.

[8] Indeed the overconvergence of $\mathrm{e}_{p^s}(\boldsymbol{f}_1^-(T) - \boldsymbol{f}_2^-(T), 1)$ is independent of the residual field; for this reason we can look for solution of the Artin–Schreier–Witt equation (7.5.4) with coefficients in the more general field k^{alg} instead of k.

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8. Quasi-unipotence and p-adic local monodromy theorem

In this section we show how to deduce the q-analog of the p-adic local monodromy theorem (cf. [And02, Ked04, Meb02]) by deformation.

Let K be a complete discrete valued field with perfect residue field (this hypothesis is necessary to have the p-adic local monodromy theorem (cf. Theorem 8.12)). Let $\mathcal{E}_K^\dagger \subset \mathcal{R}_K$ be the so-called *bounded Robba ring*,

$$\mathcal{E}_K^\dagger := \left\{ \sum_{i\in\mathbb{Z}} a_i T^i \in \mathcal{R}_K \;\middle|\; \sup|a_i| < +\infty, \lim_{i\to-\infty} |a_i| = 0 \right\}.$$

Then, since K is discrete valued, $(\mathcal{E}_K^\dagger, |\cdot|_{(0,1)})$ is a *Henselian* valued field, with residue field $k(\!(t)\!)$. It has two topologies arising from $|\cdot|_{(0,1)}$, and from the inclusion in $\mathcal{R}_K$. It is not complete with respect to either of these two topologies, but $\mathcal{E}_K^\dagger$ is dense in $\mathcal{R}_K$. One has the inclusions

$$\mathcal{H}_K^\dagger \subset \mathcal{E}_K^\dagger \subset \mathcal{R}_K. \qquad (8.0.1)$$

8.1 Frobenius functor and Frobenius structure

Let $\varphi : K \to K$ be an absolute Frobenius (i.e. a ring morphism lifting of the pth power map of k). Since $\mathcal{R}_K$ is not a local ring, and does not have a residue ring, we need a particular definition.

Definition 8.1. An absolute *Frobenius* on $\mathcal{R}_K$ (respectively $\mathcal{H}_K^\dagger$, $\mathcal{E}_K^\dagger$) is a continuous ring morphism, again denoted by $\varphi : \mathcal{R}_K \to \mathcal{R}_K$, extending φ on K and such that $\varphi(\sum a_i T^i) = \sum \varphi(a_i)\varphi(T)^i$, where $\varphi(T) = \sum_{i\in\mathbb{Z}} b_i T^i \in \mathcal{R}_K$ (respectively $\varphi(T) \in \mathcal{H}_K^\dagger$, $\varphi(T) \in \mathcal{E}_K^\dagger$) verifies $|b_i| < 1$, for all $i \neq p$, and $|b_p - 1| < 1$.

Definition 8.2. We denote by ϕ the particular absolute Frobenius on $\mathcal{R}_K$ given by the choice

$$\phi(T) := T^p, \quad \phi(f(T)) := f^\varphi(T^p), \qquad (8.1.1)$$

where $f^\varphi(T)$ is the series obtained from $f(T)$ by applying $\varphi : K \to K$ to the coefficients.

Let B be one of the rings $\mathcal{H}_K^\dagger$, $\mathcal{E}_K^\dagger$, or $\mathcal{R}_K$. For all $q \in \mathrm{D}^-(1,1)$, the following diagrams are commutative.

$$\begin{array}{ccc} \mathrm{B} & \xrightarrow{\phi} & \mathrm{B} \\ {\scriptstyle \sigma_{q^p}}\downarrow & \odot & \downarrow{\scriptstyle \sigma_q} \\ \mathrm{B} & \xrightarrow[\phi]{} & \mathrm{B} \end{array} \qquad \begin{array}{ccc} \mathrm{B} & \xrightarrow{\phi} & \mathrm{B} \\ {\scriptstyle p\cdot\delta_1}\downarrow & \odot & \downarrow{\scriptstyle \delta_1} \\ \mathrm{B} & \xrightarrow[\phi]{} & \mathrm{B} \end{array} \qquad (8.1.2)$$

Definition 8.3 (Frobenius functor). Let $S \subseteq \mathrm{D}^-(1,r)$, $0 < r \leqslant 1$. Let

$$r' := \min(r^{1/p},\ r\cdot|p|^{-1}). \qquad (8.1.3)$$

The Frobenius functor (cf. Definition 6.9)

$$\phi^* : (\sigma,\delta)\text{-}\mathrm{Mod}(\mathrm{B})_S^{[r]} \longrightarrow (\sigma,\delta)\text{-}\mathrm{Mod}(\mathrm{B})_S^{[r']}, \qquad (8.1.4)$$

respectively

$$\phi^* : \sigma\text{-}\mathrm{Mod}(\mathrm{B})_S^{[r]} \longrightarrow \sigma\text{-}\mathrm{Mod}(\mathrm{B})_S^{[r']}, \qquad (8.1.5)$$

is defined as $\phi^*(\mathrm{M}, \sigma^{\mathrm{M}}, \delta_1^{\mathrm{M}}) = (\phi^*(\mathrm{M}), \sigma^{\phi^*(\mathrm{M})}, \delta_1^{\phi^*(\mathrm{M})})$, where:

(i) $\phi^*(\mathrm{M}) := \mathrm{M} \otimes_{\mathrm{B},\phi} \mathrm{B}$ is the scalar extension of M via ϕ;

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(ii) the morphism $\sigma^{\phi^*(\mathrm{M})}$ is given by $\sigma_q^{\phi^*(\mathrm{M})} = \sigma_{q^p}^{\mathrm{M}} \otimes \sigma_q^{\mathrm{B}}$,

$$q \longmapsto \sigma_{q^p}^{\mathrm{M}} \otimes \sigma_q : S \xrightarrow{\sigma^{\phi^*(\mathrm{M})}} \mathrm{Aut}_K^{\mathrm{cont}}(\phi^*(\mathrm{M})); \tag{8.1.6}$$

(iii) the derivation is given by

$$\delta_1^{\phi^*(\mathrm{M})} = (p \cdot \delta_1^{\mathrm{M}}) \otimes \mathrm{Id}_{\mathrm{B}} + \mathrm{Id}_{\mathrm{M}} \otimes \delta_1^{\mathrm{B}}; \tag{8.1.7}$$

(iv) a morphism $\alpha : \mathrm{M} \to \mathrm{N}$ is sent into $\alpha \otimes 1 : \phi^*(\mathrm{M}) \to \phi^*(\mathrm{N})$.

Remark 8.4. The fact that the functor ϕ^* sends $(\sigma,\delta)\text{-Mod}(\mathrm{B})_S^{[r]}$ into $(\sigma,\delta)\text{-Mod}(\mathrm{B})_S^{[r']}$ with this particular value of r' (cf. (8.1.3)) follows from the fact that this result is true for *differential* equations (cf. [Pul05, Appendix] and [CM02, Proposition 7.2]), and from the confluence.

8.1.1 We observe that the pull-back $\varphi^*(\mathrm{M})$ is actually a σ-module over $S^{1/p} := \{q \in K \mid q^p \in S\}$. Indeed $\phi^*(\mathrm{M})$ is canonically endowed with the action of $\sigma_{q^{1/p}}^{\phi^*(\mathrm{M})} := \sigma_q^{\mathrm{M}} \otimes \sigma_{q^{1/p}} : \phi^*(\mathrm{M}) \to \phi^*(\mathrm{M})$, for all roots $q^{1/p}$ of q. This fact was used in [ADV04] to define the so-called confluent weak Frobenius structure (cf. Definition 8.27).

If $\mathrm{M} \in (\sigma,\delta)\text{-Mod}(\mathcal{H}_K^\dagger)_S^{[r]}$, then we can consider its Taylor solution at 1:

$$Y(T,1) = \sum_{i \geqslant 0} Y_i (T-1)^i \in GL_n(\mathcal{A}_K(1,1)), \quad Y_i \in M_n(K).$$

Then the Taylor solution of $\phi^*(\mathrm{M})$ is given by

$$Y^\phi(T^p,1) := \sum_{i \geqslant 0} \varphi(Y_i)(T^p-1)^i. \tag{8.1.8}$$

The matrices of $\phi^*(\sigma_q)$ and $\phi^*(\delta_1)$ are the following. Let $\mathbf{e} = \{\mathrm{e}_1, \dots, \mathrm{e}_n\}$ be a basis of M. Let $\sigma_q - A(q,T)$ and $\delta_1 - G(1,T)$ be the operators associated to σ_q^{M} and δ_1^{M} in this basis. Then the operators associated to $\phi^*(\mathrm{M})$ in the basis $\mathbf{e} \otimes 1$ are

$$\sigma_q - A^\varphi(q^p, T^p), \quad \delta_1 - p \cdot G^\varphi(1, T^p), \tag{8.1.9}$$

where, according to (2.1.7), one has $A(q^p,T) = A(q, q^{p-1}T) \cdots A(q,qT)A(q,T)$.

8.1.2 *Frobenius structure.* The functor $\phi^* : \delta_1\text{-Mod}(\mathcal{R}_K)^{[1]} \xrightarrow{\sim} \delta_1\text{-Mod}(\mathcal{R}_K)^{[1]}$ is an equivalence (cf. [CM02, Corollary 8.14]). By deformation ϕ^* is hence an auto-equivalence of $\sigma\text{-Mod}(\mathcal{R}_K)_S^{[1]}$ (if $S^\circ \neq \emptyset$) and $(\sigma,\delta)\text{-Mod}(\mathcal{R}_K)_S^{[1]}$ (without assuming $S^\circ \neq \emptyset$).

Definition 8.5 (Frobenius structure). Let B be one of the rings $\mathcal{H}_K^\dagger$, $\mathcal{E}_K^\dagger$, or $\mathcal{R}_K$. Let $S \subseteq \mathrm{D}^-(1,1)$ be a subset. Let M be a discrete σ-module (respectively (σ,δ)-module) over S. We will say that M has a *Frobenius structure of order* $h \geqslant 1$, if there exists a B-isomorphism $(\phi^*)^h(\mathrm{M}) \xrightarrow{\sim} \mathrm{M}$ of σ-modules over S (cf. § 8.1.1), where $(\phi^*)^h := \phi^* \circ \cdots \circ \phi^*$, h-times. We denote by

$$\sigma\text{-Mod}(\mathrm{B})_S^{(\phi)}, \quad (\text{respectively } (\sigma,\delta)\text{-Mod}(\mathrm{B})_S^{(\phi)}) \tag{8.1.10}$$

the full subcategory of $\sigma\text{-Mod}(\mathrm{B})_S^{[1]}$ (respectively $(\sigma,\delta)\text{-Mod}(\mathrm{B})_S^{[1]}$) whose objects have a Frobenius structure of some unspecified order.

If M has a Frobenius structure, then $r = r'$ (cf. (8.1.3)) and hence M is *solvable*:

$$\sigma\text{-Mod}(\mathrm{B})_S^{(\phi)} \subset \sigma\text{-Mod}(\mathrm{B})_S^{[1]}. \tag{8.1.11}$$

Hence objects in $\sigma\text{-Mod}(\mathrm{B})_S^{(\phi)}$ and $(\sigma,\delta)\text{-Mod}(\mathrm{B})_S^{(\phi)}$ are, in particular, *admissible*.

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If $Y(T,1)$ is the Taylor solution of $\mathrm{M} \in (\sigma,\delta)\text{-Mod}(\mathcal{H}_K^\dagger)_S^{[1]}$ at 1, then the fact that M has a Frobenius structure of some order $h \geqslant 1$ is equivalent to the existence of a matrix $H(T) \in GL_n(\mathcal{H}_K^\dagger)$ such that

$$Y^{\varphi^h}(T^{p^h},1) = H(T) \cdot Y(T,1). \tag{8.1.12}$$

Indeed $\mathcal{A}_K(1,1)$ is an $\mathcal{H}_K^\dagger$-discrete σ-algebra over $\mathrm{D}^-(1,1)$ trivializing M (cf. Definition 3.2). In particular the equivalences $\mathrm{Def}_{q,q'}^{\mathrm{Tay}}$ and $\mathrm{Conf}_q^{\mathrm{Tay}}$ send objects with Frobenius structure into objects with Frobenius structure.

PROPOSITION 8.6. *Let ξ be a p^nth root of unity, and let $q \in \mathcal{Q}_1 - \boldsymbol{\mu}(\mathcal{Q}_1)$. Let $\mathrm{M} \in \sigma_q\text{-Mod}(\mathcal{H}_K^\dagger)^{(\phi)}$. Then $\mathrm{Def}_{q,\xi}^{\mathrm{Tay}}(\mathrm{M}) \in \sigma_\xi\text{-Mod}(\mathcal{H}_K^\dagger)$ is trivial (i.e. isomorphic to a direct sum of copies of the unit object).*

Proof. Let $Y(T,1) \in GL_n(\mathcal{H}_K^\dagger)$ be the Taylor solution at 1 of M in some basis $\mathbf{e}$. Then, by (8.1.12), there exists $H(T)$ such that $Y^{\varphi^h}(T^{p^h},1) = H(T)\cdot Y(T,1)$. Hence, one also has $Y^{\varphi^{nh}}(T^{p^{nh}},1) = H_n(T)\cdot Y(T,1)$, for some $H_n(T) \in GL_n(\mathcal{H}_K^\dagger)$. Since $\sigma_\xi(Y^{\varphi^{nh}}(T^{p^{nh}},1)) = Y^{\varphi^{nh}}(T^{p^{nh}},1)$, it follows that in the basis $H_n(T)\cdot\mathbf{e}$ the matrix of σ_ξ is trivial: $A(\xi,T) = \mathrm{Id}$ (cf. § 3.2.1). □

8.2 Special coverings of $\mathcal{H}_K^\dagger$

We recall briefly the notions of *special coverings*. The residue field of $\mathcal{E}_K^\dagger$ is $k(\!(t)\!)$ (with respect to the norm $|\cdot|_{(0,1)}$). On the other hand, the residue ring of $\mathcal{H}_K^\dagger$ (with respect to the Gauss norm $|\cdot|_{(0,1)}$) is $k[t,t^{-1}]$. One has the following diagram.

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{H}_K^\dagger} & \subseteq & \mathcal{O}_{\mathcal{E}_K^\dagger} \\ \downarrow & \odot & \downarrow \\ k[t,t^{-1}] & \subseteq & k(\!(t)\!) \end{array} \tag{8.2.1}$$

We denote by $\mathcal{O}_K[T,T^{-1}]^\dagger$ the weak completion of$\mathcal{O}_K[T,T^{-1}]$, in the sense of Monsky and Washnitzer [MW68]. One has

$$\mathcal{H}_K^\dagger = \mathcal{O}_K[T,T^{-1}]^\dagger \otimes_{\mathcal{O}_K} K. \tag{8.2.2}$$

Let us look at the residual situation. The morphism

$$\widehat{\eta} := \mathrm{Spec}(k(\!(t)\!)) \quad \hookrightarrow \quad \mathbb{G}_{m,k} = \mathrm{Spec}(k[t,t^{-1}]) \tag{8.2.3}$$

gives rise, by pull-back, to a map

$$\left\{\begin{array}{l}\text{finite étale}\\ \text{coverings of } \widehat{\eta}\end{array}\right\} \xleftarrow{\text{pull-back}} \left\{\begin{array}{l}\text{finite étales}\\ \text{coverings of } \mathbb{G}_{m,k}\end{array}\right\}. \tag{8.2.4}$$

It is known (cf. [Kat86, 2.4.9]) that this map is surjective, and moreover that there exists a full sub-category of the right-hand category, called *special coverings of* $\mathbb{G}_{m,k}$, which is equivalent, via pull-back, to the category on the left-hand side. Special coverings are defined by the property that they are tamely ramified at ∞, and that their geometric Galois group has a unique p-Sylow subgroup (cf. [Kat86, 1.3.1]).

On the other hand, if $\pi \in \mathcal{O}_K$ is a uniformizing element, then both $(\mathcal{O}_{\mathcal{E}_K^\dagger},(\pi))$ and $(\mathcal{O}_K[T,T^{-1}]^\dagger,(\pi))$ are Henselian couples in the sense of [Ray70, ch. II] (cf. [Mat02, § 5.1]).

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One can show that the preceding situation lifts to characteristic 0. One has the equivalences

$$\begin{array}{ccccc}
\left\{\begin{array}{l}\text{special}\\ \text{extensions of } \mathcal{H}_K^\dagger\end{array}\right\} & \xrightarrow[\sim]{-\otimes\mathcal{E}_K^\dagger} & \left\{\begin{array}{l}\text{finite unramified}\\ \text{extensions of } \mathcal{E}_K^\dagger\end{array}\right\} & \xrightarrow[\sim]{-\otimes\mathcal{R}_K} & \left\{\begin{array}{l}\text{special}\\ \text{extensions of } \mathcal{R}_K\end{array}\right\} \\
{\scriptstyle -\otimes K}\uparrow\wr & \odot & \wr\uparrow{\scriptstyle -\otimes K} & & \\
\left\{\begin{array}{l}\text{special extensions}\\ \text{of } \mathcal{O}_K[T,T^{-1}]^\dagger\end{array}\right\} & \xrightarrow[\sim]{-\otimes\mathcal{O}_{\mathcal{E}_K^\dagger}} & \left\{\begin{array}{l}\text{finite unramified}\\ \text{extensions of } \mathcal{O}_{\mathcal{E}_K^\dagger}\end{array}\right\} & & \\
{\scriptstyle -\otimes k}\downarrow\wr & \odot & \wr\downarrow{\scriptstyle -\otimes k} & & \\
\left\{\begin{array}{l}\text{special}\\ \text{coverings of } \mathbb{G}_{m,k}\end{array}\right\} & \xrightarrow[\sim]{\text{pull-back}} & \left\{\begin{array}{l}\text{finite étale}\\ \text{coverings of } \hat{\eta}\end{array}\right\} & &
\end{array} \tag{8.2.5}$$

where, by special extension of $\mathcal{O}_K[T,T^{-1}]^\dagger$ (respectively $\mathcal{H}_K^\dagger$, $\mathcal{R}_K$), we mean a finite étale Galois extension of $\mathcal{O}_K[T,T^{-1}]^\dagger$ (respectively $\mathcal{H}_K^\dagger$, $\mathcal{R}_K$) coming, by Henselianity, from a special cover of $\mathbb{G}_{m,k}$.

Lemma 8.7. *Let* $\mathrm{F}/k(\!(t)\!)$ *be a finite Galois extension with Galois group* G. *Let* $\mathcal{S}^\dagger(\mathrm{F})/\mathcal{H}_K^\dagger$ *be the corresponding special extension of* $\mathcal{H}_K^\dagger$. *Then* $(\mathcal{S}^\dagger(\mathrm{F}))^{\mathrm{G}} = \mathcal{H}_K^\dagger$.

Proof. By [SGA03, Exposé V, Corollary 3.4], $(\mathcal{S}^\dagger(\mathrm{F}))^{\mathrm{G}}/\mathcal{H}_K^\dagger$ is a special extension. The assertion is then easy since, by the above equivalence, there is bijection between special sub-algebras of $\mathcal{S}^\dagger(\mathrm{F})$ over $\mathcal{H}_K^\dagger$ and sub-extensions of $\mathrm{F}/k(\!(t)\!)$. □

8.2.1 *Extension of* σ_q *to Special extensions.*

Lemma 8.8 [ADV04, § 11.3]. *Let* $F/k(\!(t)\!)$ *be a finite separable extension. Let* $\mathcal{F}^\dagger/\mathcal{O}_K[T,T^{-1}]^\dagger$ *be the corresponding special extensions. The automorphism* σ_q *of* $\mathcal{O}_K[T,T^{-1}]^\dagger$ *extends to an automorphism* $\mathcal{F}^\dagger$. *The extension is unique up to* $\mathcal{O}_K[T,T^{-1}]^\dagger$*-automorphisms of* $\mathcal{F}^\dagger$. *The same statement holds for the extensions* $(\mathcal{H}_K^\dagger)'/\mathcal{H}_K^\dagger$, $(\mathcal{E}_K^\dagger)'/\mathcal{E}_K^\dagger$, $(\mathcal{R}_K)'/\mathcal{R}_K$ *corresponding to* $F/k(\!(t)\!)$. *In particular there exists a unique extension of* σ_q *to* $\mathcal{F}^\dagger$, $(\mathcal{H}_K^\dagger)'$, $(\mathcal{E}_K^\dagger)'$, $(\mathcal{R}_K)'$ *inducing the identity on* F.

Proof. The proof results from the formal properties of Henselian couples (cf. [Ray70]). □

By uniqueness the extension of σ_q commutes with the action of $\mathrm{Gal}(k(\!(t)\!)^{\mathrm{sep}}/k(\!(t)\!))$.

Remark 8.9. Every finite extension of $\mathbb{C}(\!(T)\!)$ is of the form $\mathbb{C}(\!(T^{m/n})\!)$. Up to change of variable we have an isomorphism $\mathbb{C}(\!(T^{m/n})\!) \cong \mathbb{C}(\!(Z)\!)$. Analogously it can be seen that a finite unramified extension of $\mathcal{E}_K^\dagger$ is (non-canonically) isomorphic to $\mathcal{E}_{K'}^\dagger$ for some finite K'/K. In this case the link between the variable Z and the variable T is rather complicated and essentially unknown. *One of the problems of the theory is that the extended automorphism does not send* Z *into* qZ. The general 'confluence' theory introduced in § 4 will be crucial in solving this problem.

8.3 Quasi-unipotence of differential equations and canonical extension

In this section we recall some known facts on p-adic *differential* equations.

Definition 8.10. We denote by $\widetilde{\mathcal{H}_K^\dagger}$ (respectively $\widetilde{\mathcal{E}_K^\dagger}$, $\widetilde{\mathcal{R}_K}$) the union of all finite special (respectively unramified, special) extensions of $\mathcal{H}_K^\dagger$ (respectively $\mathcal{E}_K^\dagger$, $\mathcal{R}_K$) in an algebraically closure of the field of fractions of $\mathcal{R}_K$.

DEFINITION 8.11. Let B be one of the rings $\mathcal{H}_K^\dagger$, $\mathcal{E}_K^\dagger$, $\mathcal{R}_K$. Let $S \subseteq \mathrm{D}^-(1,1)$ be a subset (respectively $S \subseteq \mathrm{D}^-(1,1)$, with $S^\circ \neq \emptyset$). A discrete (σ,δ)-module on S (respectively discrete σ-module on S) over B is called *quasi-unipotent* if it is trivialized by the discrete (σ,δ)-algebra

$$\widetilde{\mathrm{B}}[\log(T)]. \tag{8.3.1}$$

We observe that M is trivialized by $\widetilde{\mathrm{B}}[\log(T)]$, if and only if M is trivialized by $\mathrm{B}'[\log(T)]$, where B' is a (*finite*) special extension of B. Indeed the entries of a fundamental matrix of solutions of M in $\widetilde{\mathrm{B}}[\log(T)]$ all lie in a finite extension.

THEOREM 8.12 (The p-adic local monodromy theorem, cf. [And02, Ked04, Meb02]). *Objects in δ_1-Mod$(\mathcal{R}_K)^{(\phi)}$ become quasi-unipotent possibly after a suitable extension of the field of constants K. In other words, if $\mathrm{M} \in \delta_1$-Mod$(\mathcal{R}_K)^{(\phi)}$, then there exists a finite extension K'/K such that $\mathrm{M} \otimes_K K'$ is quasi-unipotent (i.e. trivialized by $\widetilde{\mathcal{H}_{K'}^\dagger}[\log(T)]$).*

THEOREM 8.13 [Mat02, Corollary 7.10, Theorem 7.15]. *If a differential equation $\mathrm{M} \in \delta_1$-Mod$(\mathcal{R}_K)$ is quasi-unipotent, then it has a Frobenius structure. Moreover, the scalar extension functor*

$$-\otimes\mathcal{R}_K : \delta_1\text{-Mod}(\mathcal{H}_K^\dagger)^{(\phi)} \to \delta_1\text{-Mod}(\mathcal{R}_K)^{(\phi)} \tag{8.3.2}$$

is essentially surjective.

THEOREM 8.14 [Mat02, Theorem 7.15]. *There exists a full sub-category of δ_1-Mod$(\mathcal{H}_K^\dagger)^{(\phi)}$, denoted by δ_1-Mod$(\mathcal{H}_K^\dagger)^{\mathrm{Sp}}$, which is equivalent to δ_1-Mod$(\mathcal{R}_K)^{(\phi)}$ via the scalar extension functor (8.3.2). Objects in δ_1-Mod$(\mathcal{H}_K^\dagger)^{\mathrm{Sp}}$ category are trivialized by $\widetilde{\mathcal{H}_K^\dagger}[\log(T)]$.*

DEFINITION 8.15 (Canonical extension). Objects in δ_1-Mod$(\mathcal{H}_K^\dagger)^{\mathrm{Sp}}$ will be called *special objects*. We will denote by

$$\delta_1\text{-Mod}(\mathcal{R}_K)^{(\phi)} \xrightarrow[\sim]{\mathrm{Can}} \delta_1\text{-Mod}(\mathcal{H}_K^\dagger)^{\mathrm{Sp}} \subset \delta_1\text{-Mod}(\mathcal{H}_K^\dagger)^{(\phi)} \tag{8.3.3}$$

the section of the functor (8.3.2), whose image is the category of special objects (cf. Theorem 8.14). We will call it the *canonical extension functor*.

COROLLARY 8.16 [And02, Corollaire 7.1.6]. *Let $\mathrm{M} \in \delta_1$-Mod$(\mathcal{R}_K)^{(\phi)}$, then, up to replacing K by a finite extension K'/K, M decomposes in a direct sum of submodules of the form $\mathrm{N} \otimes U_m$, where N is a module trivialized by a special extension of $\mathcal{R}_K$, and U_m is the m-dimensional object defined by the operator (cf. § 7.4.1)*

$$\delta_1 - \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{8.3.4}$$

Remark 8.17. The $\log(T)$ appearing in (8.3.1) is actually added uniquely to trivialize the module of the form U_m, for $m \geqslant 2$ (cf. § 7.4.1).

LEMMA 8.18. *Let $\mathrm{N} \in \delta_1$-Mod$(\mathcal{H}_K^\dagger)^{\mathrm{Sp}}$ be a special object trivialized by $\widetilde{\mathcal{H}_K^\dagger}$. Let $\widetilde{Y} = (\widetilde{y}_{i,j}) \in GL_n(\widetilde{\mathcal{H}_K^\dagger})$ be a fundamental matrix solution of N. Let $(\mathcal{E}^\dagger)'$ (respectively $\mathcal{R}'$) be the smallest special extension of $\mathcal{E}_K^\dagger$ (respectively $\mathcal{R}_K$), such that $\mathrm{N} \otimes \mathcal{E}_K^\dagger$ is trivialized by $(\mathcal{E}^\dagger)'$ ($\mathrm{N} \otimes \mathcal{R}_K$ is trivialized by $\mathcal{R}'$). Then one has*

$$(\mathcal{E}^\dagger)' = \mathcal{E}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}], \quad \mathcal{R}' = \mathcal{R}_K[\{\widetilde{y}_{i,j}\}_{i,j}]. \tag{8.3.5}$$

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In other words, the smallest special extension of $\mathcal{E}^\dagger_K$ (respectively $\mathcal{R}_K$) trivializing N *is generated by the solutions of* N*.*

Proof. Since N is trivialized by $(\mathcal{E}^\dagger)'$, one has $\mathcal{E}^\dagger_K[\{\widetilde{y}_{i,j}\}_{i,j}] \subseteq (\mathcal{E}^\dagger)'$. Hence the differential field $\mathcal{E}^\dagger_{K'}[\{\widetilde{y}_{i,j}\}_{i,j}]$ is an unramified extension, and is then a special extension. Since $(\mathcal{E}^\dagger)'$ is minimal, $\mathcal{E}^\dagger_K[\{\widetilde{y}_{i,j}\}_{i,j}] = (\mathcal{E}^\dagger)'$. The case over $\mathcal{R}_K$ follows from the case over $\mathcal{E}^\dagger_K$. □

Corollary 8.19. *We preserve the notation of Lemma 8.18. There exists a unique automorphism of $\mathcal{E}^\dagger_K[\{\widetilde{y}_{i,j}\}_{i,j}]$ extending σ_q, and inducing the identity on the residue field. We denote it again by σ_q.*

Proof. By Lemma 8.18, $\mathcal{E}^\dagger_K[\{\widetilde{y}_{i,j}\}_{i,j}]$ is a special extension (i.e. Henselian). Hence, by § 8.2.1, the extension of σ_q to $\mathcal{E}^\dagger_K[\{\widetilde{y}_{i,j}\}_{i,j}]$ is unique. □

Corollary 8.20. *Let $S \subseteq \mathrm{D}^-_K(1,1)$ The scalar extension functor*

$$- \otimes \mathcal{R}_K : (\sigma,\delta)\text{-}\mathrm{Mod}(\mathcal{H}^\dagger_K)^{(\phi)}_S \longrightarrow (\sigma,\delta)\text{-}\mathrm{Mod}(\mathcal{R}_K)^{(\phi)}_S \tag{8.3.6}$$

is essentially surjective. Moreover there exists a full sub-category of $(\sigma,\delta)\text{-}\mathrm{Mod}(\mathcal{H}^\dagger_K)^{(\phi)}_S$, which we call $(\sigma,\delta)\text{-}\mathrm{Mod}(\mathcal{H}^\dagger_K)^{\mathrm{Sp}}_S$, equivalent via $-\otimes\mathcal{R}_K$ to $(\sigma,\delta)\text{-}\mathrm{Mod}(\mathcal{R}_K)^{(\phi)}_S$. The same statement is true for σ-modules under the assumption $S^\circ \neq \emptyset$.

Proof. By Proposition 7.13, we can assume that $S = \mathrm{D}^-(1,1)$. By Theorem 8.13 there exists a basis of M in which the matrix $G(1,T)$ of δ_1^{M} lies in $M_n(\mathcal{H}^\dagger_K)$. Moreover, $\mathrm{Can}(\mathrm{M},\delta_1^{\mathrm{M}})$ is Taylor admissible, since all solvable differential equations are Taylor admissible. By Proposition 7.13, for all $q \in \mathrm{D}^-(1,1)$, the matrix $A(q,T) := Y_G(qT,T)$ belongs also to $GL_n(\mathcal{H}^\dagger_K)$. This proves the essential surjectivity. The fully faithfulness follows by deformation of Theorem 8.14 (cf. Corollary 7.9). □

8.3.1 It is not clear to us if the smallest special extension of $\mathcal{H}^\dagger_K$ trivializing a given $\mathrm{M} \in \delta_1\text{-}\mathrm{Mod}(\mathcal{H}^\dagger_K)^{\mathrm{Sp}}$ is generated (over $\mathcal{H}^\dagger_K$) by the entries of a fundamental matrix of solution of M. So we are obliged to give the following definition.

Definition 8.21. We denote by $\widetilde{\mathrm{C}^{\text{ét}}_K}$ the sub-algebra of $\widetilde{\mathcal{H}^\dagger_K}$ generated, over $\mathcal{H}^\dagger_K$, by the entries of every fundamental solution matrix of each object in $\delta_1\text{-}\mathrm{Mod}(\mathcal{H}^\dagger_K)^{\mathrm{Sp}}$ which is trivialized by $\widetilde{\mathcal{H}^\dagger_K}$.

With the notation of Corollary 8.20, the inclusions $(\sigma,\delta)\text{-}\mathrm{Mod}(\mathcal{H}^\dagger_K)^{\mathrm{Sp}}_S \subset (\sigma,\delta)\text{-}\mathrm{Mod}(\mathcal{H}^\dagger_K)^{(\phi)}_S \subset (\sigma,\delta)\text{-}\mathrm{Mod}(\mathcal{H}^\dagger_K)^{[1]}_S$ are strict (the same holds for (σ,δ)-modules). For example the equation $\delta_1(y) = a_0 y$, with $a_0 \in \mathbb{Z}_p - \mathbb{Z}_{(p)}$, is solvable, but without Frobenius structure (cf. § 7.5). On the other hand an object with Frobenius structure could have non-zero p-adic slope at 1^+ (hence irregular at ∞), hence it is not special. Unfortunately we have no examples of *non-special* equations with Frobenius structure, but trivialized by $\widetilde{\mathrm{C}^{\text{ét}}_K}[\log(T)]$.

8.4 Quasi-unipotence of σ-modules and (σ,δ)-modules with Frobenius structure

This section is devoted to proving the following theorem.

Theorem 8.22 (The p-adic local monodromy theorem (generalized form)). *Let $S \subset \mathrm{D}^-(1,1)$ be a subset (respectively $S^\circ \neq \emptyset$). Then every object $\mathrm{M} \in (\sigma,\delta)\text{-}\mathrm{Mod}(\mathcal{R}_K)^{(\phi)}_S$ (respectively $\mathrm{M} \in \sigma\text{-}\mathrm{Mod}(\mathcal{R}_K)^{(\phi)}_S$) is quasi-unipotent, after replacing K, if necessary, by a finite extension K'/K depending on* M*.*

This result simplifies and generalizes the analogous result of [ADV04]. The proof is obtained by deformation of the p-adic local monodromy theorem of *differential* equations (cf. Theorem 8.12).

The proof is essentially the following. Assume that $S = \{q\}$, with $q \notin \boldsymbol{\mu}_{p^\infty}$. By canonical extension (cf. Corollary 8.20) M is trivialized by $\widetilde{\mathcal{R}_K}[\log(T)]$ if and only if Can(M) is trivialized by $\widetilde{\mathcal{H}_K^\dagger}[\log(T)]$ (or equivalently by $\widetilde{\mathrm{C}_K^{\text{ét}}}[\log(T)]$). Hence we can assume that $\mathrm{M} \in \sigma_q\text{-}\mathrm{Mod}(\mathcal{H}_K^\dagger)^{\mathrm{Sp}}$. Firstly apply the confluence functor to obtain a differential equation $\mathrm{Conf}_q^{\mathrm{Tay}}(\mathrm{M}, \sigma_q^{\mathrm{M}})$. We prove then in Lemma 8.23 below that $\mathrm{Conf}_q^{\mathrm{Tay}}(\mathrm{M}, \sigma_q^{\mathrm{M}})$ is $\widetilde{\mathrm{C}_K^{\text{ét}}}[\log(T)]$-extensible to $\mathrm{D}^-(1,1)$ (cf. Definition 4.4). Hence we obtain, by deformation, another q-difference module $\mathrm{Def}_{1,q}^{\widetilde{\mathrm{C}_K^{\text{ét}}}[\log(T)]}(\mathrm{Conf}_q^{\mathrm{Tay}}(\mathrm{M}, \sigma_q^{\mathrm{M}}))$ over $\widetilde{\mathcal{H}_K^\dagger}$ (cf. § 4). This q-difference module is quasi-unipotent since, by definition, it has the same solutions in $\widetilde{\mathrm{C}_K^{\text{ét}}}[\log(T)]$ of the quasi-unipotent differential equation $\mathrm{Conf}_q^{\mathrm{Tay}}(\mathrm{M}, \sigma_q^{\mathrm{M}})$. We show then that there is an embedding $\widetilde{\mathrm{C}_K^{\text{ét}}}[\log(T)] \subseteq \mathcal{A}_{K^{\mathrm{alg}}}(1,1)$ commuting with δ_1, φ, and with σ_q, for all $q \in \mathrm{D}^-(1,1)$ (cf. Lemma 8.24). This proves that the restriction of $\mathrm{Def}_{1,q}^{\mathrm{Tay}}$ to the category of objects trivialized by $\widetilde{\mathrm{C}_K^{\text{ét}}}[\log(T)]$ coincides with $\mathrm{Def}_{1,q}^{\widetilde{\mathrm{C}_K^{\text{ét}}}[\log(T)]}$ (cf. §§ 4.2.3 and 4.2.1), because $\mathrm{Def}_{1,q}^{\mathrm{Tay}} = \mathrm{Def}_{1,q}^{\mathrm{C}}$, with $\mathrm{C} = \mathcal{A}_K(1,1)$ (cf. Remark 7.5) or equivalently $\mathrm{C} = \mathcal{A}_{K^{\mathrm{alg}}}(1,1)$ (cf. Corollary 8.26). Hence

$$\mathrm{Def}_{1,q}^{\widetilde{\mathrm{C}_K^{\text{ét}}}[\log(T)]}(\mathrm{Conf}_q^{\mathrm{Tay}}(\mathrm{M}, \sigma_q^{\mathrm{M}})) = \mathrm{Def}_{1,q}^{\mathrm{Tay}}(\mathrm{Conf}_q^{\mathrm{Tay}}(\mathrm{M}, \sigma_q^{\mathrm{M}})) = (\mathrm{M}, \sigma_q^{\mathrm{M}}). \tag{8.4.1}$$

In particular $(\mathrm{M}, \sigma_q^{\mathrm{M}})$ is trivialized by $\widetilde{\mathrm{C}_K^{\text{ét}}}[\log(T)]$ and is hence quasi-unipotent.

LEMMA 8.23. *Let $\mathrm{M} \in \delta_1\text{-}\mathrm{Mod}(\mathcal{H}_K^\dagger)^{\mathrm{Sp}}$. Assume that K is sufficiently large so that M is quasi-unipotent. Let $(\mathcal{H}_K^\dagger)'$ be the smallest special extension of $\mathcal{H}_K^\dagger$ such that M is trivialized by $(\mathcal{H}_K^\dagger)'[\log(T)]$. Let $\widetilde{Y} \in GL_n(\widetilde{\mathcal{H}_K^\dagger}[\log(T)])$ be a fundamental matrix solution of the differential equation M. Then there exists a finite extension K'/K such that the matrix*

$$\widetilde{A}(q,T) := \sigma_q(\widetilde{Y}) \cdot \widetilde{Y}^{-1} \tag{8.4.2}$$

belongs to $GL_n(\mathcal{H}_{K'}^\dagger)$, for all $q \in \mathrm{D}_{K'}^-(1,1)$. In particular the operator σ_q acting on $\widetilde{\mathcal{E}_K^\dagger}$ stabilizes both $\widetilde{\mathcal{H}_K^\dagger}$ and $\widetilde{\mathrm{C}_K^{\text{ét}}}$, and hence M is $\widetilde{\mathrm{C}_K^{\text{ét}}}[\log(T)]$-extensible to the whole disk $\mathrm{D}^-(1,1)$ (cf. Definition 4.4).

Proof. We can suppose that $K = K'$. By Corollary 8.16, and by canonical extension (cf. Definition 8.15), one can assume that $\mathrm{M} = N$, or $\mathrm{M} = U_m$, where N is trivialized by a Galois étale extension $(\mathcal{H}_K^\dagger)'$ of $\mathcal{H}_K^\dagger$, and where U_m is defined over $\mathcal{H}_K^\dagger$ as in Corollary 8.16. The case $\mathrm{M} = U_m$ is trivial, since both the matrices of $\delta_1^{U_m}$ and of $\sigma_q^{U_m}$ can be described explicitly as in § 7.4.1. Let now $\mathrm{M} = \mathrm{N}$ (i.e. M is trivialized by $\widetilde{\mathcal{H}_K^\dagger}$). In this case the solution matrix $\widetilde{Y}$ lies in $GL_n((\mathcal{H}_K^\dagger)')$. The special extension $(\mathcal{H}_K^\dagger)'/\mathcal{H}_K^\dagger$ corresponds via the equivalence of § 8.2 to a finite Galois extension $\mathrm{F}/k((t))$. Let $\mathrm{G} := \mathrm{Gal}(\mathrm{F}/k((t)))$; then G acts on $(\mathcal{H}_K^\dagger)'$ by $\mathcal{H}_K^\dagger$-automorphisms, and moreover the fixed points under this action are exactly the elements of $\mathcal{H}_K^\dagger$ (cf. Lemma 8.7). After enlarging K, if necessary, for all $\gamma \in \mathrm{G}$, one has

$$\gamma(\widetilde{Y}) = \widetilde{Y} \cdot H_\gamma, \quad \text{with } H_\gamma \in GL_n(K). \tag{8.4.3}$$

Indeed by Lemma 8.18 the corresponding Galois extension $(\mathcal{E}_K^\dagger)'/\mathcal{E}_K^\dagger$ is generated by the entries of $\widetilde{Y}$. Hence $(\mathcal{E}_K^\dagger)'/\mathcal{E}_K^\dagger$ is a Picard–Vessiot extension of $\mathcal{E}_K^\dagger$ with *differential* Galois group G. It follows then by Picard–Vessiot theory that $H_\gamma \in GL_n(K)$ (cf. [vdPS03, Observation 1.26]). Since σ_q commutes with every $\gamma \in \mathrm{G}$(cf. § 8.2.1), one finds

$$\gamma(\widetilde{A}(q,T)) = \gamma(\sigma_q(\widetilde{Y}) \cdot \widetilde{Y}^{-1}) = \sigma_q(\widetilde{Y}) \cdot H_\gamma \cdot (\widetilde{Y} \cdot H_\gamma)^{-1} = \widetilde{A}(q,T). \tag{8.4.4}$$

Hence $\widetilde{A}(q,T)$ belongs to $\mathcal{H}_K^\dagger$, for all $|q-1| < 1$. □

Lemma 8.24. *Let* $\mathcal{A}_{K^{\mathrm{alg}}}(1,1) := \bigcup_{K'/K=\mathrm{finite}} \mathcal{A}_{K'}(1,1)$. *There exists an embedding* $\widetilde{\mathrm{C}^{\text{ét}}_K}[\log(T)] \subseteq \mathcal{A}_{K^{\mathrm{alg}}}(1,1)$ *commuting with the actions of* δ_1, *of* φ, *and of* σ_q, *for all* $q \in \mathrm{D}^-_{K^{\mathrm{alg}}}(1,1)$. *In other words* $\mathcal{A}_{K^{\mathrm{alg}}}(1,1)$ *is a* $\widetilde{\mathrm{C}^{\text{ét}}_K}[\log(T)]$-$(\sigma,\delta)$-*algebra over the disk* $\mathrm{D}^-_{K^{\mathrm{alg}}}(1,1)$, *and one has the following diagram of discrete* $\mathcal{H}^\dagger_K - (\sigma,\delta)$-*algebras over* $\mathrm{D}^-(1,1)$

$$\begin{array}{ccccccccc} \mathcal{A}_{K^{\mathrm{alg}}}(1,1) & \supset & \widetilde{\mathrm{C}^{\text{ét}}_K} & \subset & \widetilde{\mathcal{H}^\dagger_K} & \subset & \widetilde{\mathcal{E}^\dagger_K} & \subset & \widetilde{\mathcal{R}_K} \\ \cup & & \cup & & & & \cup & & \cup \\ \mathcal{A}_K(1,1) & \supset & \mathcal{H}^\dagger_K & & \subset & & \mathcal{E}^\dagger_K & \subset & \mathcal{R}_K \end{array} \tag{8.4.5}$$

Proof. In the following we assume K to be sufficiently large in order that every special object appearing in the proof is quasi-unipotent. Let $\mathrm{M} \in \delta_1\text{-Mod}(\mathcal{H}^\dagger_K)^{\mathrm{Sp}}$ be a special differential equation trivialized by $\widetilde{\mathcal{H}^\dagger_K}$. Let $(\mathrm{C}^{\text{ét}}_K)'$ be the smallest sub-$\mathcal{H}^\dagger_K$-algebra of $\widetilde{\mathcal{H}^\dagger_K}$ trivializing M. By definition $(\mathrm{C}^{\text{ét}}_K)'$ is generated over $\mathcal{H}^\dagger_K$ by the entries $\{\widetilde{y}_{i,j}\}_{i,j}$ of a matrix solution $\widetilde{Y}$ of M in $\widetilde{\mathcal{H}^\dagger_K}$. We consider $\mathcal{H}^\dagger_K[\log(T)]$, $(\mathrm{C}^{\text{ét}}_K)'[\log(T)]$, $\widetilde{\mathrm{C}^{\text{ét}}_K}[\log(T)]$, $\mathcal{A}_K(1,1)$ as differential algebras (we forget the actions of σ_q in a first time). We have an embedding $\mathcal{H}^\dagger_K[\log(T)] \subset \mathcal{A}_K(1,1)$ commuting with δ_1 sending the symbol $\log(T)$ into the power series $\sum_{n\geqslant 1}(-1)^{n-1}(T-1)^n/n \in \mathcal{A}_K(1,1)$. We extend this embedding to $(\mathrm{C}^{\text{ét}}_K)'[\log(T)]$ as follows. Since the differential equation M has its coefficients in $\mathcal{H}^\dagger_K$ we can consider its Taylor solutions $Y(T,1)$ at the point 1. Since M is solvable, then $Y(T,1) \in GL_n(\mathcal{A}_K(1,1))$. Let now $\mathcal{F}_K := \mathrm{Frac}(\mathcal{H}^\dagger_K)$ be the field of fractions of $\mathcal{H}^\dagger_K$. Since $\mathcal{F}_K$ is a field, then (up to enlarged K) we can apply the Picard–Vessiot theory to obtain an isomorphism $\mathcal{F}_K[\{\widetilde{y}_{i,j}\}_{i,j}] \xrightarrow{\sim} \mathcal{F}_K[\{y_{i,j}(T,1)\}_{i,j}]$, sending $\widetilde{y}_{i,j}$ into $y_{i,j}(T,1)$, and commuting with δ_1. Clearly this isomorphism identifies $(\mathrm{C}^{\text{ét}}_K)' = \mathcal{H}^\dagger_K[\{\widetilde{y}_{i,j}\}_{i,j}]$ with $\mathcal{H}^\dagger_K[\{y_{i,j}(T,1)\}_{i,j}] \subset \mathcal{A}_K(1,1)$. If M' is another differential equation, and if $\mathcal{H}^\dagger_K[\{\widetilde{y}'_{i,j}\}_{i,j}]$ is the corresponding Picard–Vessiot extension identified with $\mathcal{H}^\dagger_K[\{y'_{i,j}(T,1)\}_{i,j}] \subset \mathcal{A}_K(1,1)$, then the embedding corresponding to $\mathrm{M}\oplus\mathrm{M}'$ extends these two embeddings since the entries of a solution of $\mathrm{M}\oplus\mathrm{M}'$ are the families $\{\widetilde{y}_{i,j},\widetilde{y}'_{h,k}\}_{i,j,h,k}$ and $\{y_{i,j}(T,1),y'_{h,k}(T,1)\}_{i,j,h,k}$ respectively. It is hence clear that this family of embeddings are compatible, so that we obtain an embedding $\widetilde{\mathrm{C}^{\text{ét}}_K} \subseteq \mathcal{A}_K(1,1)$ commuting with δ_1, and consequently $\widetilde{\mathrm{C}^{\text{ét}}_K}[\log(T)] \subseteq \mathcal{A}_K(1,1)$ also commutes with δ_1. Notice that $\log(T)$ is algebraically free over $\mathcal{H}^\dagger_K$ and hence over $\widetilde{\mathrm{C}^{\text{ét}}_K}$ which is union of finite algebras over $\mathcal{H}^\dagger_K$ (cf. Lemma 8.7). We can now check that this embedding commutes with σ_q (respectively φ), by looking to its action on the entries $\{\widetilde{y}_{i,j}\}_{i,j}$ and $\{y_{i,j}(T,1)\}_{i,j}$. Hence it is enough to prove that the isomorphism $\mathcal{E}^\dagger_K[\{\widetilde{y}_{i,j}\}_{i,j}] \xrightarrow{\sim} \mathcal{E}^\dagger_K[\{y_{i,j}(T,1)\}_{i,j}]$ commutes with σ_q (respectively φ). Observe that, if we fix an embedding of $\mathcal{F}_K[\{y_{i,j}(T,1)\}_{i,j}]$ in a fixed algebraic closure of $\mathcal{E}^\dagger_K$, then $\mathcal{E}^\dagger_K[\{y_{i,j}(T,1)\}_{i,j}]$ is, by definition, the smallest field containing $\mathcal{E}^\dagger_K$ and $\{y_{i,j}(T,1)\}_{i,j}$. The actions of $\delta_1,\sigma_q,\varphi$ are defined on $\mathcal{E}^\dagger_K[\{y_{i,j}(T,1)\}_{i,j}]$ as the extensions of $\delta_1,\sigma_q,\varphi$ on $\mathcal{F}_K[\{y_{i,j}(T,1)\}_{i,j}]$. This extension exists since $\mathcal{H}^\dagger_K[\{y_{i,j}(T,1)\}_{i,j}]\cap\mathcal{E}^\dagger_K = \mathcal{H}^\dagger_K[\{\widetilde{y}_{i,j}\}_{i,j}]\cap\mathcal{E}^\dagger_K = \mathcal{H}^\dagger_K$ (cf. Lemma 8.7), and since $\delta_1,\sigma_q,\varphi$ act on $Y(T,1)$ by multiplication by matrices with coefficients in $\mathcal{H}^\dagger_K$ (cf. [Bou59, § 6, no. 1, Proposition 1]). By Lemma 8.19, there exists a unique extension of σ_q to $\mathcal{E}^\dagger_K[\{\widetilde{y}_{i,j}\}_{i,j}]$, and of course a unique extension of φ since $\mathcal{E}^\dagger_K[\{\widetilde{y}_{i,j}\}_{i,j}]/\mathcal{E}^\dagger_K$ is unramified. Hence the isomorphism $\mathcal{E}^\dagger_K[\{\widetilde{y}_{i,j}\}_{i,j}] \xrightarrow{\sim} \mathcal{E}^\dagger_K[\{y_{i,j}(T,1)\}_{i,j}]$ commutes with σ_q and φ. □

Remark 8.25. The same statement holds for $\mathcal{A}_K(c,1)$ instead of $\mathcal{A}_K(1,1)$, providing that $|c|=1$, $c\in K$, and $\varphi(c)=c$.

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Proof of Theorem 8.22. By Proposition 7.13, one has $(\sigma,\delta)\text{-Mod}(\mathcal{R}_K)^{(\phi)}_S = (\sigma,\delta)\text{-Mod}(\mathcal{R}_K)^{(\phi)}_{\mathrm{D}^-(1,1)}$, (respectively $\sigma\text{-Mod}(\mathcal{R}_K)^{(\phi)}_S = \sigma\text{-Mod}(\mathcal{R}_K)^{(\phi)}_{\mathrm{D}^-(1,1)}$). On the other hand, $(\sigma,\delta)\text{-Mod}(\mathcal{R}_K)^{(\phi)}_{\mathrm{D}^-(1,1)} = \sigma\text{-Mod}(\mathcal{R}_K)^{(\phi)}_{\mathrm{D}^-(1,1)}$ (cf. (2.4.3)). Moreover, if $q \in \mathrm{D}^-(1,1) - \boldsymbol{\mu}_{p^\infty}$, then $(\sigma,\delta)\text{-Mod}(\mathcal{R}_K)^{(\phi)}_{\mathrm{D}^-(1,1)} = (\sigma_q,\delta_q)\text{-Mod}(\mathcal{R}_K)^{(\phi)}$ (respectively $\sigma\text{-Mod}(\mathcal{R}_K)^{(\phi)}_{\mathrm{D}^-(1,1)} = \sigma_q\text{-Mod}(\mathcal{R}_K)^{(\phi)}$). Hence, without loss of generality, we can assume that M is a Taylor admissible (σ_q,δ_q)-module, with Frobenius structure. The proof follows now by the discussion after Theorem 8.22. □

COROLLARY 8.26. *Let $S \subset \mathrm{D}^-(1,1) - \boldsymbol{\mu}_{p^\infty}$ (respectively $S \subset \mathrm{D}^-(1,1)$). We have the equalities*

$$\mathrm{Conf}_q^{\mathrm{Tay}} \overset{\textit{Rem. } 7.5}{=} \mathrm{Conf}_q^{\mathcal{A}_K(1,1)} = \mathrm{Conf}_q^{\mathcal{A}_{K^{\mathrm{alg}}}(1,1)} \overset{\textit{Lemma } 8.24}{=} \mathrm{Conf}_q^{\widetilde{\mathrm{C}^{\text{ét}}_K}[\log(T)]}, \tag{8.4.6}$$

where the first three equalities hold for these functors on $\sigma\text{-Mod}(\mathcal{H}^\dagger_K)^{[1]}_S$ (respectively (σ,δ)-Mod$(\mathcal{H}^\dagger_K)^{[1]}_S$), while, in the last equality, one considers the restrictions of these functors to the full subcategory of $\sigma\text{-Mod}(\mathcal{H}^\dagger_K)^{[1]}_S$ (respectively $(\sigma,\delta)\text{-Mod}(\mathcal{H}^\dagger_K)^{[1]}_S$) of objects trivialized by $\widetilde{\mathrm{C}^{\text{ét}}_K}[\log(T)]$. In particular the last equality holds on $\sigma\text{-Mod}(\mathcal{H}^\dagger_K)^{\mathrm{Sp}}_S$ (respectively $(\sigma,\delta)\text{-Mod}(\mathcal{H}^\dagger_K)^{\mathrm{Sp}}_S$). The same relation holds for deformation functors.

Proof. By Remark 7.5 the restriction of $\mathrm{Conf}_q^{\mathrm{Tay}}$ to the category of *solvable* objects coincides with $\mathrm{Conf}_q^{\mathcal{A}_K(1,1)}$. A *solvable* object over $\mathcal{H}^\dagger_K$ is trivialized by $\mathcal{A}_{K^{\mathrm{alg}}}(1,1)$ if and only if it is trivialized by $\mathcal{A}_K(1,1)$. Indeed both these conditions are verified if and only if its Taylor solution at 1 converges on $\mathrm{D}^-(1,1)$. Hence $\mathrm{Conf}_q^{\mathcal{A}_K(1,1)} = \mathrm{Conf}_q^{\mathcal{A}_{K^{\mathrm{alg}}}(1,1)}$ on solvable objects. Now, by Theorem 8.22, special objects are trivialized by $\widetilde{\mathrm{C}^{\text{ét}}_K}[\log(T)]$ hence by $\mathcal{A}_{K^{\mathrm{alg}}}(1,1)$ (cf. Lemma 8.24). □

8.5 The confluence of André–Di Vizio

In this last section we prove that the restriction of $\mathrm{Conf}_q^{\mathrm{Tay}}$ to $\sigma_q\text{-Mod}(\mathcal{R}_K)^{(\phi)}$ is isomorphic to the functor 'Conf' defined in [ADV04, § 15.1]. In all this last section $q \in \mathrm{D}^-(1,1) - \boldsymbol{\mu}_{p^\infty}$.

We recall that an *antecedent* of a σ_q-module M over $\mathcal{R}_K$ is a σ_{q^p}-module M_1 such that $\phi^*(\mathrm{M}_1)$ is isomorphic to M as σ_q-module. The antecedent is unique up to isomorphisms, because this fact is true for differential equations (cf. Remark 8.4). In order to preserve the notation of [ADV04], we fix an $s \geqslant 1$, and we call M_1 the sth antecedent of M, i.e. $\Phi : (\phi^*)^s(\mathrm{M}_1) \xrightarrow{\sim} \mathrm{M}$.

The following definition was given in [ADV04] under the assumption $|q-1| < |p|^{1/(p-1)}$. The same definition holds for $q \in \mathrm{D}^-(1,1) - \boldsymbol{\mu}_{p^\infty}$.

DEFINITION 8.27 [ADV04, Definition 12.11]. Let $s \in \mathbb{N}_{>0}$. A *confluent weak frobenius structure* (CWFS) on a σ_q-module $\mathrm{M}_0 := (\mathrm{M}_0, \sigma_q^{\mathrm{M}_0}) \in \sigma_q\text{-Mod}(\mathcal{R}_K)$ is a sequence $\{\sigma^{\mathrm{M}_m}_{q^{p^{sm}}}\}_{m\geqslant 0}$ of $q^{p^{sm}}$-difference operators on M_0, together with a family of isomorphisms

$$\Phi_m : ((\phi^*)^s(\mathrm{M}_0), (\phi^*)^s(\sigma^{\mathrm{M}_{m+1}}_{q^{p^{s(m+1)}}})) \xrightarrow{\sim} (\mathrm{M}_0, \sigma^{\mathrm{M}_m}_{q^{p^{sm}}}), \tag{8.5.1}$$

of $q^{p^{sm}}$-difference modules (identifying $(\mathrm{M}_0, \sigma^{\mathrm{M}_m}_{q^{p^{sm}}})$ to the sth antecedent of $(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$), such that:

(i) the operators $\Delta^{\mathrm{M}_m}_{q^{p^{sm}}} := (\sigma^{\mathrm{M}_m}_{q^{p^{sm}}} - \mathrm{Id}^{\mathrm{M}_0})/(q^{p^{sm}} - 1)$ converge to a derivation $\Delta^{\mathrm{M}_\infty}$ on M_0;

(ii) if $\mathrm{M}_\infty := (\mathrm{M}_0, \Delta^{\mathrm{M}_\infty})$ is this differential module, then the sequence of isomorphisms (8.5.1) converges to a Frobenius isomorphism $\Phi_\infty : \phi^*(\mathrm{M}_\infty) \xrightarrow{\sim} \mathrm{M}_\infty$.

We denote by

$$\sigma_q\text{-Mod}(\mathcal{R}_K)^{\mathrm{conf}(\phi)} \tag{8.5.2}$$

the category whose objects are families of operators $(\mathrm{M}_0, \{\sigma^{\mathrm{M}_m}_{q^{p^{sm}}}\}_{m\geqslant 0})$ on M_0 admitting the existence of a family $\{\Phi_m\}_{m\geqslant 0}$ making it on a confluent weak Frobenius structure on $(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$. A morphism

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$\alpha : (\mathrm{M}_0, \{\sigma_{q^{p^m}}^{\mathrm{M}_m}\}_{m\geqslant 0}) \longrightarrow (\mathrm{N}_0, \{\sigma_{q^{p^m}}^{\mathrm{N}_m}\}_{m\geqslant 0})$ is an $\mathcal{R}_K$-linear morphism $\alpha : \mathrm{M}_0 \to \mathrm{N}_0$ verifying simultaneously $\alpha \circ \sigma_{q^{p^{sm}}}^{\mathrm{M}_m} = \sigma_{q^{p^{sm}}}^{\mathrm{N}_m} \circ \alpha$, for all $m \geqslant 0$.

8.5.1 *Construction of CWFSs.* A q-difference module $(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$ admits infinitely many confluent weak Frobenius structures (CWFS), even if $(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$ admits a (strong) Frobenius structure. Indeed if a CWFS $(\mathrm{M}_0, \{\sigma_{q^{p^{sm}}}^{\mathrm{M}_m}\}_{m\geqslant 0}, \{\Phi_m\}_{m\geqslant 0})$ on $(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$ is given, we now give an algorithm to produce infinitely many CWFSs on $(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$. Let $\{\psi_m : \mathrm{M}_0 \xrightarrow{\sim} \mathrm{M}_0\}_{m\geqslant 0}$ be a sequence of $\mathcal{R}_K$-linear automorphisms of M_0 such that $\lim_m \psi_m = \mathrm{Id}^{\mathrm{M}_0}$. Define

$$\sigma_{q^{p^{sm}}}^{\mathrm{M}'_m} := \psi_m \circ \sigma_{q^{p^{sm}}}^{\mathrm{M}_m} \circ \psi_m^{-1}, \quad \Phi'_m := \psi_m \circ \Phi_m \circ [\phi^*(\psi_{m+1})]^{-1}. \tag{8.5.3}$$

One easily checks that $(\mathrm{M}_0, \{\sigma_{q^{p^{sm}}}^{\mathrm{M}'_m}\}_m, \{\Phi'_m\}_m)$ is again a CWFS on $(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$. Notice that this new CWFS is not always isomorphic to the first one (even if $\psi_0 = \mathrm{Id}^{\mathrm{M}_0}$). Indeed, by definition, an isomorphism is a single arrow $\alpha : \mathrm{M}_0 \to \mathrm{M}_0$ satisfying *simultaneously* $\alpha \circ \sigma_{q^{p^{sm}}}^{\mathrm{M}_m} = \sigma_{q^{p^{sm}}}^{\mathrm{M}'_m} \circ \alpha$, for all $m \geqslant 0$. Nevertheless, since $\lim_m \psi_m = \mathrm{Id}^{\mathrm{M}_0}$, the limit differential equation is the same for all CWFSs defined in this way (cf. Remark 8.29). We observe moreover that ψ_m defines an isomorphism of $q^{p^{sm}}$-difference modules between $(\mathrm{M}_0, \sigma_{q^{p^{sm}}}^{\mathrm{M}_m})$ and $(\mathrm{M}_0, \sigma_{q^{p^{sm}}}^{\mathrm{M}'_m})$; this agrees with the uniqueness of the antecedent by Frobenius. If $\widetilde{Y}_m$ is the solution of $(\mathrm{M}_0, \sigma_{q^{p^{sm}}}^{\mathrm{M}_m})$ in $GL_n(\widetilde{\mathcal{R}_K[\log(T)]})$, and if $B_m(T) \in GL_n(\mathcal{R}_K)$ is the matrix of ψ_m, then the solution of $(\mathrm{M}_0, \sigma_{q^{p^{sm}}}^{\mathrm{M}'_m})$ is given by $B_m(T)\widetilde{Y}_m$.

Remark 8.28. Assume that M_0 admits a (strong) Frobenius structure. The constancy of the solution does not follow from the preview definition. Indeed a solution of $(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$ with values in C is a morphism $\alpha : \mathrm{M}_0 \to \mathrm{C}$ satisfying $\alpha \circ \sigma_q^{\mathrm{M}_0} = \sigma_q^{\mathrm{C}} \circ \alpha$ (cf. § 3). The fact that α is a solution of $(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$ does not imply that α commutes also with $\sigma_{q^{p^{sm}}}^{\mathrm{M}_m}$. Indeed the data $(\mathrm{M}_0, \{\sigma_{q^{p^{sm}}}^{\mathrm{M}_m}\}_{m\geqslant 0})$ is *not necessarily* a discrete σ-module over $S = \{q^{p^{sm}}\}_{m\geqslant 0}$, because the map $S \to \mathrm{Aut}^{\mathrm{cont}}(\mathrm{M}_0)$ sending q into $\sigma_{q^{p^{sm}}}^{\mathrm{M}_m}$ is not supposed to have any coherency (cf. Remark 2.5(3)). To obtain the constancy of the solutions we need to *rigidify* these constructions by introducing the notion of C-*constant* σ-module (cf. Remarks 0.2 and 0.1, and Example 2.6).

8.5.2 We have an evident fully faithful functor

$$\chi_q^{(\phi)} : \sigma_q\text{-Mod}(\mathcal{R}_K)^{(\phi)} \longrightarrow \sigma_q\text{-Mod}(\mathcal{R}_K)^{\mathrm{conf}(\phi)} \tag{8.5.4}$$

defined by

$$\chi_q^{(\phi)}(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0}) := (\mathrm{M}_0, \{(\sigma_q^{\mathrm{M}_0})^{p^{sm}}\}_{m\geqslant 0}), \tag{8.5.5}$$

where s is sufficiently large to have an isomorphism $\Phi : (\phi^*)^s(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0}) \xrightarrow{\sim} (\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$, and $\Phi_m := \Phi$, for all $m \geqslant 0$. On the other hand we have another functor (cf. [ADV04, § 12.3])

$$Lim_\infty^{(\phi)} : \sigma_q\text{-Mod}(\mathcal{R}_K)^{\mathrm{conf}(\phi)} \longrightarrow \delta_1\text{-Mod}(\mathcal{R}_K)^{(\phi)} \tag{8.5.6}$$

sending $(\mathrm{M}_0, \{\sigma_{q^{p^{sm}}}^{\mathrm{M}_m}\}_{m\geqslant 0}, \{\Phi_m\}_{m\geqslant 0})$ into its limit differential equation $(\mathrm{M}_0, \Delta^{\mathrm{M}_\infty})$. We have actually

$$Lim_\infty^{(\phi)} \circ \chi_q^{(\phi)} = \mathrm{Conf}_q^{\mathrm{Tay}}. \tag{8.5.7}$$

Indeed if $(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$ has a (strong) Frobenius structure, then $(\mathrm{M}_0, \{(\sigma_q^{\mathrm{M}_0})^{p^{sm}}\}_{m\geqslant 0})$ is a (solvable) Taylor admissible σ-module over $S := \{q^{p^{sm}}\}_{m\geqslant 0}$ (cf. Definition 7.4). Hence, by § 7.2.2, the differential equation $\mathrm{Conf}_q^{\mathrm{Tay}}(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$ is given by the limit $\Delta^{\mathrm{M}_\infty} := \lim_{m\to\infty} \Delta_{q^{p^{sm}}}^{\mathrm{M}_m}$ of Definition 8.27. Moreover, since the operator $\sigma_{q^{p^{sm}}}^{\mathrm{M}_m}$ is determined by the knowledge of the solutions of $(\mathrm{M}_0, \sigma_{q^{p^{sm}}}^{\mathrm{M}_m})$ in

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$GL_n(\widetilde{\mathcal{R}_K}[\log(T)])$, then $\chi_q^{(\phi)}(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$ is the *unique* CWFS on $(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$ such that the fundamental matrix solution of $(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$ in $GL_n(\widetilde{\mathcal{R}_K}[\log(T)])$ (or equivalently its Taylor solution in $\mathcal{A}_K(1,1)$, cf. Lemma 8.24) is simultaneously a solution of every $(\mathrm{M}_0, \sigma_{q^{p^{sm}}}^{\mathrm{M}_m})$.

Remark 8.29. It is not clear whether the limit differential equation $(\mathrm{M}_0, \Delta^{\mathrm{M}_\infty})$ depends on the particular CWFS on $(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$ or, analogously, if there exists two non-isomorphic q-difference modules endowed with CWFS giving the same limit differential equation. Indeed both these phenomena arise in the category σ_q-Mod$(\mathcal{R}_K)^{\mathrm{conf}}$ defined below (cf. Definition 8.31).

LEMMA 8.30. *If K is algebraically closed, then the functor $\chi_q^{(\phi)}$ is isomorphic to the functor $D_{\sigma_q}^{\mathrm{conf}(\phi)} \circ V_{\sigma_q}^{(\phi)}$ of [ADV04, Corollary 14.8]. Hence the functor $\mathrm{Conf}_q^{\mathrm{Tay}}(\overset{(8.5.7)}{=} Lim_\infty^{(\phi)} \circ \chi_q^{(\phi)})$ is isomorphic to the confluence functor $\mathrm{Conf} := Lim_\infty^{(\phi)} \circ D_{\sigma_q}^{\mathrm{conf}(\phi)} \circ V_{\sigma_q}^{(\phi)}$ as it was defined in [ADV04, § 15.1].*

Proof. As explained in the introduction, $V_{\sigma_q}^{(\phi)}(\mathrm{M}, \sigma_q^{\mathrm{M}})$ (respectively $V_d^{(\phi)}(\mathrm{M}, \delta_1^{\mathrm{M}})$) is the (dual of the) space of solutions of $(\mathrm{M}, \sigma_q^{\mathrm{M}})$ (respectively $(\mathrm{M}, \delta_1^{\mathrm{M}})$) in $\widetilde{\mathcal{R}_K}[\log(T)]$. By definition $D_d^{(\phi)} \circ V_d^{(\phi)} \cong \mathrm{Id}$, and $D_{\sigma_q}^{(\phi)} \circ V_{\sigma_q}^{(\phi)} \cong \mathrm{Id}$. Then $D_d^{(\phi)} \circ V_{\sigma_q}^{(\phi)} = \mathrm{Conf}_q^{\widetilde{\mathcal{R}_K}[\log(T)]}$ is the functor sending $(\mathrm{M}, \sigma_q^{\mathrm{M}})$ into the differential equation having the same solutions in $\widetilde{\mathcal{R}_K}[\log(T)]$. By definition (cf. [ADV04, Proposition 12.17]) one has $D_{\sigma_q}^{\mathrm{conf}(\phi)} \cong \chi_q^{(\phi)} \circ D_{\sigma_q}^{(\phi)}$. This proves that the functor Conf of [ADV04] is equal to $\mathrm{Conf}_q^{\widetilde{\mathcal{R}_K}[\log(T)]}$. By Corollary 8.26 we conclude. □

8.5.3 Lemma 8.30 clarifies the nature of the functor Conf of [ADV04] (cf. Corollary 8.26). Indeed Conf is equal to $\mathrm{Conf}_q^{\mathrm{Tay}}$, and sends a q-difference equation into the differential equation having the same Taylor solutions (or equivalently having the same 'étale' solutions in $\widetilde{\mathcal{R}_K}[\log(T)]$, cf. Lemma 8.24 and Corollary 8.26). This functor actually does not depend on the existence of a Frobenius structure and exists in the more general context of *admissible* modules. This generalizes the constructions of [ADV04] to all $q \in \mathrm{D}^-(1,1) - \boldsymbol{\mu}_{p^\infty}$, removing also the assumption $K = K^{\mathrm{alg}}$. Notice that the equivalence provided by the propagation theorem requires only the definition and the formal properties of the Taylor solution $Y(x,y)$. For this reason the equivalences $\mathrm{Conf}_q^{\mathrm{Tay}}$ and $\mathrm{Def}_{q,q'}^{\mathrm{Tay}}$ are not a consequence of the previously developed theory. Conversely our confluence implies the main results of [ADV04] and also of [DV04].

8.5.4 *A conjecture of [ADV04].* Section 8.5.1 proves that the fully faithful functor $\chi_q^{(\phi)}$ is not an equivalence. This answers a question asked in [ADV04, Corollary 14.8, and after]. Nevertheless observe that the existence of a CWFS on $(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$ is equivalent to the existence of a *strong* Frobenius structure on it. This was first proved for rank one equations (cf. [ADV04, Proposition 7.3]; the case with rational coefficient follows actually from § 7.5; indeed every rank one equation with rational exponent has a (strong) Frobenius structure). The general case is proved as follows (cf. Corollary 8.33).

DEFINITION 8.31. We define

$$\sigma_q\text{-Mod}(\mathcal{R}_K)^{\mathrm{conf}} \tag{8.5.8}$$

as the category whose objects are $\mathcal{R}_K$-modules M together with a family of σ_q-semi-linear automorphisms $\{\sigma_{q^{p^{sm}}}^{\mathrm{M}} : \mathrm{M} \xrightarrow{\sim} \mathrm{M}\}_{m \geqslant 0}$ (without any condition of compatibility) such that the limit

$$\delta_1^{\mathrm{M}} := \lim_{m \to \infty} \frac{\sigma_{q^{p^{sm}}}^{\mathrm{M}} - \mathrm{Id}}{q^{p^{sm}} - 1} \tag{8.5.9}$$

converges to a connection δ_1^{M} on M. Morphisms between $(\mathrm{M},\{\sigma^{\mathrm{M}}_{q^{p^{sm}}}\}_{m\geqslant 0})$ and $(\mathrm{N},\{\sigma^{\mathrm{N}}_{q^{p^{sm}}}\}_{m\geqslant 0})$ are $\mathcal{R}_K$-linear morphisms $\alpha:\mathrm{M}\to\mathrm{N}$ satisfying simultaneously $\alpha\circ\sigma^{\mathrm{M}}_{q^{p^{sm}}}=\sigma^{\mathrm{N}}_{q^{p^{sm}}}\circ\alpha$, for all $m\geqslant 0$.

Remark 8.32. We have a functor

$$Lim_\infty:\sigma_q\text{-Mod}(\mathcal{R}_K)^{\mathrm{conf}}\longrightarrow\delta_1\text{-Mod}(\mathcal{R}_K) \tag{8.5.10}$$

sending $(\mathrm{M},\{\sigma^{\mathrm{M}}_{q^{p^{sm}}}\}_{m\geqslant 0})$ into its limit differential equation. Indeed if $\alpha:\mathrm{M}\to\mathrm{N}$ satisfies simultaneously $\alpha\circ\sigma^{\mathrm{M}}_{q^{p^{sm}}}=\sigma^{\mathrm{N}}_{q^{p^{sm}}}\circ\alpha$, for all $m\geqslant 0$, then, by passing to the limit, one has $\alpha\circ\delta_1^{\mathrm{M}}=\delta_1^{\mathrm{N}}\circ\alpha$. We have then the following commutative diagram of categories:

$$\begin{array}{ccccc}\sigma_q\text{-Mod}(\mathcal{R}_K)^{[r]} & \xrightarrow{\chi_q} & \sigma_q\text{-Mod}(\mathcal{R}_K)^{\mathrm{conf}} & \xrightarrow{Lim_\infty} & \delta_1\text{-Mod}(\mathcal{R}_K)\\ \cup & \odot & \cup & \odot & \cup\\ \sigma_q\text{-Mod}(\mathcal{R}_K)^{(\phi)} & \xrightarrow[\chi_q^{(\phi)}]{} & \sigma_q\text{-Mod}(\mathcal{R}_K)^{\mathrm{conf}(\phi)} & \xrightarrow[Lim_\infty^{(\phi)}]{} & \delta_1\text{-Mod}(\mathcal{R}_K)^{(\phi)}\end{array} \tag{8.5.11}$$

where $r\geqslant|q-1|$, and where χ_q sends $(\mathrm{M},\sigma_q^{\mathrm{M}})$ into $(\mathrm{M},\{(\sigma_q^{\mathrm{M}})^{p^{sm}}\}_{m\geqslant 0})$. By § 7.2.2, as above

$$Lim_\infty\circ\chi_q=\mathrm{Conf}_q^{\mathrm{Tay}}:\sigma_q\text{-Mod}(\mathcal{R}_K)^{[r]}\xrightarrow{\ \sim\ }\delta_1\text{-Mod}(\mathcal{R}_K)^{[r]}\subset\delta_1\text{-Mod}(\mathcal{R}_K). \tag{8.5.12}$$

Corollary 8.33. *Let $q\in\mathrm{D}^-(1,1)-\boldsymbol{\mu}_{p^\infty}$. Let $(\mathrm{M},\sigma_q^{\mathrm{M}})\in\sigma_q\text{-Mod}(\mathcal{R}_K)^{[r]}$, with $r\geqslant|q-1|$. Then $(\mathrm{M},\sigma_q^{\mathrm{M}})$ admits a CWFS if and only if it admits a (strong) Frobenius structure.*

Proof. Assume that $Lim_\infty\circ\chi_q(\mathrm{M},\sigma_q^{\mathrm{M}})$ lies in $\delta_1\text{-Mod}(\mathcal{R}_K)^{(\phi)}$. By (8.1.12), $\mathrm{Def}_{1,q}^{\mathrm{Tay}}\circ Lim_\infty\circ\chi_q(\mathrm{M},\sigma_q^{\mathrm{M}})$ lies in $\sigma_q\text{-Mod}(\mathcal{R}_K)^{(\phi)}$. Now since $\mathrm{Def}_{1,q}^{\mathrm{Tay}}\circ Lim_\infty\circ\chi_q=\mathrm{Def}_{1,q}^{\mathrm{Tay}}\circ\mathrm{Conf}_q^{\mathrm{Tay}}=\mathrm{Id}$, then $\mathrm{Def}_{1,q}^{\mathrm{Tay}}\circ Lim_\infty\circ\chi_q(\mathrm{M},\sigma_q^{\mathrm{M}})$ is isomorphic to $(\mathrm{M},\sigma_q^{\mathrm{M}})$, and hence has (strong) Frobenius structure. □

8.6 The theory of slopes

In a sequence of papers, Christol and Mebkhout developed a theory of slopes for p-adic differential equations over the Robba ring. We summarize the main properties in the following theorem.

Theorem 8.34 (cf. [CM02]). *Let M be a solvable differential module over $\mathcal{R}_K$. There exists a unique decomposition of M, called* break decomposition

$$\mathrm{M}=\bigoplus_{x\in\mathbb{R}_{\geqslant 0}}\mathrm{M}(x), \tag{8.6.1}$$

satisfying the following properties. Let t_ρ be a generic point for the norm $|\cdot|_\rho$ (cf. (6.1.1)). Then there exists $\varepsilon>0$ such that:

(i) *for all $\rho\in]1-\varepsilon,1[$, $\mathrm{M}(x)$ is the biggest submodule of M trivialized by $\mathcal{A}_K(t_\rho,\rho^{x+1})$;*

(ii) *for all $\rho\in]1-\varepsilon,1[$, and for all $y<x$, $\mathrm{M}(x)$ has no solutions in $\mathcal{A}_K(t_\rho,\rho^{y+1})$.*

The number $\mathrm{Irr}(\mathrm{M}):=\sum_{x\geqslant 0}x\cdot\mathrm{rank}_{\mathcal{R}_K}(\mathrm{M}(x))$ is called the p-adic irregularity *of M, and it lies in $\mathbb{N}$.*

The fact that Irr(M) is integer is known as the *Hasse–Arf property*. This theorem has an analogous in the theory of representations of the Galois group of a local field.

Proposition 8.35 (cf. [Kat88]). *Let $\mathcal{I}$, $\mathcal{P}$ be the inertia and the wild inertia subgroups of $\mathrm{G}:=\mathrm{Gal}(k((t))^{\mathrm{sep}}/k((t)))$. Denote by $\{\mathcal{I}^{(x)}\}_{x\geqslant 0}$ the 'upper numbering filtration' of $\mathcal{I}$. Let V be a $\mathbb{Z}[1/p]$-representation of G, such that $\mathcal{P}$ acts through a finite discrete quotient. Then V admits a* break decomposition *$\mathrm{V}=\bigoplus_{x\geqslant 0}\mathrm{V}(x)$ of G-submodules $\mathrm{V}(x)$ such that $\mathrm{V}(0)=\mathrm{V}^{\mathcal{P}}$, and for all $x>0$:*

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(i) $(\mathrm{V}(x))^{\mathcal{I}^{(x)}} = 0$;

(ii) *for all* $y > x$, $(\mathrm{V}(x))^{\mathcal{I}^{(y)}} = \mathrm{V}(x)$.

The number $\mathrm{Swan}(\mathrm{V}) := \sum_{x \geqslant 0} \mathrm{rank}_{\mathbb{Z}[1/p]} \mathrm{V}(x)$ *is called the* Swan conductor *of* V, *and it lies in* $\mathbb{N}$.

For a very inspiring overview about this analogy we refer to [And04].

Different authors (cf. [Tsu98b, Mat02, Cre00]) proved that the equivalence functor introduced by Fontaine (cf. [Fon90, Tsu98a]), associating to a finite representation of G, a (φ, ∇)-module over $\mathcal{E}_K^\dagger$ (and hence a differential module over $\mathcal{R}_K$), preserves the break decompositions. *The Swan conductor of a representation equals the irregularity of the corresponding differential equation.*

In [And02] André stated a family of axiomatic conditions in a general Tannakian category in order to have a *theory of slopes*. The previous two cases respect the formalism of [And02].

In a second paper he conjectured (cf. [And04, Conjecture 4.2]) that a similar theory of slopes should exist also for $\sigma_q\text{-Mod}(\mathcal{R}_K)^{(\phi)}$ and asked if this 'new' theory of slopes is compatible with that of Christol and Mebkhout on $\delta_1\text{-Mod}(\mathcal{R}_K)^{(\phi)}$ (via the confluence), and hence with the ramification theory on $\underline{\mathrm{Rep}}_{K^{\mathrm{alg}}}(\mathcal{I}_{k^{\mathrm{alg}}((t))} \times \mathbb{G}_a)$ (via the Fontaine's functor T_1 of the introduction). He suggested to proceed in analogy with the theory of Christol and Mebkhout (cf. [CM02]), reproducing their proofs in the context of q-difference equations in order to obtain a statement analogous to Theorem 8.34. Finally he asked whether this 'new' theory of slopes on $\sigma_q\text{-Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$ is compatible or not with the theory of slopes of Christol and Mebkhout in $\delta_1\text{-Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$ via the equivalence Conf that he obtained in [ADV04] for $|q-1| < |p|^{1/(p-1)}$.

Afterwards, at the end of 2005, he actually obtained such a theory of slopes for $\sigma_q\text{-Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$, with $|q-1| < |p|^{1/(p-1)}$, and established the two corollaries below in this case. These verifications will be included in a forthcoming paper of André. This part was given by André at the *24th Nordic and 1st Franco-Nordic Congress of Mathematicians* (6 to 9 January 2006, Reykjavik, Iceland).

The next corollaries prove the above conjecture in the more general context of σ-modules. We prove it for all $|q-1| < 1$, without any assumptions about the Frobenius structure, and without assuming $K = K^{\mathrm{alg}}$. The equivalence established by Corollary 7.14 gives in fact the following analog of Theorem 8.34 for σ-modules and (σ, δ)-modules. Thanks to Proposition 7.13, without loss of generality, we can reduce this statement to the case $S = \{q\}$.

COROLLARY 8.36. *Let* $|q-1| < 1$, $q \in K$ *(respectively* $q \notin \boldsymbol{\mu}_{p^\infty}$*). Let* $\mathrm{M} \in (\sigma_q, \delta_q)\text{-Mod}(\mathcal{R}_K)^{[1]}$ *(respectively* $\mathrm{M} \in \sigma_q\text{-Mod}(\mathcal{R}_K)^{[1]}$*). Then* M *admits* a break decomposition $\mathrm{M} = \bigoplus_{x \geqslant 0} \mathrm{M}(x)$, *where* $\mathrm{M}(x)$ *is characterized by the following properties (analogs to (i) and (ii) of Theorem 8.34). There exists* $\varepsilon > 0$ *such that:*

(i) *for all* $\rho \in\,]1-\varepsilon, 1[$, $\mathrm{M}(x)$ *is the biggest submodule of* M *trivialized by* $\mathcal{A}_K(t_\rho, \rho^{x+1})$;

(ii) *for all* $\rho \in\,]1-\varepsilon, 1[$, *and for all* $y < x$, $\mathrm{M}(x)$ *has no solutions in* $\mathcal{A}_K(t_\rho, \rho^{y+1})$.

This decomposition is compatible with the confluence, i.e. $\mathrm{M}(x) = \mathrm{Def}_{1,q}^{\mathrm{Tay}}(\mathrm{Conf}_q^{\mathrm{Tay}}(\mathrm{M})(x))$. *In particular the irregularity* $\mathrm{Irr}_{\sigma_q}(\mathrm{M}) := \sum_{x \geqslant 0} x \cdot \mathrm{rank}_{\mathcal{R}_K} \mathrm{M}(x)$ *is a natural number.*

Proof. The *slopes* and the *irregularity* are defined, by Christol and Mebkhout [CM02], by means of the generic *radius of the Taylor solutions*. The K-linear equivalences $\mathrm{Conf}_q^{\mathrm{Tay}}$ and $\mathrm{Def}_{1,q}^{\mathrm{Tay}}$ preserve, by definition, the generic Taylor solution. It follows immediately that the q-difference equation inherits then, via the equivalence $\mathrm{Conf}_q^{\mathrm{Tay}}$, the *slopes* of the attached differential equation, together with their formal properties (break decomposition, Hasse–Arf property, ...). □

COROLLARY 8.37. *With the notation of [ADV04] and [And04], if* $K = K^{\mathrm{alg}}$ *is algebraically closed, the functor* $D_{\sigma_q}^{(\phi)} : \sigma_q\text{-Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)} \longrightarrow \underline{\mathrm{Rep}}_{K^{\mathrm{alg}}}(\mathcal{I}_{k^{\mathrm{alg}}((t))} \times \mathbb{G}_a)$ *preserves the slopes (by Corollary 8.36 on the left-hand side, and by the Swan conductor on the right-hand side).*

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Proof. One has $D^{(\phi)}_{\sigma_q} = D^{(\phi)}_d \circ \mathrm{Conf}^{\mathrm{Tay}}_q$. Since $D^{(\phi)}_d$ and $\mathrm{Conf}^{\mathrm{Tay}}$ preserve the slopes, so does $D^{(\phi)}_{\sigma_q}$. □

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