Canad. J. Math. Vol. 75 (3), 2023 pp. 1018–1044
http://dx.doi.org/10.4153/S0008414X2200027X


# Multiplicity one theorems over positive characteristic

Dor Mezer

*Abstract.* In Aizenbud et al. (2010, *Annals of Mathematics* 172, 1407–1434), a multiplicity one theorem is proved for general linear groups, orthogonal groups, and unitary groups ($GL$, $O$, and $U$) over $p$-adic local fields. That is to say that when we have a pair of such groups $G_n \subseteq G_{n+1}$, any restriction of an irreducible smooth representation of $G_{n+1}$ to $G_n$ is multiplicity-free. This property is already known for $GL$ over a local field of positive characteristic, and in this paper, we also give a proof for $O$, $U$, and $SO$ over local fields of positive odd characteristic. These theorems are shown in Gan, Gross, and Prasad (2012, *Sur les Conjectures de Gross et Prasad. I, Société Mathématique de France*) to imply the uniqueness of Bessel models, and in Chen and Sun (2015, *International Mathematics Research Notice* 2015, 5849–5873) to imply the uniqueness of Rankin–Selberg models. We also prove simultaneously the uniqueness of Fourier–Jacobi models, following the outlines of the proof in Sun (2012, *American Journal of Mathematics* 134, 1655–1678).

By the Gelfand–Kazhdan criterion, the multiplicity one property for a pair $H \leq G$ follows from the statement that any distribution on $G$ invariant to conjugations by $H$ is also invariant to some anti-involution of $G$ preserving $H$. This statement for $GL$, $O$, and $U$ over $p$-adic local fields is proved in Aizenbud et al. (2010, *Annals of Mathematics* 172, 1407–1434). An adaptation of the proof for $GL$ that works over of local fields of positive odd characteristic is given in Mezer (2020, *Mathematische Zeitschrift* 297, 1383–1396). In this paper, we give similar adaptations of the proofs of the theorems on orthogonal and unitary groups, as well as similar theorems for special orthogonal groups and for symplectic groups. Our methods are a synergy of the methods used over characteristic 0 (Aizenbud et al. [2010, *Annals of Mathematics* 172, 1407–1434]; Sun [2012, *American Journal of Mathematics* 134, 1655–1678]; and Waldspurger [2012, *Astérisque* 346, 313–318]) and of those used in Mezer (2020, *Mathematische Zeitschrift* 297, 1383–1396).

## 1 Introduction

Let $\mathbb{F}$ be a local field of positive characteristic different from 2. Let $\mathbb{K}$ be either equal to $\mathbb{F}$ or an extension of it of degree 2. Let $V$ be a vector space of dimension $n$ over $\mathbb{K}$. Let $W := V \oplus \mathbb{K}v_{n+1}$ be an $(n+1)$-dimensional vector space containing it. Assume that we have a nondegenerate Hermitian (symmetric in the case $\mathbb{K} = \mathbb{F}$) form on $W$, with respect to which $V$ is orthogonal to $v_{n+1}$. Note that this implies, in particular, that $\langle v_{n+1}, v_{n+1}\rangle \neq 0$. We will denote $G$ to be either the group $O$ or $SO$ in the case $\mathbb{K} = \mathbb{F}$ or $U$ in the case $\mathbb{K} \neq \mathbb{F}$. Consider the group $G(V)$ as a subgroup of $G(W)$.

The following theorem is among the main theorems proved in this paper.

Received by the editors July 1, 2021; revised February 1, 2022; accepted June 13, 2022.
Published online on Cambridge Core June 22, 2022.
D.M. was partially supported by ERC StG Grant No. 637912.
AMS subject classification: **22E50**, 20G05, 20G25, 46F10.
Keywords: Distribution, multiplicity one, Gelfand pair, invariant distribution.

***Theorem 1.1*** *Let $\pi$ and $\rho$ be irreducible smooth representations of $G(W)$ and $G(V)$, respectively. Then*

$$\dim \mathrm{Hom}_{G(V)}(\pi, \rho) \leq 1.$$

Let us define an anti-involution $\sigma$ of $G(W)$ for all three families of classical groups:

In the case of $O$, define $\sigma : g \mapsto g^{-1}$.

In the case of $SO$, choose $T \in G(V)$ of order 2 with $\det T = (-1)^{\lfloor \frac{n+1}{2} \rfloor}$: One may choose such an element by taking a basis of $V$ with respect to which the symmetric form is diagonal, and in this basis, take a diagonal matrix of $\pm 1$ with the appropriate parity of $-1$ entries. Define $\sigma : g \mapsto Tg^{-1}T$.

In the case of $U$, choose a basis of $W$ for which the Hermitian product of all pairs lies in $\mathbb{F}$ (for example, by choosing a basis that diagonalizes the Hermitian form). Then we have an involution $T : v \mapsto \bar{v}$ (writing $v$ as a vector in this basis). Define an anti-involution of $G(W)$ by $\sigma : g \mapsto Tg^{-1}T$.

Consider the action of $G(V)$ on $G(W)$ by conjugation. The following theorem implies Theorem 1.1 using the Gelfand–Kazhdan criterion.

***Theorem 1.2*** *Any $G(V)$-invariant distribution on $G(W)$ is also invariant under $\sigma$.*

The proof of this implication in zero characteristic is given in [12, Appendix B], [3, Section 1], and [14], and the same proofs apply verbatim in arbitrary odd characteristic.

We also prove another theorem which we shall now describe, given in [12] for characteristic 0. One may also look there for more extensive explanations about the basic notations and definitions used. This theorem will be related to the uniqueness of Fourier–Jacobi models, and will regard all the previous families of classical groups, as well as $Sp$.

Let $A$ be a finite-dimensional commutative involutive algebra over $\mathbb{F}$, and let $V$ be a finitely generated $A$-module. Let $\varepsilon = \pm 1$, and let $\tau$ be the involution of $A$. Assume that $V$ is equipped with a nondegenerate $\varepsilon$-Hermitian form, i.e., a nondegenerate $\mathbb{F}$-bilinear map $\langle \cdot, \cdot \rangle : V \times V \to A$ satisfying $A$-linearity in the first argument, and $\langle v, u \rangle = \varepsilon \langle u, v \rangle^{\tau}$. Denote by $S$ the group of all $A$-module automorphisms of $V$ which preserve this form. It is a finite product of general linear groups, unitary groups, orthogonal groups, and symplectic groups. Denote by $A^{\tau=-\varepsilon}$ the subset of $A$ of elements $a$ satisfying $a^{\tau} = -\varepsilon a$. Let $H$ be the Heisenberg group defined as $\{(v, t) | v \in V, t \in A^{\tau=-\varepsilon}\}$ with multiplication

$$(v, t)(v', t') = \left( v + v', t + t' + \frac{\langle v, v' \rangle}{2} - \frac{\langle v', v \rangle}{2} \right).$$

We have a natural action of $S$ on $H$. Denote by $J := H \rtimes S$ the semidirect product of $H$ and $S$ with respect to this action. We prove in this paper the following theorem.

***Theorem 1.3*** *Let $\pi$ and $\rho$ be irreducible smooth representations of $J$ and $S$, respectively. Then*

$$\dim \mathrm{Hom}_{S}(\pi, \rho) \leq 1.$$

Again, we shall use the method of Gelfand and Kazhdan. Choose an $\mathbb{F}$-linear involution $\sigma$ of $V$ such that $\langle \sigma u, \sigma v \rangle = \langle v, u \rangle$. To see that it is always possible, write

https://doi.org/10.4153/S0008414X2200027X Published online by Cambridge University Press

$G$ as a product of unitary, orthogonal, symplectic, and general linear groups with the corresponding orthogonal decomposition of $V$ over $\mathbb{F}$. In each of these cases, we are able to construct such an involution, and we can take the product of these involutions as $\sigma$. The involution $\sigma$ extends to an anti-involution of $H$ by $(v, t) \mapsto (-\sigma v, t)$, which together with the anti-involution $g \mapsto \sigma g^{-1}\sigma$ of $S$ gives an anti-involution of $J$ (which we shall also call $\sigma$). Let $S$ act on $J$ by conjugation. The following theorem implies Theorem 1.3 using the Gelfand–Kazhdan criterion.

***Theorem 1.4*** *Any $S$-invariant distribution on $J$ is also invariant under $\sigma$.*

The implication is proved in [12, Appendix B] in zero characteristic, and the same proof applies verbatim in arbitrary odd characteristic.

Sections 3–8 are dedicated to the simultaneous proof of Theorems 1.2 and 1.4.

***Remark 1.5*** One could ask whether an analog of Theorem 1.1 holds for the subgroup $SU$ of Hermitian operators with determinant 1. However, the answer turns out to be negative even for the most simple case of $\dim V = 1$. In this case, $SU(V) = \{1\}$, while $SU(W)$ is noncommutative, showing that the two groups do not satisfy a multiplicity one property.

***Remark 1.6*** The assumption of $\mathbb{F}$ having positive characteristic is not used anywhere in this paper, and is given mainly to distinguish the work in this paper from the already existing proofs of the discussed theorems when the characteristic of $\mathbb{F}$ is 0.

## 1.1 Corollaries of the main theorems

There are corollaries of Theorems 1.1 and 1.3 which were shown over characteristic 0, and the proofs of implication still apply over positive characteristic. We list them here with references.

An analog of the following theorem appears in [12] as Theorem B, and its implication from the analog of Theorem 1.3 is shown in Appendix A of the same.

***Theorem 1.7*** *Let $G$ denote one of the groups $GL(n), U(n)$, and $Sp(2n)$ (note that $U(n)$ is dependent on the choice of a Hermitian form), regarded as a subgroup of $Sp(2n)$ as usual. Let $\widetilde{G}$ be the double cover of $G$ induced by the metaplectic cover $\widetilde{Sp}(2n)$ of $Sp(2n)$. Denote by $\omega_\psi$ the smooth oscillator representation of $\widetilde{Sp}(2n)$ corresponding to a nontrivial character $\psi$ of $\mathbb{F}$. Then, for any irreducible smooth representation $\pi$ of $G$, and for any genuine irreducible smooth representation $\pi'$ of $\widetilde{G}$, one has that*

$$\dim \mathrm{Hom}_G(\pi' \otimes \omega_\psi \otimes \pi, \mathbb{C}) \leq 1.$$

An analog of the following theorem appears in [7], along with its implication from the analog of the above theorem (see [7] for the definitions and notation used).

***Theorem 1.8*** (Uniqueness of Rankin–Selberg models) *For all irreducible smooth representations $\pi$ of $GL(n)$ and $\sigma$ of $GL(r)$, and for all generic characters $\chi$ of the rth Rankin–Selberg subgroup $R_r$ of $GL(n)$, one has that*

$$\dim \mathrm{Hom}_{R_r}(\pi \otimes \sigma, \chi) \leq 1.$$

Analogs of the following two theorems appear in [8, Chapters 12–16], along with their implications from the analogs of Theorems 1.1 and 1.3 and the multiplicity

https://doi.org/10.4153/S0008414X2200027X Published online by Cambridge University Press

one property for $GL(n+1), GL(n)$, which is proved for positive characteristic in [9, Theorem 1.2] and in [1].

***Theorem 1.9*** (Uniqueness of Bessel models) *Let $V$ be a linear space with a symmetric or Hermitian (including the case $\mathbb{K} = \mathbb{F} \times \mathbb{F}$) form. Denote the respective orthogonal or unitary group $G(V)$. Let $W$ be a subspace of odd codimension on which the form is nondegenerate and so that $W^\perp$ is split. Let $H$ be the Bessel group corresponding to $W$, considered as a subgroup of $G(V) \times G(W)$, and let $\nu$ be a generic character of $H$. Then, for any irreducible smooth representations $\pi$ of $G(V)$ and $\pi'$ of $G(W)$, one has*

$$\dim \mathrm{Hom}_H(\pi \otimes \pi', \nu) \leq 1.$$

***Theorem 1.10*** (Uniqueness of Fourier–Jacobi models) *Let $V$ be a linear space with a skew-symmetric or skew-Hermitian (including the case $\mathbb{K} = \mathbb{F} \times \mathbb{F}$) form. Denote the respective symplectic or unitary group $G(V)$. Let $W$ be a subspace of even codimension on which the form is nondegenerate and so that $W^\perp$ is split. Let $H$ be the Fourier–Jacobi group corresponding to $W$, considered as a subgroup of $G(V) \times G(W)$. Let $\widetilde{H}$ be its appropriate double cover, and let $\nu$ be the representation of $\widetilde{H}$ constructed in [8, Chapter 12] (depending on some choices of characters). Take either $\pi$ to be an irreducible smooth representations of $\widetilde{G}(V)$ (an appropriate double cover of $G(V)$) and $\pi'$ to be such a representation of $G(W)$, or the other way around, i.e., $\pi$ to be an irreducible smooth representation of $G(V)$ and $\pi'$ to be such a representation of an appropriate $\widetilde{G}(W)$. Then one has*

$$\dim \mathrm{Hom}_{\widetilde{H}}(\pi \otimes \pi', \nu) \leq 1.$$

## 1.2 Comparison with previous works

In [9], the proof of a multiplicity one theorem for $GL_n$ in characteristic 0 is extended to include also positive odd characteristic. The premise of this paper is to use these methods to extend the proof of additional multiplicity one theorems from characteristic 0 to positive odd characteristic. The proofs for characteristic 0 on which we base this paper are given in [3, 12, 14]. Let us give an overview of the methods and steps of this paper, explaining which ones are taken from [3, 12, 14], which ones were introduced in [9], and which ones are new to this paper.

In Section 3, we give reformulations of the problems in a way identical to the ones given in [3, 12, 14].

In Section 4, we use a certain analog of the Harish–Chandra descent method for positive characteristic, that gives weaker results than in the zero characteristic case. The entirety of this method as used in [3, 12, 14] fails over positive characteristic fields, due to nonseparable extensions, and in fact, this is the crucial point in which these proofs fail for a positive characteristic.

In Section 5, we pass from the group to its Lie algebra using Cayley transform. The difference from the analogous linearization in [3, 12, 14] is that, in these papers, linearization is done after using the method of Harish–Chandra descent to restrict the possible support to the unipotent cone, whereas we only have a weaker restriction on the support.

In Section 6, we adapt the main new ideas of [9] to the unitary, orthogonal, and symplectic settings, introducing a new family $\rho$ of automorphisms playing the same role as $\rho$ in [9].

Section 7 uses the method of stratification to reduce the problem to a problem on a single orbit. The contents of this section are completely analogous to what is done in [3, 12, 14], only without the restriction of nilpotency, which is not truly needed, as was the case in [9].

In Section 8, we solve the previous problem on a single orbit by repeating the arguments and ideas used in [3, 12, 14], sometimes giving slight generalizations of them.

The archimedean version of Theorems 1.1 and 1.4 can be found in [13]. Special cases of Theorems 1.1 and 1.2 can be found in [4].

## 2 Preliminaries and notation

Most of this section is borrowed from the preliminaries sections of [2, 3], and also of [9] (which was also mostly borrowed from the previous two).

Let us now introduce a uniform notation for all the groups $O, SO, U$, and $Sp$. Note that the case $G = Sp$ was not included in Theorems 1.1 and 1.2, but it will be relevant for Theorems 1.3 and 1.4. Let $\mathbb{F}$ be a local field of characteristic different from 2. Let $\mathbb{K}$ be a field which is either equal to $\mathbb{F}$ or a quadratic field extension of it. Let $\lambda \mapsto \bar{\lambda}$ be either the nontrivial automorphism of $\mathbb{K}/\mathbb{F}$ or the identity automorphism if $\mathbb{K} = \mathbb{F}$. Let $V$ be a $\mathbb{K}$-linear space of dimension $n$. Assume that we have on $V$ a nondegenerate sesquilinear form $B$ which is either symmetric, Hermitian, or symplectic (in the Hermitian case $\mathbb{K} \neq \mathbb{F}$ and in the other cases $\mathbb{K} = \mathbb{F}$). Denote by $G = G(V)$ one of the groups $O(V), SO(V), U(V)$, or $Sp(V)$. Denote by $\mathfrak{g} = \mathfrak{g}(V)$ the Lie algebra of $G$, which is either $\mathfrak{o}(V), \mathfrak{u}(V)$, or $\mathfrak{sp}(V)$, i.e., linear transformations $A$ satisfying $A^* = -A$ with respect to the symmetric, Hermitian, or symplectic form. In the $O, SO, U$ cases, assume that we have $W \supseteq V$ of dimension $n+1$ with an extension of $\langle \cdot, \cdot \rangle$ to a form on $W$ of the same type. In these cases, we have also $G(W)$, and we may consider $G$ as a subgroup of it.

Let $\widetilde{G}$ denote the subgroup of $\mathrm{Aut}_{\mathbb{F}}(V) \times \{\pm 1\}$ consisting of all $(T, \delta)$ such that $\langle Tu, Tv \rangle = \langle u, v \rangle$ if $\delta = 1$ and $\langle Tu, Tv \rangle = \langle v, u \rangle$ if $\delta = -1$. In the case that $G = SO$, we also require that $\det T = \delta^{\lfloor \frac{n+1}{2} \rfloor}$. This group contains $G$ as a subgroup of index 2. Denote by $\chi : \widetilde{G} \to \pm 1$ the character $(T, \delta) \mapsto \delta$. We have natural actions of $G$ on $G(W), G, \mathfrak{g}, V$ (by conjugation on all but $V$, on which we let $G$ act in the usual way). This action extends to an action of $\widetilde{G}$ by $(T, \delta).A := TA^{\delta}T^{-1}$ on $G(W)$ and $G$, by $(T, \delta).A := \delta TAT^{-1}$ on $\mathfrak{g}$, and by $(T, \delta).v := \delta Tv$ on $V$.

***Notation 2.1*** Let $\Delta : G \to \mathbb{K}[x]$ be the characteristic polynomial map. We shall also consider it as a map from $G \times V$, by first projecting onto $G$.

We shall use the standard terminology of $l$-spaces introduced in [6, Section 1]. We denote by $\mathcal{S}(Z)$ the space of Schwartz functions on an $l$-space $Z$, and by $\mathcal{S}^*(Z)$ the space of distributions on $Z$ equipped with the weak topology.

***Notation 2.2*** (Fourier transform) Let $W$ be a finite-dimensional vector space over $\mathbb{F}$ with a nondegenerate bilinear form $B$ on $W$. We denote by $\mathcal{F}_B : \mathcal{S}^*(W) \to \mathcal{S}^*(W)$ the

https://doi.org/10.4153/S0008414X2200027X Published online by Cambridge University Press

Fourier transform defined using $B$ and the self-dual Haar measure on $W$. If $W$ is clear from the context, we sometimes omit it from the notation and denote $\mathcal{F} = \mathcal{F}_W$.

***Remark 2.3*** In the Hermitian case, we take for Fourier transform the $\mathbb{F}$-bilinear form given by taking the trace of the Hermitian form.

***Theorem 2.4*** (Localization principle; see [5, Section 1.4]) *Let $q : Z \to T$ be a continuous map of l-spaces. We can consider $\mathcal{S}^*(Z)$ as an $\mathcal{S}(T)$-module. Denote $Z_t := q^{-1}(t)$. For any $M$ which is a closed $\mathcal{S}(T)$-submodule of $\mathcal{S}^*(Z)$,*

$$M = \overline{\bigoplus_{t \in T} (M \cap \mathcal{S}^*(Z_t))}.$$

Informally, it means that, in order to prove a certain property of distributions on $Z$, it is enough to prove that distributions on every fiber $Z_t$ have this property.

***Corollary 2.5*** *Let $q : Z \to T$ be a continuous map of l-spaces. Let an l-group $H$ act on $Z$ preserving the fibers of $q$. Let $\mu$ be a character of $H$. Suppose that, for any $t \in T$, $\mathcal{S}^*(q^{-1}(t))^{H,\mu} = 0$. Then $\mathcal{S}^*(Z)^{H,\mu} = 0$.*

***Corollary 2.6*** *Let $H_i \subset \widetilde{H}_i$ be l-groups acting on l-spaces $Z_i$, for $i = 1, \ldots, k$. Suppose that $\mathcal{S}^*(Z_i)^{H_i} = \mathcal{S}^*(Z_i)^{\widetilde{H}_i}$ for all i. Then $\mathcal{S}^*(\prod Z_i)^{\prod H_i} = \mathcal{S}^*(\prod Z_i)^{\prod \widetilde{H}_i}$.*

***Theorem 2.7*** (Frobenius descent [5, Section 1.5]) *Let $H$ be a unimodular l-group acting on two l-spaces $E$ and $Z$, with the action on $Z$ being transitive. Suppose that we have an $H$-equivariant map $\varphi : E \to Z$. Let $x \in Z$ be a point with a unimodular stabilizer in $H$. Denote by $F$ the fiber of $x$ with respect to $\varphi$. Then, for any character $\mu$ of $H$, the following holds:*

(i) *There exists a canonical isomorphism* $\mathrm{Fr} : \mathcal{S}^*(E)^{H,\mu} \to \mathcal{S}^*(F)^{\mathrm{Stab}_H(x),\mu}$.
(ii) *For any distribution $\xi \in \mathcal{S}^*(E)^{H,\mu}$,* $\mathrm{Supp}(\mathrm{Fr}(\xi)) = \mathrm{Supp}(\xi) \cap F$.
(iii) *Frobenius descent commutes with Fourier transform.*

To formulate (iii) explicitly, let $W$ be a finite-dimensional linear space over $\mathbb{F}$ with a nondegenerate bilinear form $B$, and suppose that $H$ acts on $W$ linearly preserving $B$. Then, for any $\xi \in \mathcal{S}^*(Z \times W)^{H,\mu}$, we have $\mathcal{F}_B(\mathrm{Fr}(\xi)) = \mathrm{Fr}(\mathcal{F}_B(\xi))$, where Fr is taken with respect to the projection $Z \times W \to Z$.

***Remark 2.8*** Let $Z$ be an $l$-space, and let $Q \subset Z$ be a closed subset. We may identify $\mathcal{S}^*(Q)$ with the space of all distributions on $Z$ supported on $Q$. In particular, we can restrict a distribution $\xi$ to any open subset of the support of $\xi$.

***Definition 2.9*** An element $A \in \mathrm{gl}(V)$ is said to be regular if its minimal polynomial is equal to its characteristic polynomial. In case that this polynomial $f$ of $A$ is a power of an irreducible polynomial, we call $A$ a *minimal* regular element.

***Theorem 2.10*** (Rational canonical form) *Any element $A \in \mathrm{gl}(V)$ can be represented as a direct sum of minimal regular elements. Moreover, the isomorphism classes of these elements are uniquely determined by the conjugacy class of $A$ inside $GL(V)$ and vice versa.*

This form is called the rational canonical form of $A$.

***Definition 2.11*** For any polynomial $f$ given by $f(x) = \sum_{i=0}^{n} a_i x^i$, define

$$f^*(x) = \sum_{i=0}^{n} (-1)^i \bar{a}_i x^i,$$

$$f^\dagger(x) = \sum_{i=0}^{n} \overline{a_{n-i}} x^i.$$

***Remark 2.12*** For any $A \in \mathfrak{g}$ (resp. $A \in G(V)$), we have $f(A)^* = f^*(A)$ (resp. $f(A)^\dagger = A^{-n} f^\dagger(A)$).

***Lemma 2.13*** *Let $A \in \mathfrak{g}$ ($A \in G$). Let $(f_i)_{i\in I}$ be the different irreducible factors in the characteristic polynomial of $A$. Let $(V_i)_{i\in I}$ be the generalized eigenspace associated with each. Take $i, j \in I$, not necessarily different. If $f_i^* \neq \pm f_j$ (resp. $f_i^\dagger \nmid f_j$), then $V_i \perp V_j$.*

**Proof** We give the proof in the $A \in \mathfrak{g}$ case (the $A \in G$ case is analogous). Take $u \in V_i, v \in V_j$. Then, for $k$ large enough,

$$0 = \langle f_i(A)^k u, v\rangle = \langle u, f_i^*(A)^k v\rangle.$$

Thus, $V_i \perp f_i^*(A)^k V_j$, but if $f_i^* \neq \pm f_j$, then they are coprime to each other, and so $f_i^*(A)^k V_j = V_j$. ∎

***Definition 2.14*** (1) An operator $A \in \mathfrak{g}$ will be called a simple split operator (or block) if the following conditions hold:

- There is a possibly nonorthogonal decomposition $V = V' \oplus \overline{V'^*}$.
- $V'$ and $\overline{V'^*}$ are isotropic, and the sesquilinear form $B$ induces the natural pairing between them.
- The action of $A$ on $V' \oplus \overline{V'^*}$ decomposes as $A' \oplus A''$.
- $\langle A'u, v\rangle = \langle u, -A''v\rangle$ for any $u \in V', v \in \overline{V'^*}$.
- $A'$ (and so also $A''$) is a minimal regular operator (see Definition 2.9).
- The irreducible factor $f$ of the minimal polynomial of $A'$ is not equal to $f^*$.

(2) An operator $A \in \mathfrak{g}$ will be called a simple nonsplit operator (or block) if:

- It is a minimal regular operator.
- Its characteristic polynomial is not equal to $x^d$ with $d$ even if $\mathfrak{g} = \mathfrak{o}$, and it is not equal to $x^d$ with $d$ odd if $\mathfrak{g} = \mathfrak{sp}$.

(3) An operator $A \in \mathfrak{o}$ will be called a simple even nilpotent operator (or block) if the following conditions hold:

- Its minimal polynomial is $x^d$ for some even $d$.
- $V$ has a basis of the form $e, Ae, \dots, A^{d-1}e, f, Af, \dots, A^{d-1}f$.
- For all $i, j$ we have $\langle A^i e, A^j e\rangle = \langle A^i f, A^j f\rangle = 0$.
- For all $i, j$ we have $\langle A^i e, A^j f\rangle = \begin{cases} (-1)^j, & \text{if } i + j = d - 1, \\ 0, & \text{otherwise.} \end{cases}$

(4) An operator $A \in \mathfrak{sp}$ will be called a simple odd nilpotent operator (or block) if the following conditions hold:

- Its minimal polynomial is $x^d$ for some odd $d$.
- $V$ has a basis of the form $e, Ae, \dots, A^{d-1}e, f, Af, \dots, A^{d-1}f$.
- For all $i, j$ we have $\langle A^i e, A^j e\rangle = \langle A^i f, A^j f\rangle = 0$.

- For all $i, j$ we have $\langle A^i e, A^j f\rangle = \begin{cases} (-1)^j, & \text{if } i+j=d-1,\\ 0, & \text{otherwise.}\end{cases}$

The following useful proposition will be proved in Appendix A.

***Proposition 2.15*** *Each $A \in \mathfrak{u}$ decomposes as an orthogonal sum of simple split blocks and simple nonsplit blocks. Each $A \in \mathfrak{o}$ decomposes as an orthogonal sum of simple split blocks, simple nonsplit blocks, and simple even nilpotent blocks. Each $A \in \mathfrak{sp}$ decomposes as an orthogonal sum of simple split blocks, simple nonsplit blocks, and simple odd nilpotent blocks.*

***Remark 2.16*** The contents of Proposition 2.15 are contained in known papers and books such as [11, 15]. However, for clarity and completeness, we formulated only the propositions we need and give short proofs of them in Appendix A.

***Proposition 2.17*** [9, Appendix A] *Let $V$ be a linear space of finite dimension $n$ over a field $\mathbb{K}$, $A \in \mathrm{gl}(V)$, $v \in V$, and $\phi \in V^*$. The following are equivalent.*

(1) $\forall k \geq 0,\ \phi A^k v = 0.$
(2) *For all* $\lambda \in \mathbb{K}$, $\mathrm{ch}(A + \lambda v \otimes \phi) = \mathrm{ch}(A)$.
(3) *There exists* $\lambda \in \mathbb{F}^\times$ *such that* $\mathrm{ch}(A + \lambda v \otimes \phi) = \mathrm{ch}(A)$.
(4) $\frac{\partial}{\partial \lambda}\mathrm{ch}(A + \lambda v \otimes \phi)|_{\lambda=0} = 0.$

Let $V$ be a vector space over a local field $\mathbb{F}$ of characteristic different from 2. Let $\mathrm{GL}(V)$ act on $\mathfrak{gl}(V) \times V \times V^*$ by conjugation, and the natural actions on $V, V^*$. Consider also the transposition involution, which involves a choice of an isomorphism $t : V \to V^*$, and sends $(A, v, \phi)$ to $A^t, \phi^t, v^t$. As immediate corollaries of [9, Theorem 3.1] (and of the proof that it implies Theorem 1.1 of the same paper), we have the following theorems.

***Theorem 2.18*** *Any* $\mathrm{GL}(V)$*-invariant distribution on $GL(V) \times V \times V^*$ is also invariant to transposition.*

***Theorem 2.19*** *Any* $\mathrm{GL}(V)$*-invariant distribution on $\mathfrak{gl}(V) \times V \times V^*$ is also invariant to transposition.*

## 3 Reformulations of the problem

Let $V, G, \widetilde{G}, \chi$ be as in Section 2. Both Theorems 1.2 and 1.4 follow from the following theorem.

***Theorem 3.1*** *Any $(\widetilde{G}, \chi)$-equivariant distribution on $G \times V$ is 0.*

**Proof that Theorem 3.1 implies Theorem 1.2** This proof is the same as the proof of [3, Proposition 5.1]. Use the notations given in the introduction (e.g., $V, W, G(V), \widetilde{G}(V)$). We actually prove that Theorem 3.1 for $W$ implies Theorem 1.2 for $V$. The idea is to consider the set

$$Y := \{w \in W | \langle w, w\rangle = \langle v_{n+1}, v_{n+1}\rangle\},$$

and use Frobenius descent (Theorem 2.7) on the projection $G(W) \times Y \to Y$. The group $\widetilde{G}(W)$ acts on $Y$ with centralizer $\widetilde{G}(V)$. The fiber of the projection is $G(W)$.

https://doi.org/10.4153/S0008414X2200027X Published online by Cambridge University Press

So we get a bijection between $(\widetilde{G}(V), \chi)$-equivariant distributions on $G(W)$ and $(\widetilde{G}(W), \chi)$-equivariant distributions on $G(W) \times Y$. The latter of which has no nonzero elements by the assumption on $W$. Thus, any $(\widetilde{G}(V), \chi)$-equivariant distribution on $G(W)$ is 0, which immediately implies Theorem 1.2. ∎

**Proof that Theorem 3.1 implies Theorem 1.4** The group $S$ in the formulation of Theorem 1.4 decomposes as a product of groups of types $O, SO, U, Sp$, with $\sigma$ a product of appropriate anti-involutions. By Corollary 2.5 of the localization principle, it follows that Theorem 3.1 implies an analogous claim for $S$, which can be seen to easily imply Theorem 1.4. ∎

The proof of the theorem is by induction on $\dim V$, proving simultaneously the following theorem.

***Theorem 3.2*** *Any $(\widetilde{G}, \chi)$-equivariant distribution on $\mathfrak{g} \times V$ is 0.*

For $n = 0$, Theorems 3.1 and 3.2 are trivial.

***Remark 3.3*** In [14], the needed induction basis was $n = 1, n = 2$, as the proof given there used the triviality of the center of $G$ (up to $\pm 1$). However, we do not use this fact and so the trivial case $n = 0$ suffices for us as a basis for the induction.

Theorem 3.2 (along with Theorem 3.1) is proved in the end of Section 7.

## 4 Harish–Chandra descent

In this section, we use the technique of Harish–Chandra descent to restrict the support of an equivariant distribution as discussed in Theorems 3.1 and 3.2. For the course of this section assume that Theorems 3.1 and 3.2 hold for all smaller dimensions, over all finite field extensions of $\mathbb{K}$.

Let $(A, v)$ be a point in the support of a $(\widetilde{G}, \chi)$-equivariant distribution either on $G \times V$ (the group case) or on $\mathfrak{g} \times V$ (the Lie-algebra case). Let $g(X)$ be the characteristic polynomial of $A$. Consider also the characteristic polynomial map $\Delta : G \times V \to \mathbb{K}[x]$ (or $\Delta : \mathfrak{g} \times V \to \mathbb{K}[x]$ in the Lie algebra case). Note that $g \mid g^\dagger$ in the group case, and $g = \pm g^*$ in the Lie algebra case (recall Definition 2.11 of $g^\dagger$ and $g^*$).

***Theorem 4.1*** *Unless we are in the group case and $G = SO$, the polynomial $g$ cannot be factorized into two coprime factors $g_1, g_2$ satisfying $g_1 \mid g_1^\dagger$ and $g_2 \mid g_2^\dagger$ (resp. $g_1 = \pm g_1^*$ and $g_2 = \pm g_2^*$ in the Lie algebra case). In the case $G = SO$, it is still true that it is impossible for $g$ to be divisible by both $x - 1$ and $x + 1$.*

**Proof** We give the proof for the group case and for the Lie algebra case simultaneously. By the localization principle (Corollary 2.5), it is enough to show that there is no $(\widetilde{G}, \chi)$-equivariant distribution $\xi$ on any of the fibers of $\Delta$ which is above a polynomial not satisfying the condition we gave on $g$. Let $F$ be such a fiber lying above a polynomial $g(x) = g_1(x)g_2(x)$ with $g_1, g_2$ coprime and of positive degree, satisfying $g_1 \mid g_1^\dagger$ and $g_2 \mid g_2^\dagger$ ($g_1 = \pm g_1^*$ and $g_2 = \pm g_2^*$ in the Lie algebra case). If we are in the group case and $G = SO$, we further assume that $g_1(x) = (x-1)^k$ for some $k > 0$. Let $d_1, d_2$ be the degrees of $g_1, g_2$. Given $A$ with characteristic polynomial $g(x)$, one may consider $V_1, V_2$, its generalized eigenspaces associated with $g_1(x), g_2(x)$, respectively.

By Lemma 2.13, $V_1, V_2$ are perpendicular to each other. Consider

$$\Lambda = \{V_1, V_2 \subset V | V = V_1 \oplus V_2,\ V_1 \perp V_2,\ \dim V_i = d_i\}$$

to be the space of decompositions of $V$ as an orthogonal sum $V_1 \oplus V_2$ to subspaces of dimensions $d_1, d_2$. There is a natural $\widetilde{G}$-equivariant map $\rho : F \to \Lambda$. Consider the stratification on $\Lambda$ given by $G$-orbits. Note that these are the same as $\widetilde{G}$-orbits. To show that this is indeed a stratification, we must show that there are finitely many $G$-orbits. Recall that there are finitely many isomorphism classes of sesquilinear forms of the same type as $B$ (symmetric, Hermitian, or symplectic) on a $\mathbb{K}$-vector space of a given dimension. If two elements in $\Lambda$ share the isomorphism classes of the restrictions of $B$ to $V_1, V_2$, then these isomorphisms can be extended orthogonally to an element of $G$ (in the case $G = SO$, it will only be an element of $O$. However, it is enough to prove that there are finitely many $O$ orbits). This implies that the two elements we had in $\Lambda$ are in the same $G$-orbit ($O$-orbit if $G = SO$). It follows that there are indeed finitely many $G$-orbits, and so partition into orbits is a stratification.

Let $S$ be the union of strata intersecting $\rho(\mathrm{supp}(\xi))$, and let $\Omega$ be a stratum of the largest dimension in it (we assume by contradiction that $\xi \neq 0$, i.e., $S$ is nonempty). It is open in $S$, and so we may restrict $\xi$ to $\rho^{-1}(\Omega)$. Since $\Omega \subseteq S$, this restriction is not the zero distribution.

For the following, assume that we are not in the groups case where $G = SO$. The action of $\widetilde{G}$ on $\Omega$ is transitive by definition, and the stabilizer of a point in $\Omega$ (call it $H$) is a subgroup of index 2 of $\widetilde{G}(V_1) \times \widetilde{G}(V_2)$, which is a unimodular group (thus, it is also unimodular). $H$ also contains $G(V_1) \times G(V_2)$ as a subgroup of index 2. Using Frobenius descent (Theorem 2.7) on $\xi$, we get an $(H, \chi)$-equivariant distribution on the fiber, which is a closed subspace of $(G(V_1) \times V_1) \times (G(V_2) \times V_2)$. In particular, this distribution is $G(V_1) \times G(V_2)$-invariant. Hence, this distribution is $\widetilde{G}(V_1) \times \widetilde{G}(V_2)$-invariant by the induction hypothesis and Corollary 2.6 to the Localization Principle. In particular, it is also $H$-invariant; thus, it is 0, in contradiction to our assumption.

In the case $G = SO$, we have a similar situation. The action of $\widetilde{SO}(V)$ on $\Omega$ is transitive, and the stabilizer of a point in $\Omega$, which we will call $H$, is a unimodular subgroup of index 4 inside $\widetilde{O}(V_1) \times \widetilde{O}(V_2)$. This group $H$ contains $SO(V_1) \times SO(V_2)$ as a subgroup of index 4, on which the character $\chi$ is trivial. Since the determinant of an operator acting on $V_1$ with characteristic polynomial $g_1(x) = (x-1)^k$ and on $V_2$ with characteristic polynomial $g_2(x)$ is 1, we get that $g_2(0) = (-1)^{\dim V_2}$. If $\dim V_2$ was odd, it would imply that $g_2^\dagger = -g_2$, and in particular $g_2(1) = 0$. By assumption, this is not the case, and so we have that $\dim V_2$ must be even. It follows that

$$(-1)^{\lfloor \frac{\dim V_1 + 1}{2} \rfloor} (-1)^{\lfloor \frac{\dim V_2 + 1}{2} \rfloor} = (-1)^{\lfloor \frac{\dim V_1 + 1}{2} \rfloor + \frac{\dim V_2}{2}} = (-1)^{\lfloor \frac{\dim V + 1}{2} \rfloor}.$$

Take elements $(g_1, -1) \in \widetilde{SO}(V_1) \backslash SO(V_1)$ and $(g_2, -1) \in \widetilde{SO}(V_2) \backslash SO(V_2)$. From the above, it follows that $\gamma := (g_1 \oplus g_2, -1)$ is an element of $(\widetilde{SO}(V_1) \times \widetilde{SO}(V_2)) \cap H$, on which $\chi$ gives $-1$. The fiber of $\rho$ above $(V_1, V_2)$ is $SO(V_1) \times SO(V_2)$, because any element in it acts with characteristic polynomial $(x-1)^k$ on $V_1$, and thus has determinant 1 when restricted to it. It follows that the restriction to $V_2$ also has determinant 1. As before, we get that any $(H, \chi)$-equivariant distribution on

https://doi.org/10.4153/S0008414X2200027X Published online by Cambridge University Press

$SO(V_1) \times SO(V_2)$ is invariant to $\widetilde{SO}(V_1) \times \widetilde{SO}(V_2)$ (using the localization principle and the induction hypothesis). In particular, it is $\gamma$-invariant, and thus it is 0, since $\chi(\gamma) = -1$. Using Frobenius descent, we get that this implies $\xi = 0$, giving a contradiction. ■

We give the following theorem only in the Lie algebra case as this is what will be used. However, it also holds in the group case, with the same proof.

***Theorem 4.2*** *Assume that we are in the Lie algebra case. The polynomial g must be a power of an irreducible polynomial.*

**Proof** Again, we use the localization principle. Let $F$ be the fiber above a polynomial of the form $g(x) = g_1(x)g_2(x)$ with $g_2 = \pm g_1^*$, and the two are coprime to each other. (By Theorem 4.1, it is enough to consider this case.) Given $A$ with characteristic polynomial $g(x)$, one may consider $V_1, V_2$, its generalized eigenspaces associated with $g_1(x), g_2(x)$, respectively. By Lemma 2.13, $V_1, V_2$ are both isotropic. Consider

$$\Lambda = \left\{ V_1, V_2 \subset V | V = V_1 \oplus V_2,\ B|_{V_1} = 0,\ B|_{V_2} = 0,\ \dim V_1 = \dim V_2 = \frac{\dim V}{2} \right\}.$$

There is a natural $\widetilde{G}$-equivariant map $\rho : F \to \Lambda$.

To see that $G$ acts transitively on $\Lambda$, take $(V_1, V_2), (V_1', V_2') \in \Lambda$. Choose arbitrary bases $E_1, E_1'$ of $V_1, V_1'$. We may take $E_2$ to be the basis of $V_2$ dual to $E_1$ with respect to the pairing between $V_1, V_2$ induced by $B$. Similarly, we may take $E_2'$. The linear transformation which sends $E_1$ to $E_1'$ and $E_2$ to $E_2'$ preserves $B$, and thus it is an element of $G$. So the actions of both $G$ and $\widetilde{G}$ on $\Lambda$ are transitive, and the stabilizer inside $\widetilde{G}$ of a point in $\Lambda$ is isomorphic to $\widetilde{\mathrm{GL}}(V_1)$, which is a unimodular group. Using Frobenius descent (Theorem 2.7) on $\xi$, we get a $(\widetilde{\mathrm{GL}}(V_1), \chi)$-equivariant distribution on the fiber, which is isomorphic to

$$\mathfrak{gl}(V_1) \times V_1 \times V_2 \cong \mathfrak{gl}(V_1) \times V_1 \times V_1^*.$$

By Theorem 2.19, this distribution must be equal to 0, and so is the original one. ■

We formulate the next theorem only for the Lie algebra case, and $\mathfrak{g} = \mathfrak{sp}$, although again it is also true for all the other cases.

***Theorem 4.3*** *Consider the Lie algebra case of $\mathfrak{g} = \mathfrak{sp}$. In this case, the irreducible factor of g is either linear or inseparable.*

**Proof** Again, we use the localization principle. Let $F$ be the fiber above a polynomial $g(x) = f(x)^s$ with $f$ irreducible, separable, of degree $d > 1$, and satisfying $f^* \neq \pm f$. Given $A$ with characteristic polynomial $g(x)$, we may consider its additive Jordan decomposition into semisimple and unipotent parts, $A_s$ and $A_u$ (that is in virtue of the characteristic polynomial being separable). Let $F_s$ be the space of possible $A_s$'s, that is the space of semisimple elements of $\mathfrak{g}$ with characteristic polynomial $g(x)$. We have a $\widetilde{G}$-equivariant map $\theta : F \to F_s$. By [11], $F_s$ is a disjoint union of finitely many $G$-orbits, all of the same dimension. By [10, Chapter 4, Proposition 1.2], for each point $A \in F_s$, there is an element in $\widetilde{G} \backslash G$ which centralizes it (see the details of this implication in Lemma 8.5). Thus, $G$-orbits in $F_s$ are $\widetilde{G}$-invariant. So it is enough to show that, for any orbit $\mathfrak{O} \subseteq F_s$, any $(\widetilde{G}, \chi)$-equivariant distribution

https://doi.org/10.4153/S0008414X2200027X Published online by Cambridge University Press

on $\theta^{-1}(\mathfrak{O})$ is 0. By Frobenius descent, this is equivalent to showing that, for some $A \in \mathfrak{O}$, any $(\widetilde{G}_A, \chi)$-equivariant distribution on $\theta^{-1}(A)$ is 0 ($\widetilde{G}_A$ being the stabilizer of $A$ in $\widetilde{G}$).

In order to prove this, let us describe the stabilizer of a point $A$ in $F_s$. Let $m := \mathbb{F}[T]/f(T)$. We define an $\mathbb{F}$-linear involution of $m$ by $\sigma : h(T) \mapsto h^*(T) = h(-T)$ (that is the same as saying $T \mapsto -T$). Let $m_0$ be the fixed subfield of this involution. It is a subfield of $m$ of index 2. Fix a nonzero $\mathbb{F}$-linear functional $\ell : m \to \mathbb{F}$ which satisfies that $\ell(\sigma h) = \ell(h)$ for all $h \in \mathbb{F}$. Any other $\mathbb{F}$-linear functional can be written as $h \mapsto \ell(\lambda h)$ for some unique $\lambda \in m$. Being semisimple, $f(A)$ must be equal 0. Thus, $V$ can be given the structure of a linear space over $m$, by $hv := h(A)v$. Given $v, v' \in V$, the map $h \mapsto \langle h(A)v, v'\rangle$ is an $\mathbb{F}$-linear functional $m \to \mathbb{F}$, and so can be written as $\langle h(A)v, v'\rangle = \ell(S(v, v')h)$ for some $S(v, v') \in m$. One may check that $S(v, v')$ is $m$-sesquilinear (with respect to $\sigma$) considering $V$ as a linear space over $m$ by $h \cdot v := h(A)v$. The form $S$ is also nondegenerate, and satisfies $S(v', v) = -\sigma S(v, v')$. Fix $a \in m$ such that $\sigma a = -a$ (e.g., $a = T \in \mathbb{F}[T]/f(T)$). Then it follows from the above that $aS(\cdot, \cdot)$ is a nondegenerate Hermitian form on $V_m$ (with respect to the involution $\sigma$), where $V_m$ is $V$ as a linear space over $m$. To say that a linear automorphism of $V$ commutes with $A$ is to say that it is $m$ linear, and for such an automorphism to say that it is in $G(V)$ is to say that it preserves $aS$. Thus, we have that the centralizer of $A$ in $G(V)$ can be described as $U(V_m)$. Moreover, the stabilizer of $A$ in $\widetilde{G}(V)$ can be described as $\widetilde{U}(V_m)$. Moreover, the centralizer of $A$ inside $\mathfrak{g}(v)$ can be described as $\mathfrak{u}(V_m)$.

Recall that we need to show that any $(\widetilde{U}(V_m), \chi)$-equivariant distribution on $\theta^{-1}(A)$ is 0. The space $\theta^{-1}(A)$ is identified with $\mathfrak{u}_n(V_m) \times V_m$, $\mathfrak{u}_n(V_m)$ being the space of nilpotent elements in $\mathfrak{u}(V_m)$. This in turn is a closed subspace of $\mathfrak{u}(V_m) \times V$. Thus, our claim follows from the fact that any $(\widetilde{U}(V_m), \chi)$-equivariant distribution on $\mathfrak{u}(V_m)$ is 0, which follows from our induction hypothesis of Theorem 3.1, as $\dim V_m < n$. ■

## 5 Separation of $1, -1$ as eigenvalues, and passage to the Lie algebra

In this section, we pass from the group case to the Lie algebra case, showing that Theorem 3.2 implies Theorem 3.1.

***Definition 5.1*** Let $G_{(1)}$ be the open subset of $G$ consisting of elements of which 1 is not an eigenvalue. Similarly, define $G_{(-1)}$ to be the open subset of elements of which $-1$ is not an eigenvalue. Define also $\Xi := G\backslash(G_{(1)} \cup G_{(-1)})$, i.e., elements of which both $1, -1$ are eigenvalues.

The following proposition is an immediate corollary of Theorem 4.1.

***Proposition 5.2*** *To prove Theorem 3.1, it is enough to show that any $(\widetilde{G}, \chi)$-equivariant distribution on $G_{(\pm1)} \times V$ is 0.*

***Definition 5.3*** (Cayley transform) Define $C_1 : G_{(1)}(V) \to \mathfrak{g}$ by $C_1(A) = \frac{I+A}{I-A}$. Similarly, define $C_{-1} : G_{(-1)}(V) \to \mathfrak{g}$ by $C_{-1}(A) = \frac{I-A}{I+A}$.

https://doi.org/10.4153/S0008414X2200027X Published online by Cambridge University Press

This definition makes sense since, for $A \in G_{(1)}(V)$,

$$\left(\frac{I+A}{I-A}\right)^* = \frac{I+A^*}{I-A^*} = \frac{I+A^{-1}}{I-A^{-1}} = \frac{A+I}{A-I} = -\frac{I+A}{I-A},$$

and similarly for $C_{-1}$.

***Definition 5.4*** Let $\mathfrak{g}_0$ be the subspace of $\mathfrak{g}$ not having $\pm 1$ as eigenvalues.

***Proposition 5.5*** *The maps $C_{\pm 1}$ are $\widetilde{G}$-homeomorphisms from $G_{(\pm 1)}(V)$ (respectively) to $\mathfrak{g}_0$, unless we are in the case $G = SO$, considering $C_1$, and* dim $V$ *is even. In this case, $SO_{(1)}(V) = \varnothing$.*

**Proof** First, exclude the case $G = SO$. We will give the proof for $C_1$. The proof for $C_{-1}$ is similar. First, notice that indeed $C_1(A)$ does not have $\pm 1$ as eigenvalues. If it did have, then $(I+A)v = \pm(I-A)v$, which leads to either $v = 0$ or $Av = 0$, and $A$ is invertible. Second, we construct an inverse, $B \mapsto \frac{B-I}{B+I}$: One can see that the inverse map is indeed into $G_{(1)}$, as

$$\left(\frac{B-I}{B+I}\right)^* = \frac{B^*-I}{B^*+I} = \frac{-B-I}{-B+I} = \frac{B+I}{B-I} = \left(\frac{B-I}{B+I}\right)^{-1}$$

and also $\frac{B-I}{B+I}$ cannot have 1 as an eigenvalue, as then $(B-I)v = (B+I)v$, and hence $v = 0$. To see that we indeed constructed an inverse map,

$$\frac{I + \frac{B-I}{B+I}}{I - \frac{B-I}{B+I}} = \frac{B+I+B-I}{B+I-(B-I)} = \frac{2B}{2I} = B,$$

$$\frac{\frac{I+A}{I-A} - I}{\frac{I+A}{I-A} + I} = \frac{I+A-(I-A)}{I+A+I-A} = \frac{2A}{2I} = A.$$

There are similar arguments for $C_{-1}$ showing it is a $\widetilde{G}$-isomorphism to $\mathfrak{g}_0$.

For the case $G = SO$, it simply holds that $SO_{(-1)} = O_{(-1)}, SO_{(1)} = O_{(1)}$ when dim $V$ is even, and $SO_{(1)} = \varnothing$ when dim $V$ is odd. ∎

***Proposition 5.6*** *To prove Theorem 3.1, it suffices to show that any $(\widetilde{G}, \chi)$-equivariant distribution on $\mathfrak{g}_0 \times V$ is 0.*

**Proof** Use $C_{\pm 1}$. ∎

***Proposition 5.7*** *Theorem 3.1 follows from Theorem 3.2.*

**Proof** Take $\xi$ to be a $(\widetilde{G}, \chi)$-equivariant distribution on $\mathfrak{g}_0 \times V$. Assume by contradiction that it is not 0, and let $(A_0, v_0)$ be a point in its support. Let $t = \det((A_0 - I)(A_0 + I)) \neq 0$. One can choose $f \in \mathcal{S}(\mathbb{K})$ s.t. $f(t) \neq 0, f(0) = 0$. Note that $\mathfrak{g}_0$ is an open subset of $\mathfrak{g}$, and $g(A) := f(\det((A-I)(A+I)))$ is a locally constant function, compactly supported inside $\mathfrak{g}_0$. Thus, we can extend $g \cdot \xi$ to a $(\widetilde{G}, \chi)$-equivariant distribution on $\mathfrak{g} \times V$ with $(A_0, v_0)$ in its support. In particular, this distribution is not 0, which creates a contradiction to our assumption. ∎

## 6 An important lemma and automorphisms

***Lemma 6.1*** *Any $(\widetilde{G}, \chi)$-equivariant distribution on $\mathfrak{g} \times V$ is supported on $\mathfrak{g} \times \Gamma$, where* $\Gamma := \{v \in V | \langle v, v \rangle = 0\}$.

**Proof** We do not consider in the following the case $\mathfrak{g} = \mathfrak{sp}$, as in this case $\Gamma = V$ and there is nothing to prove. This proof is the same as the proof of [3, Proposition 5.2]. The idea is to consider the map $\mathfrak{g} \times V \to \mathbb{K}$ given by $(A, v) \mapsto \langle v, v \rangle$, and apply the localization principle (Corollary 2.5) to it to restrict to a fiber. Then apply Frobenius descent (Theorem 2.7) on the projection on the second coordinate, to reach a point where it is enough to show that any $(\widetilde{G}(V'), \chi)$-equivariant distribution on $\mathfrak{g}(V')$ is 0, for some subspace $V' \subseteq V$ of codimension 1. We have a decomposition $\mathfrak{g} = \mathfrak{g}(V') \oplus V' \oplus E$, with $E$ being either a zero- or one-dimensional vector space over $\mathbb{F}$ with trivial $\widetilde{G}(V')$-action, and so we can use the induction hypothesis to finish. ∎

Denote by $\phi_v$ the linear transformation $u \mapsto \langle u, v \rangle v$.

The following definition will be relevant for the cases of $\mathfrak{u}$ and $\mathfrak{sp}$.

***Definition 6.2*** For any $\lambda \in \mathbb{K}$, requiring $\bar{\lambda} = -\lambda$ if $\mathfrak{g} = \mathfrak{u}$, we define an automorphism of $\mathfrak{g}$ by $\nu_\lambda(A, v) := (A + \lambda\phi_v, v)$. This is an automorphism of $\mathfrak{g}$ as a space with a $\widetilde{G}$ action.

The following definition will be relevant only for case of $\mathfrak{o}$.

***Definition 6.3*** For any $\lambda \in \mathbb{F}$, define an automorphism of $\mathfrak{g} \times V$ by

$$\mu_\lambda(A, v) := (A + \lambda A\phi_v + \lambda\phi_v A, v).$$

This is an automorphism of $\mathfrak{g} \times V$ as a space with a $\widetilde{G}$ action.

Fix a fiber $F$ of $\Delta : \mathfrak{g} \times V \to \mathbb{K}[x]$ at a polynomial $f$. Recall that we must have $f^*(x) = (-1)^n f(x)$. Choose a polynomial $g \in \mathbb{K}[x]$ coprime to $f$ that also satisfies $g^*(x) = g(x) \bmod f(x)$. Then we can define the following definition.

***Definition 6.4*** Define an automorphism of $F$ by $\rho_g(A, v) = (A, g(A)v)$.

To show that it is invertible, notice that there is an "inverse" polynomial $g^{-1}$ such that $gg^{-1} = 1 \bmod f$. It also satisfies $(g^{-1})^*(x) = g^{-1}(x) \bmod f(x)$, as for some polynomial $a$,

$$\begin{aligned} 1 &= ((g^{-1}g + af))^*(x) = (g^{-1})^*(x)g^*(x) + a^*(x)f^*(x) \\ &= (g^{-1})^*(x)g^*(x) + (-1)^n a^*(x)f(x). \end{aligned}$$

The last being equal to $(g^{-1})^*(x)g(x)$ modulo $f(x)$. This implies that we have $\rho_{g^{-1}}$ which is inverse to $\rho_g$. To show that $\rho_h$ commutes with the action of $\widetilde{G}$, the only nontrivial part is to show that it commutes with the action of an element $x \in \widetilde{G} \backslash G$. Consider $x$ as an element of $\mathrm{End}_{\mathbb{F}}(V)$ satisfying $\langle xu, xw \rangle = \langle w, u \rangle$ for any $u, w \in V$. In particular, $x$ satisfies $axu = x\bar{a}u$ for any $a \in \mathbb{K}, u \in V$. To show commutation, we need to show that $-xg(A)v = g(-xAx^{-1})(-xv)$. This is true as

$$g(-xAx^{-1})(-xv) = -xg^*(A)x^{-1}(xv) = -xg^*(A)v = -xg(A)v.$$

https://doi.org/10.4153/S0008414X2200027X Published online by Cambridge University Press

For the last equation, we used the condition imposed on $g$, and the fact that $f(A) = 0$.

Thus, we get that $\rho_g$ is a $\widetilde{G}$-automorphism of $F$.

In the case $\mathfrak{g} = \mathfrak{sp}$, we give the following lemma by using the automorphisms $\nu_\lambda$ to amplify the restriction of Theorem 4.3.

***Lemma 6.5*** *Assume that $\mathfrak{g} = \mathfrak{sp}$. Then any $(\widetilde{G}, \chi)$-equivariant distribution on $\mathfrak{g} \times V$ is supported on $\mathfrak{g} \times \Gamma_1$, where $\Gamma_1 := \{v \in V | \langle Av, v\rangle = 0\}$.*

**Proof** Given $A \in \operatorname{End}(V)$, write the characteristic polynomial of $A$ as

$$x^n + c_1(A)x^{n-1} + c_2(A)x^{n-2} + \cdots$$

with the convention that $c_0(A) = 1$. Let $(A, v)$ be a point in the support of a $(\widetilde{G}, \chi)$-equivariant distribution on $\mathfrak{g} \times V$. From Theorem 4.3, we deduce that we have $c_1(A) = n\alpha$ and $c_2(A) = \binom{n}{2}\alpha^2$ for some $\alpha \in \mathbb{F}$ (if the characteristic polynomial is a power of a linear polynomial this is clear, and if it is a power of an inseparable polynomial, then we indeed have $c_1(A) = c_2(A) = 0$). Note that using this condition, $c_1(A)$ determines uniquely $c_2(A)$. Now, we may translate by $\nu_\lambda$ and then apply the above condition to get $c_1(A + \lambda\phi_v) = n\beta, c_2(A + \lambda\phi_v) = \binom{n}{2}\beta^2$. However, by [9, Theorem A.2], we have

$$c_1(A + \lambda\phi_v) = c_1(A) - \langle v, v\rangle$$

and

$$c_2(A + \lambda\phi_v) = c_2(A) - c_1(A)\langle v, v\rangle - \langle Av, v\rangle.$$

Since $\langle v, v\rangle = 0$, this is saying that $c_1(A + \lambda\phi_v) = c_1(A)$, and hence $\beta = \alpha$ and so also $c_2(A + \lambda\phi_v) = c_2(A)$, and from the second equation $c_2(A + \lambda\phi_v) = c_2(A) - \langle Av, v\rangle$. Thus, $\langle Av, v\rangle = 0$. ∎

***Definition 6.6*** For any $A \in \mathfrak{g}$, denote

$$R_A := \{v \in V | \forall k \geq 0, \langle A^k v, v\rangle = 0\}.$$

Denote also

$$R = \{(A, v) \in \mathfrak{g} \times V | v \in R_A\}.$$

***Proposition 6.7*** *Any $(\widetilde{G}, \chi)$-equivariant distribution on $\mathfrak{g} \times V$ is supported on $R$.*

**Proof** By the localization principle (Corollary 2.5), it is enough to show that any $(\widetilde{G}, \chi)$-equivariant distribution on a fiber $F$ of $\Delta$ at a polynomial $f$ is supported on $R \cap F$ (note that $R$ is $\widetilde{G}$-invariant). Let $\xi$ be such a distribution, and let $(A, v)$ be a point in $\operatorname{supp}(\xi)$. Let us start with the case of $\mathfrak{u}$.

Choose $\omega \in \mathbb{K}^\times$ with $\bar{\omega} = -\omega$. Let $g \in \mathbb{F}[x]$ and consider $g_1(x) = g(x^2)$ and $g_2(x) = \omega x g(x^2)$. They satisfy $\overline{g_1}(-x) = g_1(x), \overline{g_2}(-x) = g_2(x)$. Choose $g$ such that $g_1$ will be coprime to $f$. We can apply $\rho_{g_1}$ to $\xi$ and extend back to $\mathfrak{g} \times V$ to get that by Lemma 6.1, $\langle g_1(A)v, g_1(A)v\rangle = 0$. We know this for a Zariski dense subset of polynomials

https://doi.org/10.4153/S0008414X2200027X Published online by Cambridge University Press

$g \in \mathbb{F}[x]$, and so for all $g \in \mathbb{F}[x]$. The same goes for $g_2$. So, in particular,

$$\langle A^{2k}v, v\rangle = \frac{\langle (A^{2k}+I)v, (A^{2k}+I)v\rangle - \langle A^{2k}v, A^{2k}v\rangle - \langle v, v\rangle}{2} = 0$$

$$\langle A^{2k+1}v, v\rangle = \omega^{-1}\langle \omega A^{2k+1}v, v\rangle = \frac{\omega^{-1}}{2}(\langle (\omega A^{2k+1}+I)v, (\omega A^{2k+1}+I)v\rangle - \langle \omega A^{2k+1}v, \omega A^{2k+1}v\rangle - \langle v, v\rangle) = 0.$$

Note that indeed it follows from $A^* = -A$ that $\langle A^{2k}v, v\rangle = \langle v, A^{2k}v\rangle$ and that $\langle \omega A^{2k+1}v, v\rangle = \langle v, \omega A^{2k+1}v\rangle$.

Now, for the case of $\mathfrak{o}$, we still have $g_1$, and the same proof as before shows that $\langle A^{2k}v, v\rangle = 0$. However, it is always true that

$$\langle A^{2k+1}v, v\rangle = \langle v, -A^{2k+1}v\rangle = -\langle A^{2k+1}v, v\rangle,$$

and hence $\langle A^{2k+1}v, v\rangle = 0$.

For the case of $\mathfrak{sp}$, we use the same technique but to the condition imposed from Lemma 6.5. This way we get that for a Zariski dense subset of $\mathbb{F}[x]$ (and thus for all $g \in \mathbb{F}[x]$) that $\langle Ag(A^2)v, g(A^2)v\rangle = 0$. From this, we are able to get

$$\langle A^{2k+1}v, v\rangle = \frac{1}{2}(\langle A(A^{2k}+I)v, (A^{2k}+I)v\rangle - \langle A\cdot A^{2k}v, A^{2k}v\rangle - \langle Av, v\rangle) = 0.$$

Moreover, it is always true that

$$\langle A^{2k}v, v\rangle = \langle v, A^{2k}v\rangle = -\langle A^{2k}v, v\rangle,$$

and hence $\langle A^{2k}v, v\rangle = 0$. ∎

***Lemma 6.8*** *Given $(A, v) \in R$, we have:*

(i) *For any $\lambda \in \mathbb{K}$ for which $\nu_\lambda$ is defined, $\Delta(\nu_\lambda(A, v)) = \Delta(A, v)$.*
(ii) *For any $\lambda \in \mathbb{F}$, $\Delta(\mu_\lambda(A, v)) = \Delta(A, v)$.*

**Proof** For $\Delta(\nu_\lambda(A, v))$, this follows directly from Proposition 2.17. For $\Delta(\mu_\lambda(A, v))$, this also follows from Proposition 2.17, but with an iterative use.

Since $\langle A^kAv, v\rangle = 0$ for all $k \ge 0$, $\Delta(A) = \Delta(A + A\phi_v)$. To prove that we also have $\Delta(A + A\phi_v) = \Delta(A + A\phi_v + \phi_v A)$, we must check that $\langle (A + A\phi_v)^k v, A^*v\rangle = 0$ for all $k \ge 0$. Now, for any $k \ge 0$, we have $\phi_v A^k v = \langle A^k v, v\rangle v = 0$, so

$$\langle (A + A\phi_v)^k v, A^*v\rangle = \langle A^k v, A^*v\rangle = \langle A^{k+1}v, v\rangle = 0.$$

∎

## 7 Stratification

For any $g \in \mathbb{K}[x]$ which is a power of an irreducible polynomial, let $Y_g$ be the subspace of $\mathfrak{g}$ consisting of elements with characteristic polynomial $g$. By the localization principle (Corollary 2.5), the previous reformulations, and Theorem 4.2, it is enough to prove that any $(\widetilde{G}, \chi)$-equivariant distribution on $\Delta^{-1}(g) = Y_g \times V$ is 0, for any $g$ as above. Let us fix $g$ and prove this claim for it.

We proceed similarly to [3, 9]. The strategy will be to stratify $Y_g$ and restrict stratum by stratum the possible support for a $(\widetilde{G}, \chi)$-equivariant distribution (note that $Y_g$ is a union of finitely many $\widetilde{G}$ orbits). For the unitary case, choose $\omega \in \mathbb{K}$ s.t. $\bar{\omega} = -\omega$ (in the symplectic case denote $\omega = 1$). For $\lambda \in \mathbb{F}$, denote by $\eta_\lambda$ either $\nu_{\lambda\omega}$ or $\mu_\lambda$, depending on which case we are in.

***Notation 7.1*** Denote by $P_i(g)$ the union of all $\widetilde{G}$-orbits of $Y_g$ of dimension at most $i$, and let $R_i(g) := R \cap (P_i(g) \times V)$. Moreover, for any open $\widetilde{G}$-orbit $O$ of $P_i(g)$, set

$$\widetilde{O} := (O \times V) \cap \bigcap_{\lambda \in \mathbb{F}} \eta_\lambda^{-1}(R_i(g)).$$

Note that $P_i(g)$ are Zariski closed inside $Y_g$, $P_k(g) = Y_g$ for $k$ big enough, and $P_{-1}(g) = \varnothing$.

We denote by $\mathcal{F}_V$ the Fourier transform on $V$ with respect to the nondegenerate $\mathbb{F}$-bilinear form $(u, v) \mapsto \mathrm{tr}_{\mathbb{K}/\mathbb{F}}(\langle u, v\rangle)$. It will also be used to denote the partial Fourier transform on $V$ when applied to $X \times V$ for some space $X$. In the cases of $\mathfrak{g} = \mathfrak{o}$ and $\mathfrak{g} = \mathfrak{u}$, $\mathcal{F}_V$ commutes with the action of $\widetilde{G}$. In the case of $\mathfrak{g} = \mathfrak{sp}$, it is not true. Instead, the action on $\widetilde{G}$ after applying $\mathcal{F}_V$ is compatible with the action of $\widetilde{G}$ on $V$ by $(g, \delta).v := gv$ (recall that the usual action of $\widetilde{G}$ on $V$ is by $(g, \delta).v := \delta g v$). Since $-1 \in Sp$, we still have that Fourier transform maps $\mathcal{S}^*(X \times V)^{H,\tau}$ into itself for any $X \subseteq \mathfrak{sp}$, any subgroup $H$ of $\widetilde{Sp}$ containing $-1$, and any $\tau \in \{1, \chi\}$.

***Claim 7.2*** Let $g \in \mathbb{K}[x]$ be a polynomial which is a power of an irreducible polynomial $f$ satisfying $f = \pm f^*$. Let $O$ be an open $\widetilde{G}$-orbit of $P_i(g)$. Suppose that $\xi$ is a $(\widetilde{G}, \chi)$-equivariant distribution on $O \times V$ such that

$$\mathrm{supp}(\xi) \subseteq \widetilde{O}$$

and

$$\mathrm{supp}(\mathcal{F}_V(\xi)) \subseteq \widetilde{O}.$$

Then $\xi = 0$.

This claim will be proved in the next section.

Let us now show how it implies the main theorems. Recall that Theorem 3.2, which states that any $(\widetilde{G}, \chi)$-equivariant distribution on $\mathfrak{g} \times V$ is 0, implies Theorems 1.2 and 1.4. This is by virtue of Theorem 5.7 and what is shown in Section 3.

**Proof of Theorem 3.2** We prove the following claim by downward induction—any $(\widetilde{G}, \chi)$-equivariant distribution on $\Delta^{-1}(g)$ is supported inside $R_i(g)$. This claim for $i$ big enough follows from Proposition 6.7, and the claim for $i = -1$ implies Theorem 3.2 by the localization principle (Corollary 2.5) and Proposition 4.2, as already explained in the top of this section. For the induction step, take such a distribution $\xi$. As $P_i(g) \backslash P_{i-1}(g)$ is a disjoint union of open orbits, it is enough to show that the restriction of $\xi$ to any $O \times V$, where $O$ is an open orbit of $P_i(g)$, is zero. Let $\zeta = \xi|_{O \times V}$ be such a restriction. By the induction hypothesis applied to $\eta_\lambda(\xi)$, we know that $\mathrm{supp}(\zeta) \subseteq \widetilde{O}$ and similarly $\mathrm{supp}(\mathcal{F}_V(\zeta)) \subseteq \widetilde{O}$. Hence, by Claim 7.2, $\zeta = 0$. ∎

https://doi.org/10.4153/S0008414X2200027X Published online by Cambridge University Press

## 8 Handling a single stratum—proof of Claim 7.2

### 8.1 Nice operators

This subsection closely follows [3, Section 6] and [9, Section 4.3], but we give it here for completeness.

***Notation 8.1*** For $A \in \mathrm{gl}(V)$, set, in the cases $\mathfrak{g} = \mathfrak{u}, \mathfrak{sp}$,

$$Q_A := \{v \in V | \phi_v \in [A, \mathfrak{g}(V)]\}.$$

In the case $\mathfrak{g} = \mathfrak{o}$, set

$$Q_A := \{v \in V | A\phi_v + \phi_v A \in [A, \mathfrak{g}(V)]\}.$$

Here, $[B, C] := BC - CB$ is the Lie bracket, and $[A, \mathfrak{g}(V)] = \{[A, B] | B \in \mathfrak{g}(V)\}$.

***Proposition 8.2*** *If $(A, v) \in \widetilde{O}$, then $v \in Q_A$.*

**Proof** Consider a point $(A, v) \in \widetilde{O}$. The Zariski tangent space to $O$ at $A$ is $[A, \mathfrak{g}(V)]$. Denote by $A_\lambda$ the operator $A + \lambda\omega\phi_v$ in the unitary case, $A + \lambda\phi_v$ in the symplectic case, or $A + \lambda(A\phi_v + \phi_v A)$ in the orthogonal case. Since $A_\lambda$ is contained in $O$ for $\lambda$ small enough (as $\eta_\lambda$ keeps $A$ inside $P_i(g)$, in which $O$ is open), we get that $\phi_v \in [A, \mathfrak{g}(V)]$ (or $A\phi_v + \phi_v A \in [A, \mathfrak{g}(V)]$ in the orthogonal case). ∎

***Theorem 8.3*** *Unless we are in the case $\mathfrak{g} = \mathfrak{o}$ and the characteristic polynomial of $A$ is equal to $x^n$, we have $Q_A \subseteq R_A$. In this case, we still have $\langle A^k v, v\rangle = 0$ for any $v \in Q_A$ and $k \geq 1$.*

**Proof** For the unitary and symplectic cases: Assume that $\phi_v = [A, B]$, for some $B \in G$. Then

$$\langle A^k v, v\rangle = \mathrm{tr} A^k \phi_v = \mathrm{tr}[A, A^k B] = 0.$$

For the orthogonal case: Assume that $A\phi_v + \phi_v A = [A, B]$, for some $B \in G$. Then

$$\mathrm{tr} A^k A\phi_v = \mathrm{tr} A^k \phi_v A = \frac{\mathrm{tr}(A^{k+1}\phi_v + A^k\phi_v A)}{2} = \frac{\mathrm{tr} A^k(AB - BA)}{2} = \frac{\mathrm{tr}[A, A^{k+1}B]}{2}$$
$$= 0.$$

Now, $\mathrm{tr} A^k A\phi_v = \langle A^{k+1} v, v\rangle$, so we know for any $k \geq 1$ that $\langle A^k v, v\rangle = 0$. If the characteristic polynomial $g$ (which is a power of an irreducible polynomial) is not a power of $x$, then there is a polynomial $h(A)$ s.t. $Ah(A) = \mathrm{Id}$. This implies that $\langle v, v\rangle = \langle Ah(A)v, v\rangle = 0$, and so $v \in R_A$. ∎

***Notation 8.4*** Let $A \in \mathfrak{g}$. We denote by $C_A$ the stabilizer of $A$ in $G$ and by $\widetilde{C}_A$ the stabilizer of $A$ in $\widetilde{G}$.

***Lemma 8.5*** *For any $A \in \mathfrak{g}$, $\widetilde{C}_A \neq C_A$.*

**Proof** We give here the proof for all cases except $G = SO$. The proof for $G = SO$ is given in Appendix B. By [10, Chapter 4, Proposition 1.2], There exists an $\mathbb{F}$-linear map $T : V \to V$ which satisfies $TAT^{-1} = -A$, such that, for any $u, v \in V$, we have that $\langle Tu, Tv\rangle = \langle v, u\rangle$ (this condition implies that $s\lambda u = \overline{\lambda u}$). Consider $s = (T, -1)$ as an element of $\widetilde{G}$. Then $s.A = -TAT^{-1} = A$. Thus, $s \in \widetilde{C}(A) \backslash C_A$. ∎

Thus, the $\widetilde{G}$-orbit of $A$ is equal to its $G$-orbit. It is known that $C_A$ is unimodular, and hence $\widetilde{C}_A$ is also unimodular. Claim 7.2 follows now from Frobenius descent (Theorem 2.7), Proposition 8.2, and the following proposition.

***Proposition 8.6*** *Let $A \in \mathfrak{g}$. Let $\eta \in \mathcal{S}^*(V)^{C_A}$. Suppose that both $\eta$ and $\mathcal{F}_V(\eta)$ are supported in $Q_A$. Then $\eta \in \mathcal{S}^*(V)^{\widetilde{C}_A}$.*

***Definition 8.7*** Call an element $A \in \mathfrak{g}$ "nice" if the previous proposition holds for $A$. Namely, $A$ is "nice" if any distribution $\eta \in \mathcal{S}^*(V)^{C_A}$ such that both $\eta$ and $\mathcal{F}(\eta)$ are supported in $Q_A$ is also $\widetilde{C}_A$-invariant.

***Lemma 8.8*** *Let $A_1 \in \mathfrak{g}(V_1)$ and $A_2 \in \mathfrak{g}(V_2)$ be nice. Then $A_1 \oplus A_2 \in \mathfrak{g}(V_1 \oplus V_2)$ is nice.*

**Proof** See the proof of [3, Lemma 6.3]. ∎

## 8.2 A "simple" operator is nice

Using the classification of Proposition 2.15, we need to check that simple nonsplit, simple even nilpotent, and simple odd nilpotent blocks are nice (recall that we assumed the characteristic polynomial of our original operator to be a power of an irreducible polynomial, and thus we need not check simple split operators). Let $A$ be a block of one of these types. Let $s = (T, -1)$ be an element of $\widetilde{C}_A$ with $\chi(s) = -1$. We have $A = s.A = -TAT^{-1}$, and so $TA = -AT$. We need to prove the following claim for each of the possible block types.

***Claim 8.9*** Let $\xi$ be a $C_A$-invariant distribution on $V$, such that both $\xi$ and $\mathcal{F}(\xi)$ are supported on $Q_A$. Then $\xi$ is also $s$-invariant.

We shall prove this claim in the following subsections. This claim implies Claim 7.2.

### 8.2.1 Simple nonsplit blocks

Assume that $A$ is a simple nonsplit block with minimal polynomial $f^d, f$ irreducible, and $f^* = \pm f$. If we are in the case $\mathfrak{g} = \mathfrak{o}$, assume also that $f(x) \neq x$. We know by Proposition 8.3 that $Q_A \subseteq R_A$. Consider the self-dual increasing filtration $V^i = \ker f(A)^i$. One can easily see that $R_A = V^{\lfloor d/2 \rfloor}$. The fact that $\mathcal{F}(\xi)$ is supported on $V^{\lfloor d/2 \rfloor}$ means that $\xi$ is invariant to shifts by $(V^{\lfloor d/2 \rfloor})^\perp = V^{\lceil d/2 \rceil}$. Now, consider two cases:

(1) $d$ is odd. Then $V^{\lfloor d/2 \rfloor} \subsetneq V^{\lceil d/2 \rceil}$. Choosing a vector $v \in V^{\lceil d/2 \rceil} \setminus V^{\lfloor d/2 \rfloor}$, we get that $\xi$ is the same as $\xi$ shifted by $v$, and that it is supported on $V^{\lfloor d/2 \rfloor} \cap (v + V^{\lfloor d/2 \rfloor}) = \varnothing$, and thus $\xi = 0$.

(2) $d$ is even. Then $V^{\lfloor d/2 \rfloor} = V^{\lceil d/2 \rceil} = V^{d/2}$. Thus, $\xi$ is the extension by 0 of $\xi|_{V^{d/2}}$, which is a shift invariant distribution on $V^{d/2}$. Thus, it is a multiple of the Lebesgue measure on $V^{d/2}$. So it is left to check that if the Lebesgue measure $\zeta$ on $V^{d/2}$ is $C_A$ invariant it is also $s$-invariant. For this, first check that $V^{d/2}$ is $T$-invariant (and so $s$ invariant):

$$TV^{d/2} = Tf(A)^{d/2}V = \bar{f}(TAT^{-1})^{d/2}TV = \bar{f}(-A)^{d/2}V = f(A)^{d/2}V = V^{d/2}.$$

So $s$ multiplies $\zeta$ by a constant $c$, which is positive, because $s$ preserves the positivity of the Lebesgue measure. Since $s^2 \in C_A$, we have by assumption $s^2\zeta = \zeta$. So unless $\zeta = 0$, $c^2 = 1$, hence by positivity $c = 1$, and we are done.

### 8.2.2 Simple nonsplit nilpotent blocks in the orthogonal case

Note that a simple nonsplit nilpotent block is such that $V$ has a basis of the form $e, Ae, \dots, A^{d-1}e$, the minimal polynomial of $A$ is equal to $x^d$, and for some nonzero constant $c \in \mathbb{F}$,

$$\langle A^i e, A^j e\rangle = \begin{cases} (-1)^j c, & i + j = d - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Note that this implies that $d$ must be odd. Let $A$ be such a block. Denote

$$\begin{aligned} V_1 :=& \operatorname{Span}(e, \dots, A^{(d-3)/2}e), \\ V_2 :=& \operatorname{Span}(A^{(d-1)/2}e), \\ V_3 :=& A^{(d+1)/2}V = \operatorname{Span}(A^{(d+1)/2}e, \dots, A^{d-1}e). \end{aligned}$$

Then, by Theorem 8.3, $Q_A \subseteq V_2 \oplus V_3$. So $\xi$ is supported on $V_2 \oplus V_3$, and is invariant to shifts by $(V_2 \oplus V_3)^\perp = V_3$. So it is of the form $\delta_1 \otimes R \otimes dv_3$, where $dv_3$ is the Lebesgue measure on $V_3$, $\delta_1$ is the Dirac measure at 0 on $V_1$, and $R$ is some distribution on $V_2$. Since it is enough to prove our claim for any valid choice of $s = (T, -1) \in \widetilde{C}_A$, we may simply take $T(A^i e) = (-1)^{(d+1)/2+i}A^i e$. Then $s$ acts on $V$ by $A^i e \mapsto (-1)^{(d-1)/2+i}A^i e$. The spaces $V_1, V_2, V_3$ are $s$ invariant, and $s$ acts on $V_2$ by identity. It is also clear that $dv_3, \delta_1$ are $s$-invariant. Thus, $\xi$ is $s$ invariant, and we are done.

### 8.2.3 Simple even nilpotent blocks

Let $A \in \mathfrak{o}$ be a simple even nilpotent block. Denote

$$E := \operatorname{Span}(e, Ae, \dots, A^{d-1}e)$$

and

$$F := \operatorname{Span}(f, Af, \dots, A^{d-1}f).$$

Denote also

$$E_1 := \operatorname{Span}(e, Ae, \dots, A^{d/2-1}e), E_2 := \operatorname{Span}(A^{d/2}e, A^{d/2+1}e, \dots, A^{d-1}e)$$

and

$$F_1 := \operatorname{Span}(f, Af, \dots, A^{d/2-1}f), F_2 := \operatorname{Span}(A^{d/2}f, A^{d/2+1}f, \dots, A^{d-1}f).$$

Let $P : V \to V$ be defined by $PA^i e = A^i f, PA^i f = A^i e$. So $PA = AP$. For any two vectors $u, w \in V$, define the linear operator $\phi_{u,w}v := \langle v, w\rangle u$. For any $X \in \operatorname{End}(V)$, denote by $X|_E$ the linear operator from $E$ to itself which sends $v \in E$ to $u_E$, where $Xv = u_E + u_F$ and we have $u_E \in E, u_F \in F$. Let $v \in Q_A$. By definition, we have

https://doi.org/10.4153/S0008414X2200027X Published online by Cambridge University Press

$A\phi_{v,v} + \phi_{v,v}A = [A, B]$ for some $B \in \mathfrak{g}$. We have

$$\begin{aligned}(P(A\phi_{v,v} + \phi_{v,v}A))|_E =& (P[A,B])|_E = (PAB - PBA)|_E = (APB - PBA)|_E \\ =& [A|_E, (PB)|_E].\end{aligned}$$

The last equation following from the fact that $A$ preserves $E$ and $F$. From this, it follows that, for any $k \geq 0$,

$$\begin{aligned}\mathrm{tr}((P(A\phi_{v,v} + \phi_{v,v}A)A^k)|_E) =& \mathrm{tr}([A|_E, (PB)|_E](A|_E)^k) \\ =& \mathrm{tr}([A|_E, (PB)|_E(A|_E)^k]) = 0.\end{aligned}$$

However,

$$\begin{aligned}\mathrm{tr}((PA\phi_{v,v}A^k)|_E) =& \mathrm{tr}((AP\phi_{v,v}A^k)|_E) = \mathrm{tr}(A|_E(P\phi_{v,v}A^k)|_E) \\ =& \mathrm{tr}((P\phi_{v,v}A^k)|_E A|_E) = \mathrm{tr}((P\phi_{v,v}A^{k+1})|_E).\end{aligned}$$

So this expression is equal to half of the left-hand side of the previous equation, and so

$$\mathrm{tr}((P\phi_{v,v}A^{k+1})|_E) = 0.$$

However,

$$\mathrm{tr}((P\phi_{v,v}A^{k+1})|_E) = \mathrm{tr}(\phi_{Pv,(-A)^{k+1}v}|_E) = \mathrm{tr}(\phi_{Pv_F,(-A)^{k+1}v_F})\langle Pv_F, (-A)^{k+1}v_F\rangle,$$

where $v = v_E + v_F$, $v_E \in E$, $v_F \in F$. Thus, for any $k \geq 1$, we have $\langle A^k P v_F, v_F\rangle = 0$. Similarly, $\langle A^k P v_E, v_E\rangle = 0$. This implies $v \in E_2 \oplus F_2$. Thus, if $\xi$ is a distribution as in the statement of Claim 8.9, it must be supported on $E_2 \oplus F_2$ and invariant to translations by $(E_2 \oplus F_2)^\perp = E_2 \oplus F_2$. Thus, it is equal to a multiple of the Lebesgue measure on $E_2 \oplus F_2$. As it is enough to prove the claim for any specific choice of $s = (T, -1) \in \widetilde{C}_A$, we can choose $T$ to be

$$A^i e \mapsto (-1)^i A^i e, A^i f \mapsto (-1)^{i+1} A^i f.$$

For this choice, $s$ acts by $A^i e \mapsto (-1)^{i+1} A^i e, A^i f \mapsto (-1)^i A^i f$, and so fixes $\xi$ as desired.

### 8.2.4 Simple odd nilpotent blocks

Let $A \in \mathfrak{sp}$ be a simple odd nilpotent block. Denote

$$E := \mathrm{Span}(e, Ae, \ldots, A^{d-1}e)$$

and

$$F := \mathrm{Span}(f, Af, \ldots, A^{d-1}f).$$

Denote also

$$\begin{aligned}E_1 :=& \mathrm{Span}(e, Ae, \ldots, A^{\frac{d-3}{2}}e), \\ E_2 :=& \mathrm{Span}(A^{\frac{d-1}{2}}e), \\ E_3 :=& \mathrm{Span}(A^{\frac{d+1}{2}}e, \ldots, A^{d-1}e),\end{aligned}$$

and

$$\begin{aligned} F_1 :=& \mathrm{Span}(f, Af, \ldots, A^{\frac{d-3}{2}} f), \\ F_2 :=& \mathrm{Span}(A^{\frac{d-1}{2}} f), \\ F_3 :=& \mathrm{Span}(A^{\frac{d+1}{2}} f, \ldots, A^{d-1} f). \end{aligned}$$

The following is done similarly to the previous case, of simple even nilpotent blocks. Let $P : V \to V$ be defined by $PA^i e = A^i f, PA^i f = A^i e$. So $PA = AP$. For any two vectors $u, w \in V$, define the linear operator $\phi_{u,w} v := \langle v, w\rangle u$. For any $X \in \mathrm{End}(V)$, denote by $X|_E$ the linear operator from $E$ to itself which sends $v \in E$ to $u_E$, where $Xv = u_E + u_F$ and we have $u_E \in E, u_F \in F$. Let $v \in Q_A$. By definition, we have $\phi_{v,v} = [A, B]$ for some $B \in \mathfrak{g}$. We have

$$\begin{aligned} (P\phi_{v,v})|_E =& (P[A,B])|_E = (PAB - PBA)|_E = (APB - PBA)|_E \\ =& [A|_E, (PB)|_E]. \end{aligned}$$

From this, it follows that, for any $k \geq 0$,

$$\begin{aligned} \mathrm{tr}((P\phi_{v,v}A^k)|_E) =& \mathrm{tr}([A|_E, (PB)|_E](A|_E)^k) \\ =& \mathrm{tr}([A|_E, (PB)|_E(A|_E)^k]) = 0. \end{aligned}$$

However,

$$\mathrm{tr}((P\phi_{v,v}A^k)|_E) = \mathrm{tr}(\phi_{Pv,(-A)^k v}|_E) = \langle Pv_F, (-A)^k v_F\rangle,$$

where $v = v_E + v_F$, $v_E \in E, v_F \in F$. Thus, for any $k \geq 0$, we have $\langle A^k P v_F, v_F\rangle = 0$. Similarly, $\langle A^k P v_E, v_E\rangle = 0$. This implies $v \in E_3 \oplus F_3$. Thus, if $\xi$ is an equivariant distribution as in the statement of Claim 8.9, it must be supported on $E_3 \oplus F_3$ and invariant to translations by $(E_3 \oplus F_3)^\perp = E_2 \oplus E_3 \oplus F_2 \oplus F_3$. This clearly implies that $\xi = 0$.

# A Conjugacy classes and "simple" elements in the orthogonal and unitary groups

Our goal in this appendix is to prove Proposition 2.15. We will focus on the classification of $G$-conjugacy classes inside $\mathfrak{g}$, although most of the work, if not all of it, applies also for conjugacy classes in $G(V)$. The classification will be in three parts. The first is, using Lemma 2.13, to separate the different nonrelated eigenvalues, the second is to separate the different sizes of rational (or Jordan) blocks, and the third is to separate blocks of the same size one from the other.

## A.1 Separating nonrelated eigenvalues

Recall Definition 2.11.

Lemma 2.13 has the following corollary, using also Theorem 2.10.

***Corollary A.1*** *Any $A \in \mathfrak{g}$ splits to a direct orthogonal sum of block of two types:*

(A) *simple split blocks;*

https://doi.org/10.4153/S0008414X2200027X Published online by Cambridge University Press

(B) *blocks of which the characteristic polynomial is* $f^d$, *and* $f$ *is irreducible and satisfies* $f^* = \pm f$.

Now, we continue with the classification of blocks of Type (B).

## A.2 Classifying blocks of Type (B)

Let $A \in \mathfrak{g}(V)$ be a block of Type (B), with minimal polynomial $f^d$, $f$ irreducible and satisfying $f = \pm f^*$.

***Definition A.2*** An operator $X \in \mathfrak{g}(W)$ (for some $W$) which is of Type (X) will be called homogeneous if, for any $0 \le j \le d$, we have $f(X)^j V = \ker f(X)^{d-j}$. That is equivalent to saying that the rational canonical form of $X$ consists of blocks all of the same size.

We dedicate this subsection to proving the following proposition.

***Proposition A.3*** *$V$ can be decomposed as an orthogonal sum $V = \bigoplus_{i=1}^d U_i$, and accordingly $A = \bigoplus_{i=1}^d A_i$, such that each $A_i$ is homogeneous.*

Consider the decreasing filtration $f(A)^i V$ of $V$, and the increasing filtration $V_i = \ker f(A)^i$. We have $f(A)^i V \subseteq V_{d-i}$, and $V_i^\perp = f(A)^i V$. Let $m$ be the minimal integer so that $V_m \subsetneq f(A) V$. For any $0 \le i \le m-1$, we have $f(A)^i V_m = V_{m-i}$.

***Lemma A.4*** *For any* $0 \le i \le d$, $V_i^\perp = f(A)^i V$.

**Proof** Obviously, $f(A)^i V \subseteq V_i^\perp$. However,

$$\dim f(A)^i V = \dim V - \dim \operatorname{Ker}(f(A)^i) = \dim V - \dim V_i = \dim V_i^\perp.$$

Thus, we have $V_i^\perp = f(A)^i V$. ■

***Lemma A.5*** *The form $B$ on $V$ induces a nondegenerate pairing of $f(A)^i V_m / f(A)^{i+1} V_{m+1}$ with $f(A)^{m-i-1} V_m / f(A)^{m-i} V_{m+1}$.*

**Proof** Let $v \in f(A)^i V_m = V_{m-i}$, and assume that $v \perp f(A)^{m-i-1} V_m = V_{i+1}$. Then $v \in V_{i+1}^\perp \cap V_{m-i} = f(A)^{i+1} V \cap V_{m-i} = f(A)^{i+1} V_{m+1}$. The other direction is symmetric. ■

**Proof of Proposition A.3** One can naturally give $V_m / f(A) V_{m+1}$ the structure of a vector space over the field $L := \mathbb{K}[A]/f(A)$. Choose $e_1, \ldots, e_k \in V_m$ which are the lifts of a basis of $V_m / f(A) V_{m+1}$ over $L$, and let $U_m := \operatorname{Span}(A^i e_j) \subseteq V_m$. We claim that, for any relation of the form $h_1(A) e_1 + \cdots + h_k(A) e_k + f(A)^r v = 0$, with $v \in V$, all the polynomials $h_i$ are divisible by $f(A)^{\min(m,r)}$. To see this, assume otherwise, and rewrite this relation as $f(A)^\ell (f(A)^{r-\ell} v + \sum_{i=1}^k \widetilde{h}_i(A) e_i) = 0$, where at least one of the polynomials $\widetilde{h}_i(A)$ is not divisible by $f(A)$. Since $\ell < m$, we have $\sum_{i=1}^k \widetilde{h}_i(A) e_i \in V_\ell + f(A) V = f(A) V$, and thus $\sum_{i=1}^k \widetilde{h}_i(A) e_i$ is a nontrivial relation in $V_m / f(A) V_{m+1}$ over $L$, which is a contradiction. From this claim, it follows that the map $f(A)^i U_m / f(A)^{i+1} U_m \to f(A)^i V_m / f(A)^{i+1} V_{m+1}$ is an isomorphism. Thus, the form $B$ on $V$ induces a nondegenerate pairing of $f(A)^i U_m / f(A)^{i+1} U_{m+1}$ with $f(A)^{m-i-1} U_m / f(A)^{m-i} U_{m+1}$. In particular, its restriction to $U_m$ is nondegenerate.

So $U_m$ splits as an $A$-invariant orthogonal direct summand of $V$. The restriction of $A$ to $U_m$ is a homogeneous block because

$$f(A)^j U_m = f(A)^j V_m \cap U_m = V_{m-j} \cap U_m = \ker f(A|_{U_m})^{m-j}.$$

Because of the choice of $m$, $\dim U_m > 0$. Furthermore, on $V' := U_m^{\perp}$, the minimal $m'$ s.t. $V'_{m'} \subsetneq f(A)V'$ is bigger than $m$. So, by induction, we are done. ■

## A.3 Decomposing homogeneous blocks to simple nonsplit blocks

***Lemma A.6*** *Given a nonzero symmetric, Hermitian, or skew-Hermitian form on a vector space V, there is a non-isotropic vector.*

**Proof** Assume that $\langle v, v\rangle = 0$ holds for any $V$. Then

$$0 = \langle u+v, u+v\rangle - \langle u, u\rangle - \langle v, v\rangle = \langle u, v\rangle + \langle v, u\rangle.$$

Thus, the form is skew-symmetric, which contradicts our assumptions. ■

***Corollary A.7*** *Any homogeneous operator decomposes as the direct orthogonal sum of simple nonsplit blocks, unless it is in the* $\mathfrak{o}$ *case and its minimal polynomial is* $x^d$ *for some even d, or it is in the* $\mathfrak{sp}$ *case and its minimal polynomial is* $x^d$ *for some odd d. In these cases, it decomposes as the direct sum of simple even nilpotent blocks (resp. simple odd nilpotent blocks).*

**Proof** Assume that we are neither in the symplectic case nor in the even nilpotent orthogonal case which was excluded. Define a nondegenerate sesquilinear form on $U := V/f(A)V$ by $\langle u, v\rangle_U := \langle f(A)^{d-1}u, v\rangle_V$. If we are in the unitary case, then it is either Hermitian or skew-Hermitian, depending on whether $(f^{d-1})^* = f^{d-1}$ ($f^{d-1}$ is of even degree), or $(f^{d-1})^* = -f^{d-1}$ ($f^{d-1}$ is of odd degree). If we are in the orthogonal case, $(f^{d-1})^* = f^{d-1}$, and so this bilinear form is symmetric. So there is a nonisotopic vector in $U$, which means that there is a vector $v \in V$ such that $\langle f(A)^{d-1}v, v\rangle = \lambda \neq 0$. Let $V_0 = \mathrm{Span}(v, Av, A^2v, \dots)$. It is an $A$-invariant space, to which the restriction of the form $B$ is nondegenerate. It is also generated by one vector ($v$). By induction, we are done.

In the symplectic case which is not the case excluded, we have a similar proof. Again, define a bilinear form on $U := V/f(A)V$. If the minimal polynomial is $x^d$ for $d$ even, define as before $\langle u, v\rangle_U := \langle A^{d-1}u, v\rangle$, and it will be a symmetric form. Otherwise, $A$ is invertible and we set $\langle u, v\rangle_U := \langle Af(A)^{d-1}u, v\rangle$. Again this bilinear form is symmetric. The rest of the proof follows the same, with noticing that $\langle Af(A)^{d-1}v, v\rangle \neq 0$ implies that $V_0 \cap V_0^{\perp} = 0$ for $V_0 = \mathrm{Span}(v, Av, A^2v, \dots)$.

In the orthogonal case, if the minimal polynomial is $x^d$ for $d$ even (resp. the symplectic case and $d$ odd), the bilinear form $\langle u, v\rangle_U := \langle A^{d-1}u, v\rangle$ on $U$ is skew-symmetric. Take $u_1, u_2 \in U$ s.t. $\langle u_1, u_2\rangle_U = 1$. Lift them to $v_1, v_2 \in V$, and Take $V_0 = \mathrm{Span}(v_1, Av_1, A^2v_1, \dots) \oplus \mathrm{Span}(v_2, Av_2, A^2v_2, \dots)$ (note that this sum is indeed direct). $V_0$ is $A$-invariant, and $V_0^{\perp} \cap V_0 = 0$. Now, we are left to show that $V_0$ is a simple even nilpotent block (resp. simple odd nilpotent block). The first step is to show that we can alter the lifts of $u_1, u_2$ to $v_1, v_2$ (from $V_0/AV_0$ to $V_0$) such that $\langle A^j v_1, v_1\rangle = 0$ for all $j$. (We will show it for $v_1$, but it is exactly the same for $v_2$.) For odd (resp. even) $j$, this holds automatically. We assume for the following that $v_2$ is any lift of

https://doi.org/10.4153/S0008414X2200027X Published online by Cambridge University Press

$u_2$ (the important property is that $\langle A^{d-1}v_1, v_2\rangle = 1$). If $m$ is the minimal integer such that $\langle A^{d-m}v_1, v_1\rangle \neq 0$, we can add a multiple of $A^{m-1}v_2$ to $v_1$ to fix that, as

$$\begin{aligned}&\langle A^{d-m}(v_1 + \lambda A^{m-1}v_2), v_1 + \lambda A^{m-1}v_2\rangle\\ =&\langle A^{d-m}v_1, v_1\rangle + 2\lambda\langle A^{d-1}v_2, v_1\rangle - \lambda^2\langle A^{d+m-2}v_2, v_2\rangle\\ =&\langle A^{d-m}v_1, v_1\rangle - 2\lambda,\end{aligned}$$

$$\langle A^{d-i}(v_1 + \lambda A^{m-1}v_2), v_1 + \lambda A^{m-1}v_2\rangle = \langle A^{d-i}v_1, v_1\rangle = 0$$

for any $i < m$, and

$$\langle A^{d-1}(V_1 + \lambda A^{m-1}v_2), v_2\rangle = \langle A^{d-1}v_1, v_2\rangle = 1.$$

Notice that $m$ must be even and $m \geq 2$. So, by applying this consecutively, we may change the lift of $v_1$ (and similarly $v_2$) in the desired way (notice that indeed we changed it only by vectors in $AV_0$). Now, all that is left is to again change $v_1, v_2$ so that $\langle A^k v_1, v_2\rangle = 0$ for any $k < d-1$. For this, simply choose a vector $\tilde{v}_2$ which is orthogonal to $v_1, Av_1, A^2v_1, \dots, A^{d-2}v_1, v_2, Av_2, \dots, A^{d-1}v_2$ and such that $\langle A^{d-1}v_1, v_2\rangle = 1$. Note that the vector $v_2 - \tilde{v}_2$ is perpendicular to the subspace $V_2 := \mathrm{Span}(v_2, Av_2, \dots, A^{d-1}v_2)$, and so $v_2 - \tilde{v}_2 \in V_2^{\perp} = V_2$. This implies that, for all $k \geq 0$, $\langle A^k\tilde{v}_2, \tilde{v}_2\rangle = 0$. Moreover, $v_2 - \tilde{v}_2$ is perpendicular to $A^{d-1}V_0$, and so $v_2 - \tilde{v}_2 \in AV_2$. Thus, we can replace $v_2$ with $\tilde{v}_2$, and all of the needed conditions will be satisfied. ∎

The above immediately implies Proposition 2.15.

## B On the centralizers in $\widetilde{\mathrm{SO}}$

In this appendix, we prove Theorem 8.5 for the case of $G = \mathrm{SO}(V)$. We need to show that there exists an element $T \in \mathrm{O}(V)$ such that $TAT^{-1} = -A$ and $\det T = (-1)^{\lfloor \frac{n+1}{2} \rfloor}$. Assume the decomposition of Proposition 2.15. It is enough to prove Theorem 8.5 for each of the blocks, as then taking the direct sum of the elements $T_i$ found gives an element $T \in \mathrm{O}(V)$ with $TAT^{-1} = -A$. If all of the dimensions of the blocks $n_i$ are even, then $\det(\bigoplus T_i) = \prod \det T_i = (-1)^{\sum n_i/2} = (-1)^{n/2}$. Otherwise, there is an odd block, and by replacing $T_i$ by $-T_i$ if needed, we can control the sign of the determinant to be as we wish. Now, we need to check each of the simple block types.

### B.1 Simple split blocks

We have $V = V' \oplus V'^*$, with the natural symmetric bilinear form coming from the pairing. $A = \begin{bmatrix} A' & 0 \\ 0 & -A'^* \end{bmatrix}$. By the well-known theorem claiming that any square matrix (over any field and of any dimension) is conjugate to its transpose, there is an isomorphism $B : V'^* \to V'$ such that $BA'^*B^{-1} = A'$. Taking $T = \begin{bmatrix} 0 & B \\ (B^*)^{-1} & 0 \end{bmatrix} \in \mathrm{O}(V)$, we get

$$TAT^{-1} = \begin{bmatrix} 0 & B \\ (B^*)^{-1} & 0 \end{bmatrix} \begin{bmatrix} A' & 0 \\ 0 & -A'^* \end{bmatrix} \begin{bmatrix} 0 & B^* \\ B^{-1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -BA'^* \\ (B^*)^{-1}A' & 0 \end{bmatrix} \begin{bmatrix} 0 & B^* \\ B^{-1} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} -BA'^*B^{-1} & 0 \\ 0 & (B^*)^{-1}A'B^* \end{bmatrix} = \begin{bmatrix} -A' & 0 \\ 0 & A'^* \end{bmatrix} = -A.$$

Furthermore, $\det T = (-1)^{\dim V'} = (-1)^{\dim V/2}$.

## B.2 Simple nonsplit blocks

We have $V = \mathrm{Span}(e, Ae, A^2e, \dots)$ for some $e \in V$. Define $T : V \to V$ by $T(g(A)e) = g(-A)e$. It is well defined since $g$ is well defined modulo $f(A)^d$, and $f(-A) = \pm f(A)$. $T$ is an element of $\mathrm{O}(V)$, as

$$\langle g(-A)e, h(-A)e\rangle = \langle e, g(A)h(-A)e\rangle = \langle h(-A)g(A)e, e\rangle = \langle g(A)e, h(A)e\rangle.$$

Clearly, $TAT^{-1} = -A$. Moreover, $\det T = (-1)^{\lfloor n/2 \rfloor}$, which is what we wanted in the nonnilpotent case (where $n$ is even), and in the nilpotent case (where $n$ is odd), we may replace $T$ by $-T$ if needed, in order to achieve the desired sign of the determinant of $T$.

## B.3 Simple even nilpotent blocks

We can choose $T(A^ie) = (-1)^iA^ie$ and $T(A^if) = (-1)^{i+1}A^if$. It clearly satisfies $TAT^{-1} = -A$ and $T \in \mathrm{O}(V)$. It only remains to note that $\det T = (-1)^{\dim V/2}$, which is exactly what we wanted.

**Acknowledgment** I would like to deeply thank my advisor, Dmitry Gourevitch, for helping me with my work on this paper, for exposing me to this fascinating area of mathematics, and for the guidance and support along the way. I would also like to thank him for the exceptional willingness to help. I would also like to thank Guy Henniart for fruitful discussions he had with my advisor.

## References

[1] A. Aizenbud, N. Avni, and D. Gourevitch, *Spherical pairs over close local fields*. Comment. Math. Helv. **87**(2012), no. 4, 929–962. https://doi.org/10.4171/cmh/274

[2] A. Aizenbud and D. Gourevitch, *A proof of the multiplicity one conjecture for $GL_n$ in $GL_{n+1}$*. Preprint, 2007. arXiv:0707.2363v2 [math.RT].

[3] A. Aizenbud, D. Gourevitch, S. Rallis, and G. Schiffmann, *Multiplicity one theorems*. Ann. of Math. (2) **172**(2010), no. 2, 1407–1434. https://doi.org/10.4007/annals.2010.172.1413

[4] A. Aizenbud, D. Gourevitch, and E. Sayag, *$(O(V \oplus F), O(V))$ is a Gelfand pair for any quadratic space $V$ over a local field $F$*. Math. Z. **261**(2009), 239–244. https://doi.org/10.1007/s00209-008-0318-5

[5] J. Bernstein, *P-invariant distributions on GL(N) and the classification of unitary representations of GL(N) (non-archimedean case)*. In: Lie group representations, II (College Park, MD, 1982/1983), Lecture Notes in Mathematics, 1041, Springer, Berlin, 1984, pp. 50–102. https://doi.org/10.1007/bfb0073145

[6] J. Bernstein and A. V. Zelevinsky, *Representations of the group $GL(n, F)$, where F is a local non-Archimedean field*. Uspekhi Mat. Nauk **10**(1976), no. 3, 5–70.

[7] F. Chen and B. Sun, *Uniqueness of Rankin–Selberg periods*. Int. Math. Res. Not. IMRN **2015**(2015), no. 14, 5849–5873. https://doi.org/10.1093/imrn/rnu110

[8] W. T. Gan, B. H. Gross, and D. Prasad, *Symplectic local root numbers, central critical L-values, and restriction problems in the representation theory of classical groups*. In: Sur les Conjectures de Gross et Prasad. I, Société Mathématique de France, France, Astérisque, 346, 2012, 109 pp.

[9] D. Mezer, *Multiplicity one theorem for $(GL_{n+1}, GL_n)$ over a local field of positive characteristic*. Math. Z. **297**(2020), 1383–1396. https://doi.org/10.1007/s00209-020-02561-1

[10] C. Mœglin, M.-F. Vignéras, and J.-L. Waldspurger, *Correspondances de Howe Sur un Corpsp-adique*, Lecture Notes in Mathematics, 1291, Springer-Verlag, New York, 1987.

[11] T. A. Springer, and R. Steinberg, *Seminar on algebraic groups and related finite groups, E. Conjugacy Classes; Chapter IV. Conjugacy Classes in Classical Groups; 2. Unitary, orthogonal, symplectic groups*, Springer-Verlag, Berlin–Heidelberg, 1970.

[12] B. Sun, *Multiplicity one theorems for Fourier–Jacobi models*. Amer. J. Math. **134**(2012), no. 6, 1655–1678. https://doi.org/10.1353/ajm.2012.0044

[13] B. Sun and C.-B. Zhu, *Multiplicity one theorems: the Archimedean case*. Ann. of Math. (2) **175**(2012), no. 1, 23–44. https://doi.org/10.4007/annals.2012.175.1.2

[14] J. L. Waldspurger, *Une variante d'un résultat de Aizenbud, Gourevitch, Rallis et Schiffmann*. Astérisque **346**(2012), 313–318, Sur les conjectures de Gross et Prasad. I, Société Mathématique de France, France.

[15] G. E. Wall, *On the conjugacy classes in the unitary, symplectic and orthogonal groups*. J. Aust. Math. Soc. 3(1963), no. 1, 1–62. https://doi.org/10.1017/S1446788700027622

*Department of Mathematics, Weizmann Institute of Science, Rehovot, Israel*
*e-mail*: dor.mezer@gmail.com