Introduction

First, we will explain the overall plan of this book. Proofs and accurate definitions will not be given here, but will appear in subsequent chapters with more precise details.

Let X be a smooth projective complex algebraic variety, or in other words, a closed subvariety in a complex projective space. Denote $\dim X = n$ and take a positive integer m. A regular m-canonical differential form on X is defined locally by $h(x)(dx_1 \wedge \cdots \wedge dx_n)^{\otimes m}$ where $x_1, \ldots, x_n$ are local coordinates of X and h is a regular function on X. The set of regular m-canonical differential forms on X is a finite-dimensional $\mathbf{C}$-linear space denoted by $H^0(X, mK_X)$. Here K_X is the canonical divisor. For example, when $m = 0$, this space is just $\mathbf{C}$, and when $m = 1$, this space consists of all regular canonical differential forms.

For two positive integers m, m', we can define a multiplication map

$$H^0(X, mK_X) \otimes H^0(X, m'K_X) \to H^0(X, (m + m')K_X),$$

inducing a graded ring

$$R(X, K_X) = \bigoplus_{m=0}^{\infty} H^0(X, mK_X)$$

over the complex number field, which is called the *canonical ring* of X.

Two algebraic varieties X, Y are said to be *birationally equivalent* if there are non-empty Zariski open subsets $U \subset X$, $V \subset Y$ such that there is an isomorphism $U \cong V$. In this case Y is called a *birational model* of X, and the canonical ring is a *birational invariant*, that is, $R(X, K_X) \cong R(Y, K_Y)$. Birational invariants reflect intrinsic properties of algebraic varieties.

The main theorem of this book is the following proved by Birkar–Cascini–Hacon–M^c^Kernan ([16]):

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Theorem 1 (*Finite generation of canonical rings*) *For any smooth projective complex algebraic variety X, the canonical ring $R(X, K_X)$ is a finitely generated graded* $\mathbf{C}$*-algebra.*

The proof uses the *minimal model program* (*MMP*). The main part of this book is devoted to the foundation of the MMP.

If the transcendental degree of the canonical ring is $n+1$, then X is said to be of *general type*. In this case one can show that there exists a "minimal model" X' birationally equivalent to X with good properties. An MMP is a sequence of operations constructing X' starting from X. In general X' has singularities, but the singularities are mild so that the birational invariance $R(X, K_X) \cong R(X', K_{X'})$ still holds as the smooth case. The finite generation of canonical rings of minimal models is a consequence of the "*basepoint-free theorem*."

When X is not of general type, by applying the "*semipositivity theorem*" of algebraic fiber spaces, one can reduce the problem to the case of "log general type," and then derive the finite generation by the "log version" of the MMP.

The MMP is a process of changing birational models one after another. During this process, algebraic varieties with singularities naturally appear. However, those singularities are special kinds of normal singularities. The singularities in the MMP are very interesting research objects for their own sake. With the development of higher dimensional algebraic geometry, it is gradually becoming more common to consider algebraic varieties with singularities.

Proofs in the minimal model theory often use induction on integral invariants such as dimensions and Picard numbers. In order for this to work well, it is necessary to enhance the category of objects we consider. Here we extend to the *log version* and the *relative version*.

In the log version, instead of a single algebraic variety X, we consider a couple (X, B) consisting of X and an $\mathbf{R}$-divisor B on X. For historical reasons, this is called a *log pair* and B is called a *boundary divisor*. Here an $\mathbf{R}$-*divisor* $B = \sum b_j B_j$ is a formal $\mathbf{R}$-linear combination of subvarieties B_j of codimension 1 with real coefficients b_j. It is called a $\mathbf{Q}$-*divisor* if b_j are rational numbers. Instead of the canonical divisor K_X, the *log canonical divisor* $K_X + B$ plays the main role.

Conditions on singularities are imposed onto the log pair (X, B). In this book, we mainly consider the "KLT condition" (Kawamata log terminal condition) and the "DLT condition" (divisorially log terminal condition). For example, when X is smooth and the support $\sum B_j$ of B is a "normal crossing divisor," these conditions correspond to inequalities $0 < b_j < 1$ and $0 < b_j \leq 1$, respectively.

In the relative version, all objects are considered over a base variety. Instead of a single algebraic variety X, we consider a morphism $f : X \to S$ to a base variety.

In summary, we are going to consider a log pair (X, B) with the KLT or DLT condition and a projective morphism $f : X \to S$ to another algebraic variety. Sometimes we use $f : (X, B) \to S$ for short to keep in mind the log version and the relative version at the same time.

The *log canonical ring* is defined as

$$R(X/S, K_X + B) = \bigoplus_{m=0}^{\infty} f_*(\mathcal{O}_X(\llcorner m(K_X + B)\lrcorner)).$$

Here the symbol $\llcorner\ \lrcorner$ means *round down*, that is, to replace each coefficient by the nearest integer from below and f_* is the direct image of sheaves. $R(X/S, K_X + B)$ is a graded $\mathcal{O}_S$-algebra.

The log and relative version of the finite generation of canonical rings is as follows:

Theorem 2 *Let $f : (X, B) \to S$ be a projective morphism from a KLT pair defined over the complex number field, where B is a $\mathbf{Q}$-divisor. Then the log canonical ring $R(X/S, K_X + B)$ is a finitely generated graded $\mathcal{O}_S$-algebra.*

In Chapter 1, we will give basic definitions used in this book. The main idea is to associate a variety with a divisor called boundary and to consider them as a pair. Such "logarithmization" makes it possible to introduce many new methods. Log pairs are allowed to have mild singularities. Usually in algebraic geometry, nonsingular varieties are the central objects, but singularities of pairs are indispensable and play important roles in the minimal model theory. We will also explain two big theorems in characteristic 0 (the Hironaka desingularization theorem and the Kodaira vanishing theorem), both of which are main tools of this book. In particular, it is known that the vanishing theorem fails when the characteristic is not 0, so most results of this book are in characteristic 0. Then we will describe the classification theory of low-dimensional algebraic varieties. The goal of this part is to provide examples, and it is logically independent.

In Chapter 2, we will explain the outline of the minimal model theory. There are two main theorems: the basepoint-free theorem and the cone theorem. Using these theorems we formulate the MMP. The minimality of a log pair (X, B) is tested by the "numerical property" of the log canonical divisor. If the pair is not minimal, then there exists an "extremal ray" by the "cone theorem," and it induces a "contraction morphism" by the "basepoint-free theorem." There are three types of contraction morphisms: "divisorial contractions," "small contractions," and "Mori fiber spaces." For small contractions we need to consider another birational map called "flip" on the "opposite side." We will also explain new contents such as an effective version of the basepoint-free theorem and the MMP with scaling. In addition, we describe an important extension theorem developed from the theory of multiplier ideal sheaves.

In Chapter 3, we will give the proof of the finite generation of canonical rings, which is the main topic of this book. To this end, we show the existence of minimal models for varieties of general type. The "existence of flips" is proved as a special case of the finite generation of canonical rings. Furthermore, the "termination of flips" is proved under the assumption of "general type," which finishes the proof of the finite generation theorem in the general type case. In the end, we apply the semipositivity theorem of Hodge bundles, which is also a result to be held only in characteristic 0.

The content of Chapter 2 is basically the same as [76]. This book is a sequel of [76] and [67]. [76] summarized the results of the minimal model theory in its early stages, and has been cited in much literature. In there, the minimal model theory was already described in the log version and the relative version, which is consistent with the direction of the development afterward. We are proud to have played a certain role in producing a standard literature on the minimal model theory. At that stage, the basepoint-free theorem and the cone theorem were proved and the existence of minimal models was reduced to two conjectures on flips. In subsequent developments, the existence of flips was proved, along with the termination of flips in some special but important cases. The purpose of Chapter 3 is to explain those developments.

Remark 3 (1) In the proof, it is necessary to consider not only $\mathbf{Q}$-divisors, but also $\mathbf{R}$-divisors. However, the finite generation theorem only holds when B is a $\mathbf{Q}$-divisor.

(2) Although in our discussion we assumed that the base field is the complex number field $\mathbf{C}$, all proofs work for algebraically closed fields in characteristic 0. Moreover, the results can be extended to algebraically nonclosed fields after necessary modifications. On the other hand, it is expected that the same conclusions (theorems in the minimal model theory and the finite generation of canonical rings) still hold true in positive characteristics, but the arguments in this book fail for two reasons. First, the desingularization theorem will be used in many places, which is still an open problem in positive characteristics; second, the vanishing theorem is a key tool in the

proofs, which has counterexamples in positive characteristics. Therefore, there is almost no progress in positive characteristics. (Added in 2023: This claim held only at the time of the publication of the Japanese version.)

(3) In this book, all results are stated in the log and relative version. If this seems annoying to you, just take the boundary B to be 0, take S to be a point $\operatorname{Spec} k$, and replace the direct image sheaf f_*F by the space of global sections $H^0(X, F)$, but the point of the proof will not change at all. However, as the proofs in the MMP are inductive, it is indispensable to state the log and relative version. Also, when dealing with algebraic varieties of nongeneral type, even if we start from an ordinary algebraic variety without boundary, log pairs naturally appear from the structure of algebraic fiber spaces.

(4) The finite generation of canonical rings is one of the main goals of the MMP in the beginning. Even though it is proved now, the existence of minimal models still remains open in the general case.

As prerequisites, we hope the reader has some familiarity with algebraic varieties. For this, it is sufficient to have standard knowledge from the textbook of Hartshorne ([44]). In particular, the theory of cohomologies of coherent sheaves is a basic tool; the concept of linear systems of divisors and the correspondence of Cartier divisors and invertible sheaves on a normal algebraic variety are important, which will be explained in Chapter 1; also it is better to have knowledge of algebraic surface theory as in [44, Chapter V]; but it is not necessary to understand every detail in [44] because, other than Section 2.7, this book does not deal with general schemes but only deals with irreducible reduced separated schemes of finite type over an algebraically closed field (i.e. algebraic varieties). The Kodaira vanishing theorem and the Hironaka desingularization theorem are important theorems cited in this book (the statements of the theorems will be given). These are indispensable tools for the discussions in this book, but it is not necessary to understand the proofs.