SCREENABILITY, POINTWISE PARACOMPACTNESS, AND METRIZATION OF MOORE SPACES

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F. B. Jones **(6)** has shown that, if $2^{\aleph_0} < 2^{\aleph_1}$, then every separable normal Moore space is metrizable. It is not known whether this assumption is necessary, though perhaps some progress is made in **(5)**. However, it is easily seen from R. H. Bing's Example E in **(3)** that a certain condition (see (1) below) implied by $2^{\aleph_0} < 2^{\aleph_1}$ is necessary. Also in **(3)**, Bing showed that every screenable normal Moore space is metrizable. In this paper we establish that: (1) every separable normal Moore space is metrizable if and only if every uncountable subspace M of E^1 contains a subset which is not an F_σ (in M); (2) if every pointwise paracompact normal Moore space is metrizable, then so is every separable normal Moore space; (3) every screenable Moore space is pointwise paracompact but not conversely; (4) a T_3-space is a pointwise paracompact Moore space if and only if it has a uniform base (in the sense of **(1**, p. 40**)**, *not* a uniformity). Also screenability is shown to be the generalization of strong paracompactness obtained by replacing star-countable refinement by σ-star-countable refinement, and it is shown that in certain spaces screenability is similarly related to full normality.

Terms not defined are as in **(1)**, **(3)**, or **(7)**. A Moore space is one satisfying Axiom 0 and the first three parts of Axiom 1 of **(9)**.

THEOREM 1. *Every separable normal Moore space is metrizable if and only if every uncountable subspace M of E^1 contains a subset which is not an F_σ (in M).*

Proof. The necessity of the condition follows from **(3**, Example E**)**.

The condition is also sufficient. For, suppose that S is a separable normal Moore space which is not metrizable. Then, by **(6**, Lemma C**)**, S contains an uncountable subset U_2 which has no limit point. Let R be a simple one-to-one point sequence such that $R_1, R_2, \ldots$ is dense in S. For each $x \in U_2$, let $r(x)$ be some one definite sub-sequence of R which converges to x. Let L_2 be the linearly ordered space whose points are the sequences $\{r(x):x \in U_2\}$ with the lexicographic ordering: $r(x) < r(y)$ in L_2 provided that, for some n, the first n entries of $r(x)$ and $r(y)$ are the same and $r(x)_{n+1}$ precedes $r(y)_{n+1}$ in R. Note that, since L_2 is separable, L_2 is homeomorphic to a subspace of E^1. By **(9**, Theorem 73, p. 50**)** there is a subset U_1 of U_2 such that, if $L_1 = r(U_1) = \{r(x) : x \in U_1\}$, then every segment with end points in L_1 contains uncountably many points of L_1.

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It will now be shown that there is an uncountable subset U of U_1 such that, for any $x \in U$ and any n, there are points y and z of U such that $r(y) < r(x) < r(z)$ and $r(x)$, $r(y)$, and $r(z)$ all have the same first n terms. For each n, let P_n be the subset of U_1 to which x belongs only if (1) no member $r(y)$ of L_1 which precedes $r(x)$ has the same first n terms as $r(x)$ and (2) n is the smallest natural number for which (1) is true. Then since, for $x, y \in P_n$, $r(x)$ does not have the same first n terms as $r(y)$, and since there are only countably many finite sub-sequences of R, P_n is at most countable. Hence

$$P = \bigcup_{n=1}^{\infty} P_n$$

is countable. Similarly, if F is the subset of U_1 to which x belongs only if, for some n, no member $r(y)$ of $L_1 - r(P)$ such that $r(y) > r(x)$ has the same first n terms as $r(x)$, then F is countable. It is easily seen that $U = (U_1 - P) - F$ has the desired property.

Let $L = r(U)$ and let H be a subset of U such that $r(H)$ is not an F_σ (in L). Since every subset of U is closed ($U \subset U_2$ and U_2 has no limit point), and since S is normal, there are mutually exclusive open sets Q and V such that $H \subset Q$ and $U - H \subset V$. Note that, for $x \in H$, $r(x)$ converges to x. Hence, there is an n such that, for $i > n$, $r(x)_i \in Q$. Let

$$H_n = \{x : x \in H \text{ and } r(x)_i \in Q \text{ for } i > n\}$$

Note that

$$H = \bigcup_{n=1}^{\infty} H_n;$$

hence

$$r(H) = \bigcup_{n=1}^{\infty} r(H_n).$$

Suppose that $x \in U$ and that, for some n, $r(x) \in \overline{r(H_n)}$. For each m there are points y and z in U such that $r(y) < r(x) < r(z)$ and the first $n + m$ terms of $r(x)$, $r(y)$, and $r(z)$ are the same. But, since $r(x) \in \overline{r(H_n)}$, there is a $q \in H_n$ such that $r(y) < r(q) < r(z)$. Thus, for each m, there is a $q \in H_n$ such that $r(q)$ and $r(x)$ have the same first $n + m$ terms. Hence, all except possibly the first n terms of $r(x)$ are points of Q; hence (since $r(x)$ converges to x, $x \in U$ and $\bar{Q} \cap [U - H] = \emptyset$) $x \in H$. Thus,

$$r(H) = \bigcup_{n=1}^{\infty} \overline{r(H_n)},$$

so that $r(H)$ is an F_σ contrary to the choice of H. The condition is, therefore, sufficient.

Theorem 2 (or its corollary) and Example 1 establish that in a Moore space pointwise paracompactness is a weaker condition than screenability.

THEOREM 2. *Let S be a topological space in which every closed set is a G_δ. If S is screenable, then S is pointwise paracompact.*

Proof. Let H be an open covering of S. Since S is screenable, there is a sequence $K_1, K_2, \ldots$ such that each K_i is a collection of mutually exclusive open sets and $\bigcup_{i=1}^{\infty} K_i$ is a covering of S which refines H. For each i, let $M_i = S - K_i^*$ and let $R_{i1}, R_{i2}, \ldots$ be a decreasing sequence of open sets with common part M_i. Let

$$G = \left\{ g : g = \left[\bigcap_{j=1}^{i-1} R_{ji} \right] \cap k, k \in K_i, i = 2, 3, \ldots \right\}.$$

Then $G \cup K_1$ is an open covering of S. For, if $p \in S$ and i is the smallest natural number such that $p \in k$ for some $k \in K_i$, then either $i = 1$, or, in case $i > 1$,

$$p \notin \left[\bigcup_{j=1}^{i-1} K_j \right]^*$$

so that

$$p \in \bigcap_{j=1}^{i-1} (S - K_j^*) \subset \bigcap_{j=1}^{i-1} R_{ji},$$

in which case

$$p \in \left[\bigcap_{j=1}^{i-1} R_{ji} \right] \cap k \subset G^*$$

for some $k \in K_i$. Clearly $G \cup K_1$ refines H. Finally, $G \cup K_1$ is point-finite. For, if $p \in S$ and i is the smallest natural number such that $p \in k$ for some $k \in K_i$, then there is a natural number N such that, for $j \geqslant N$, $p \notin R_{ij}$, hence such that, for $j > N$ and any $k \in K_j$,

$$p \notin \left[\bigcap_{n=1}^{j-1} R_{nj} \right] \cap k.$$

Then, since p belongs to at most one member of each of the collections $K_1, K_2, \ldots, K_N$, p belongs to at most N elements of $G \cup K_1$. Thus $G \cup K_1$ is a point-finite covering of S which refines H and S is pointwise paracompact.

COROLLARY. *Every screenable Moore space is pointwise paracompact.*

Proof. By (**9**, Theorem 118, p. 81) every closed set in a Moore space is a G_δ (i.e., an inner limiting set).

Note that every closed set is a G_δ in a semi-metric space (**8**) (a more general class than Moore spaces, although still not the most general to which Theorem 2 applies).

Example 1. A pointwise paracompact Moore space S which is not screenable. The space S consists of all points of the plane on or above the x-axis with a

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basis G as follows: (1) for p above the x-axis, $\{p\} \in G$; (2) for each x and each natural number n,

$$\{(t, y) : t = x + y \text{ or } t = x - y, 0 \leqslant y \leqslant 1/n\} \in G$$

(every "V" with vertex on the x-axis, sides of slope 1 and -1, height $1/n$). Clearly S is a Moore space, and, for every open covering H of S, there is an open covering K of S such that K refines H and no point of S belongs to more than two members of K. That S is not screenable follows readily by a category argument or by observing that, if S were screenable, then Example 3.3 of **(8)** would be screenable and hence metrizable (which it is not).

The space of Example 2 has all of the properties stated in Example B of **(3)** but is considerably simpler.

Example 2. A screenable non-normal Moore space S with an open covering H with respect to which the star of each point is metrizable.

The points of S are all points of the plane on or above the x-axis. A basis G for S is as follows: (1) for p above the x-axis, $\{p\} \in G$; (2) for each rational r and natural number n,

$$\{(r, y) : 0 \leqslant y \leqslant 1/n\} \in G$$

(vertical segment with its lower end point at $(r, 0)$); and (3) for each irrational x and natural number n,

$$\{(t, y) : t = x + y, 0 \leqslant y \leqslant 1/n\} \in G$$

(segment with slope 1 and an end point at $(x, 0)$). By a category argument (the irrationals on the x-axis being second category) S is easily seen to be non-normal. The other properties follow immediately.

The author conjectures that every pointwise paracompact normal Moore space is metrizable; it is suggested that some modification of the proof of **(3**, Theorem 3**)** might be used to prove this. By Theorem 2 and Example 1 that would generalize **(3**, Theorem 8**)**. The following theorem shows that the conjectured proposition might also be useful in determining whether $2^{\aleph_0} < 2^{\aleph_1}$ is a necessary condition for every separable normal Moore space to be metrizable.

THEOREM 3. *If every pointwise paracompact normal Moore space is metrizable, then so is every separable normal Moore space.*

Proof. Suppose that there exists a non-metrizable separable normal Moore space. Then by Theorem 1 there is an uncountable subspace M of E^1 such that every subset of M is an F_σ (in M). Let the space S consist of all points of the plane either on the x-axis, with abscissa in M, or above the x-axis, with a basis G as follows: (1) for p above the x-axis, $\{p\} \in G$; (2) for $x \in M$ and n a natural number,

$$\{(t, y) : t = x + y \text{ or } t = x - y, 0 \leqslant y \leqslant 1/n\} \in G$$

(each "V" with vertex $(x, 0)$ for $x \in M$, sides of slope 1 and -1, height $1/n$). It is easy to see that S is a normal Moore space (**3**, Example E). That S is not screenable (hence not metrizable) follows by an elementary argument, or by observing that, if S is screenable, then so is Example E of (**3**) (with $X = M$), which leads to a contradiction.

Theorem 4 shows that Aleksandrov's Metrization Theorem in (**1**, p. 40) is a corollary of (**3**, Theorems 10 and 12); see also (**3**, footnote 10, p. 183).

Definition. (**1**, p. 40). A base B for a space S is *uniform* if, for $x \in S$, any infinite subset of B, each member of which contains x, is a base at x.

Note (**1**, p. 40) that S is then pointwise paracompact.

THEOREM 4. *A T_3-space S is a pointwise paracompact Moore space if and only if S has a uniform base.*

Proof. Suppose that S has a uniform base B_1. For each n, let H_n be a point-finite refinement of B_n and let B_{n+1} be a subset of B_1 which covers S and each non-degenerate element of which is properly contained in an element of H_n. For

$$G_n = \bigcup_{i=n}^{\infty} B_i,$$

$G_1, G_2, \ldots$ is clearly a development for S.
The necessity follows trivially.

Remark. See (**4**, **10**, and **1**)—in the light of Theorem 4—for some metrization theorems for pointwise paracompact Moore spaces. For other related theorems and further references see (**7**, p. 171), in which the term "metacompact" is used in place of "pointwise paracompact."
Theorem 5 gives a relationship between screenability and strong paracompactness (**1**, pp. 36–39).

Definition (**1**, footnote 1, p. 38). A collection H of sets is said to be *star-countable* provided that every element of H intersects at most countably many elements of H.

Definition (**2**, p. 511). The collection G is *coherent* means that each proper subcollection G' of G contains an element which intersects an element of $G - G'$.

LEMMA 1. *If H is a star-countable collection of sets, then any coherent subcollection K of H is countable.*

Proof. Let $g \in K$. Define a sequence $K_1, K_2, \ldots$ of subcollections of K as follows: $K_1 = \{g\}$, and, for each n, K_{n+1} is the star of K_n^* with respect to K. Since H is star-countable, $\bigcup_{n=1}^{\infty} K_n$ is countable. Moreover, $K = \bigcup_{n=1}^{\infty} K_n$. For, if not, then ($K$ being coherent) some member h of $K - \bigcup_{n=1}^{\infty} K_n$ intersects a member of $\bigcup_{n=1}^{\infty} K_n$. Hence, h intersects a member of K_n for some n, so that $h \in K_{n+1}$ contrary to $h \in K - \bigcup_{n=1}^{\infty} K_n$.

THEOREM 5. *A necessary and sufficient condition that a space S be screenable is that every open covering of S have a σ-star-countable* (**1**, pp. 35–38) *refinement which is an open covering of S.*

Proof. The condition is sufficient. For, suppose that H is an open covering of S which has a refinement $L = \bigcup_{n=1}^{\infty} L_n$ such that L is an open covering of S, and each L_n is star-countable. By Lemma 1, for each n, every maximal coherent subcollection of L_n is countable. Order each such subcollection of L_n in a simple sequence, and, for each i, let L_{ni} consist of all ith terms of the maximal coherent subcollections of L_n. Then, for each n, $L_n = \bigcup_{i=1}^{\infty} L_{ni}$ such that each L_{ni} is a collection of mutually exclusive open sets. Hence if

$$K = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} L_{ni},$$

then K is a countable set of collections of mutually exclusive open sets, K covers S, and K refines H. Thus, S is screenable.

The necessity follows trivially.

Theorems 6 and 7 give a similar relationship between screenability and full normality (**3**, footnote 10, p. 183) in certain spaces.

Definition. The collection K is a *point-star-refinement* of the collection H means that, for each point p, the star of p with respect to K is contained in an element of H. See (**1**, p. 35) for the meaning of the prefix "σ."

THEOREM 6. *A necessary and sufficient condition that a pointwise paracompact topological space S be screenable is that every open covering H of S have a σ-point-star-refinement which is an open covering of S.*

Proof. The condition is sufficient. For suppose that H is an open covering of S. Let K_1 be a point-finite open covering of S which refines H; let $J_1 = \bigcup_{i=1}^{\infty} L_i$ be an open covering of S such that each L_i is a point-star-refinement of K_1; and let M_1 be the set of all points each of which belongs to exactly one member of K_1. For each $k \in K_1$ such that $k \cap M_1 \neq \emptyset$ and each i, let

$$R_1(k, i) = \bigcup \{g\colon g \in L_i, g \cap k \cap M_1 \neq \emptyset\}.$$

If $h, k \in K_1$ and $h \neq k$, then $R_1(h, i) \cap R_1(k, i) = \emptyset$. For, if $p \in R_1(h, i) \cap R_1(k, i)$, then the point-star with respect to L_i of p intersects both $k \cap M_1$ and $h \cap M_1$. Hence some member d of K_1 intersects both $k \cap M_1$ and $h \cap M_1$ (L_i being a point-star refinement of K_1); and therefore $h = d = k$, since every point of M_1 belongs to exactly one member of K_1. Thus, for each i, the members of

$$G(1, i) = \{R_1(h, i) : h \in K_1\}$$

are mutually exclusive and $\bigcup_{i=1}^{\infty} G(1, i)$ covers M_1.

Let M_2 be the set of all points each of which belongs to exactly two members of K_1, and let

$$K_2 = \{g : g = h \cap k \text{ for } h \neq k \text{ and } h, k \in K_1\} \cup \left[\bigcup_{i=1}^{\infty} G(1, i) \right].$$

If $p \in S - M_1$, then p belongs to at least two members of K_1. Hence K_2 covers S. Note that if $p \in M_2$, then p belongs to exactly one member of K_2. Let J_2 be a covering of S which is a σ-point-star refinement of K_2, and, for each i, define the collection $G(2, i)$ in the same way that $G(1, i)$ was defined. For each $n > 2$, let M_n be the set of all points belonging to exactly n members of K_1, let

$$K_n = \left\{ g : g = \bigcap_{i=1}^{n} h_i, h_i \neq h_j \text{ for } i \neq j, h_i \in K_1 \text{ for } i = 1, \dots n \right\} \cup \left[\bigcup_{j=1}^{n-1} \bigcup_{i=1}^{\infty} G(j, i) \right],$$

and for each i define $G(n, i)$ as $G(1, i)$ was defined. Since

$$S = \bigcup_{i=1}^{\infty} M_i,$$

$$G = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} G(n, i)$$

covers S. By definition, G refines H and is the union of countably many collections of mutually exclusive open sets. Thus S is screenable.

The necessity follows trivially.

THEOREM 7. *A necessary and sufficient condition that a developable* (**3**, p. 180) *space S be screenable is that every open covering of S have a σ-point-star-refinement which is an open covering of S.*

Proof. The condition is sufficient. For suppose that H is an open covering of S. Let $G_1, G_2, \dots$ be a development for S such that each G_i refines H. By the proof of (**3**, Theorem 9), there is a sequence $X_1, X_2, \dots$ of collections of subsets of S such that: (1) for each i, no member of G_i intersects two elements of X_i; (2) for each i and each $M \in X_i$, the star of M with respect to G_i is contained in some member of H; and (3) $\bigcup_{i=1}^{\infty} X_i$ covers S. For each i, let $L_i = \bigcup_{n=1}^{\infty} L_{in}$ be an open covering of S such that each L_{in} is a point-star-refinement of G_i. For each i and n and $M \in X_{i+1}$, let $K_{in}(M)$ be the star of M with respect to L_{in}. Note that for M, $N \in X_i$ and $M \neq N$, $K_{in}(M) \cap K_{in}(N) = \emptyset$. For, if not, there are members h and k of L_{in} such that $h \cap k \neq \emptyset$, $h \cap M \neq \emptyset$, and $k \cap N \neq \emptyset$. But, since L_{in} is a point-star-refinement of G_i, there is then a $g \in G_i$ such that $h \cup k \subset g$, so that g intersects both M and N contrary to (1) above. Thus, for each i and n, $\{K_{in}(M) : M \in X_i\}$ is a collection of mutually exclusive open sets. Since

$$\bigcup_{i=1}^{\infty}\bigcup_{n=1}^{\infty}\{K_{in}(M) : M \in X_i\}$$

clearly covers S and refines H, S is screenable.

The necessity follows trivially.

In conclusion, it appears that an answer to any of the following questions would probably also settle: (a) whether $2^{\aleph_0} < 2^{\aleph_1}$ is a necessary condition for Jones' metrization theorem in **(6)**, and (b) to what extent Aleksandrov's Metrization Theorem **(1**, p. 40**)** can be generalized.

1. Is every normal pointwise paracompact Moore space (or normal space with a uniform base **(1)**) metrizable?

2. What is a sufficient condition for a pointwise paracompact Moore space to be screenable?

3. Does the condition that every uncountable subspace M of E^1 contains a set which is not an F_σ (in M) imply that $2^{\aleph_0} < 2^{\aleph_1}$ or is it a consequence of the other axioms of the real numbers?

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