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Inverse Pressure Estimates and the Independence of Stable Dimension for Non-Invertible Maps

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Abstract. We study the case of an Axiom A holomorphic non-degenerate (hence non-invertible) map $f\colon \mathbb{P}^2\mathbb{C} \to \mathbb{P}^2\mathbb{C}$, where $\mathbb{P}^2\mathbb{C}$ stands for the complex projective space of dimension 2. Let Λ denote a basic set for f of unstable index 1, and x an arbitrary point of Λ; we denote by $\delta^s(x)$ the Hausdorff dimension of $W^s_r(x) \cap \Lambda$, where r is some fixed positive number and $W^s_r(x)$ is the local stable manifold at x of size r; $\delta^s(x)$ is called *the stable dimension at x*. Mihailescu and Urbański introduced a notion of inverse topological pressure, denoted by P^-, which takes into consideration preimages of points. Manning and McCluskey studied the case of hyperbolic diffeomorphisms on real surfaces and give formulas for Hausdorff dimension. Our non-invertible situation is different here since the local unstable manifolds are not uniquely determined by their base point, instead they depend in general on whole prehistories of the base points. Hence our methods are different and are based on using a sequence of inverse pressures for the iterates of f, in order to give upper and lower estimates of the stable dimension. We obtain an estimate of the oscillation of the stable dimension on Λ. When each point x from Λ has the same number d' of preimages in Λ, then we show that $\delta^s(x)$ is independent of x; in fact $\delta^s(x)$ is shown to be equal in this case with the unique zero of the map $t \to P(t\phi^s - \log d')$. We also prove the Lipschitz continuity of the stable vector spaces over Λ; this proof is again different than the one for diffeomorphisms (however, the unstable distribution is not always Lipschitz for conformal non-invertible maps). In the end we include the corresponding results for a real conformal setting.

1 Introduction and Notations. Inverse Topological Pressure

In the case of C^2 Axiom A diffeomorphisms of real surfaces, Manning and McCluskey [2] proved that the Hausdorff dimension of a basic set Λ is given by the formula $HD(\Lambda) = \delta_u + \delta_s$, with δ_u, δ_s being the unique zeros of the pressure functions of the potentials $-t\log|Df_u|$, $t\log|Df_s|$ respectively, considered on Λ. For the case of hyperbolic automorphisms on $\mathbb{C}^2$ (Henon maps), Verjovsky and Wu [10] showed that the Hausdorff dimension of the intersection between local stable manifolds and the Julia set is given also as the unique zero of a pressure function. For non-invertible conformal maps f (for example holomorphic maps on the projective complex space $\mathbb{P}^2$) which are hyperbolic on a basic set Λ, the situation is completely different, and as shown in [3, 5], this stable dimension (the precise definition will be given later) is not equal to the unique zero of the pressure function. At the same time, we do not have a uniquely determined unstable manifold going through a given point of the basic set Λ. In order to deal with the non-invertible case, Mihailescu and

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Urbanski have introduced a notion of inverse pressure [6], which takes into consideration all the inverse iterates of points (instead of the forward iterates from the case of usual topological pressure). In this paper we will obtain a theorem (Theorem 2.1) giving lower estimates of the stable dimension by using zeros of inverse pressures of iterates of f. As a corollary we obtain an estimate of the maximum possible oscillation of the stable dimension on Λ.

Then, when the map is open on the basic set Λ, we will prove (Theorem 3.1) that the stable dimension is independent of the point; in the proof we use again ideas and concepts related to inverse pressure.

Most of these proofs and results work for a more general setting (finite-to-one conformal maps with hyperbolic structure on a basic set, and with the dimension of the stable vector spaces equal to 2), but we prefer to state them first in the case of holomorphic Axiom A maps on $\mathbb{P}^2$, and we include a section at the end of the paper with the theorems in the more general case.

Note also that in Theorem 1.5 we actually use the holomorphicity at the end of the proof.

In this section we recall some definitions and properties of inverse pressure which will be used later. We consider the setting where X is a compact metric space, $f\colon X \to X$ is a continuous surjective map on X, and $Y \subseteq X$ is a subset of X. Due to the surjectivity of f, for any point y of X, and any positive integer m, there exists $y_{-m} \in X$ such that $f^m(y_{-m}) = y$. By *prehistory of length m* (or *m-prehistory*, or *branch of length m*) of y, we will understand a collection of consecutive preimages of y, $C = (y, y_{-1}, \dots, y_{-m})$, where $f(y_{-i}) = y_{-i+1}, i = 1, \dots, m, y_0 = y$. Given a prehistory C, we shall denote by $n(C)$ its length. Fix $\varepsilon > 0$. Denote by $\mathcal{C}_m$ the set of all m-prehistories of points from X. For such an m-prehistory C, let $X(C, \varepsilon)$ be the set of points ε-*shadowed* by C (in backward time) *i.e.,*

$$X(C,\varepsilon) := \Big\{ z \in B(y_0,\varepsilon) : \text{for } j = 1,\dots,m,\ \exists z_{-j} \in f^{-1}(z_{-j+1}) \text{ and } z = z_0 \\ \text{such that } d(z_{-j}, y_{-j}) < \varepsilon \Big\}$$

Given the m-prehistory of y, $C = (y, y_{-1}, \dots, y_{-m})$ and a real continuous function ϕ on X, one can define the *consecutive sum* of ϕ on C, $S_m^-\phi(C) = \phi(y) + \phi(y_{-1}) + \cdots + \phi(y_{-m})$. We may also use the notation $S_m^-\phi(y_{-m})$ instead of $S_m^-\phi(C)$. We will define now the inverse pressure P^- by a procedure similar to that used in the case of Hausdorff outer measure. Let ϕ be an arbitrary continuous function $\phi \in \mathcal{C}(X,\mathbb{R})$ (where $\mathcal{C}(X,\mathbb{R})$ is the set of real continuous functions on X); also let λ be a real number and N a positive integer. Denote by $\mathcal{C}_* := \bigcup_{m\geq 0} \mathcal{C}_m$. We say that a subset $\Gamma \subset \mathcal{C}_*$, ε-*covers* X if $X = \bigcup_{C\in\Gamma} X(C,\varepsilon)$. Then define

$$M_f^-(\lambda,\phi,Y,N,\varepsilon) := \inf\Big\{ \sum_{C\in\Gamma} \exp(-\lambda n(C) + S_{n(C)}^-\phi(C)),\ n(C) \geq N, \forall C \in \Gamma, \\ \text{and } \Gamma \subset \mathcal{C}_* \text{ s.t } Y \subset \bigcup_{C\in\Gamma} X(C,\varepsilon) \Big\}.$$

When N increases, the set of acceptable candidates Γ which ε-cover X gets smaller, therefore the infimum increases in the previous expression. Hence

$$\lim_{N\to\infty} M_f^-(\lambda,\phi,Y,N,\varepsilon)$$

exists and will be denoted by $M_f^-(\lambda, \phi, Y, \varepsilon)$. Now, let

$$P_f^-(\phi, Y, \varepsilon) := \inf\{\lambda : M_f^-(\lambda, \phi, Y, \varepsilon) = 0\}.$$

Consider two positive numbers $\varepsilon_1 < \varepsilon_2$ and let us compare

$$P_f^-(\phi, Y, \varepsilon_1) \quad \text{and} \quad P_f^-(\phi, Y, \varepsilon_2).$$

Given any prehistory C, we have that $X(C, \varepsilon_1) \subset X(C, \varepsilon_2)$, so if $\Gamma \subset \mathcal{C}_*$ ε_1-covers Y, then Γ also ε_2-covers Y. Therefore there are more candidates Γ in the expression of $M_f^-(\lambda, \phi, Y, N, \varepsilon_2)$ than in the expression of $M_f^-(\lambda, \phi, Y, N, \varepsilon_1)$. This shows that for any N, $M_f^-(\lambda, \phi, Y, N, \varepsilon_2) \leq M_f^-(\lambda, \phi, Y, N, \varepsilon_1)$. Hence $0 \leq M_f^-(\lambda, \phi, Y, \varepsilon_2) \leq M_f^-(\lambda, \phi, Y, \varepsilon_1)$, and then from definition, $P_f^-(\phi, Y, \varepsilon_2) \leq P_f^-(\phi, Y, \varepsilon_1)$. This proves that when ε decreases to 0, $P_f^-(\phi, Y, \varepsilon)$ increases, so the limit $\lim_{\varepsilon \to 0} P_f^-(\phi, Y, \varepsilon)$ does exist and is denoted by $P_f^-(\phi, Y)$ and is called *the inverse pressure* (or *inverse upper pressure*) of ϕ on Y. We call $P_f^-(\phi, Y, \varepsilon)$ the ε-*inverse pressure* of ϕ on Y. This notion has been introduced in [6]; here we have used slightly different notations. When the map f will be clear from the context, we may drop the index f from the notations for $P_f^-(\phi, Y), P_f^-(\phi, Y, \varepsilon), M_f^-(\lambda, \phi, Y, N, \varepsilon)$, etc. Also, we will denote by $P_f^-(\phi)$, $P_f^-(\phi, \varepsilon)$, $M_f^-(\lambda, \phi, N, \varepsilon)$, etc., the quantities $P_f^-(\phi, X)$, $P_f^-(\phi, X, \varepsilon)$, $M_f^-(\lambda, \phi, X, N, \varepsilon)$, etc., respectively. The following proposition provides some properties of P^-.

Proposition 1.1 *Let $f\colon X \to X$ be a continuous surjective map on the compact metric space X, ε a positive number and ϕ a function from $\mathcal{C}(X, \mathbb{R})$.*

(i) *If $Y_1 \subset Y_2 \subset X$, then $P_f^-(\phi, Y_1) \leq P_f^-(\phi, Y_2)$ and $P_f^-(\phi, Y_1, \varepsilon) \leq P_f^-(\phi, Y_2, \varepsilon)$.*

(ii) *If $Y = \bigcup_{j \in J} Y_j$ is a finite or countable union of subsets of X, then $P_f^-(\phi, Y, \varepsilon) = \sup_{j \in J} P_f^-(\phi, Y_j, \varepsilon)$ and $P_f^-(\phi, Y) = \sup_{j \in J} P_f^-(\phi, Y_j)$.*

(iii) *If f is a homeomorphism on X, then $P_f^-(\phi) = P_f(\phi)$, where $P_f(\phi)$ denotes the usual (forward) topological pressure of ϕ with respect to the map f.*

(iv) *$P_f^-(\phi, Y)$ is invariant to topological conjugacy,* i.e., *if $f\colon X \to X$, $g\colon X' \to X'$ are continuous surjective maps and $\Psi\colon X \to X'$ is a homeomorphism such that $\Psi \circ f = g \circ \Psi$, then $P_f^-(\phi, Y) = P_g^-(\phi \circ \Psi^{-1}, \Psi(Y))$, for any subset $Y \subset X$.*

Proof We will prove only part (ii), the others are straightforward. Assume that $Y = \bigcup_{j \in J} Y_j$ is a finite or countable union of subsets of X. We will show that given some $\varepsilon > 0$, $P_f^-(\phi, Y, \varepsilon) = \sup_j P_f^-(\phi, Y_j, \varepsilon)$, for any function $\phi \in \mathcal{C}(X, \mathbb{R})$; the other equality, $P_f^-(\phi, Y) = \sup_j P_f^-(\phi, Y_j)$ will follow similarly. First, directly from the definition of P^-, it follows that $P_f^-(\phi, Y, \varepsilon) \geq \sup_j P_f^-(\phi, Y_j, \varepsilon)$. Now take $t > \sup_j P_f^-(\phi, Y_j, \varepsilon)$. Then there exists some number $\alpha > 0$ so small that $t - \alpha > P_f^-(\phi, Y_j, \varepsilon), \forall j \in J$. So $M_f^-(t - \alpha, \phi, Y_j, \varepsilon) = 0$ for all $j \in J$. But from the fact that

$M_f^-(t-\alpha,\phi,Y_j,N,\varepsilon)$ grows with N, we obtain that

$$M_f^-(t-\alpha,\phi,Y_j,N,\varepsilon)=0,\ \forall j\in J,\ \forall N>0.$$

So, if N is fixed, then for any $j \in J$ there exists a set $\Gamma_j \subset \mathcal{C}_*$ such that $Y_j \subset \bigcup_{C\in\Gamma_j} X(C,\varepsilon)$ and $n(C) \geq N, \forall C \in \Gamma_j$ and we have $\sum_{C\in\Gamma_j} \exp(-(t-\alpha)n(C) + S^-_{n(C)}\phi(C)) \leq \frac{1}{2^j}$. Now, if we consider the collection $\Gamma := \bigcup_{j\in J}\Gamma_j$, then $Y = \bigcup_{j\in J} Y_j \subset \bigcup_{C\in\Gamma} X(C,\varepsilon), n(C) \geq N, \forall C \in \Gamma$, and $\sum_{C\in\Gamma} \exp(-(t-\alpha)n(C) + S^-_{n(C)}\phi(C)) \leq 1$. This means that $M_f^-(t-\alpha,\phi,Y,N,\varepsilon) \leq 1$, hence

$$M_f^-(t,\phi,Y,N,\varepsilon)\leq e^{-\alpha N}.$$

Thus $M_f^-(t,\phi,Y,\varepsilon) = 0$ and $t \geq P_f^-(\phi,Y,\varepsilon)$. In conclusion, since t has been taken arbitrarily larger than $\sup_{j\in J} P_f^-(\phi,Y_j,\varepsilon)$, we obtain the required equality, $P_f^-(\phi,Y,\varepsilon) = \sup_{j\in J} P_f^-(\phi,Y_j,\varepsilon)$. ■

Here are also some additional properties of P^-, whose proofs can partly be found in [6]; the proofs of the properties for ε-inverse pressures are similar.

Proposition 1.2 *Let $f\colon X\to X$ be a continuous surjective map on the compact metric space X, Y a subset of X and $\phi,\psi \in \mathcal{C}(X,\mathbb{R})$. Then*

(i) $P_f^-(\phi+\alpha,Y) = P_f^-(\phi,Y)+\alpha$.

(ii) *If $\phi \leq \psi$ on Y and ε is a positive number, then $P_f^-(\phi,Y) \leq P_f^-(\psi,Y)$ and $P_f^-(\phi,Y,\varepsilon) \leq P_f^-(\psi,Y,\varepsilon)$.*

(iii) *$P_f^-(\,\cdot\,,Y)$ is either finitely valued or constantly ∞.*

(iv) *$|P_f^-(\phi,Y) - P_f^-(\psi,Y)| \leq \|\phi-\psi\|$ if $P_f^-(\,\cdot\,,Y)$ is finitely valued; a similar inequality holds for the corresponding ε-inverse pressures.*

(v) $P_f^-(\phi+\psi\circ f-\psi,Y) = P_f^-(\phi,Y)$.

(vi) *If ϕ is a strictly negative function on X, then the mapping $t \to P_f^-(t\phi,Y)$ is strictly decreasing if $P_f^-(\,\cdot\,,Y)$ is finitely valued. Furthermore, the mapping $t \to P_f^-(t\phi,Y,\varepsilon)$ is strictly decreasing.*

The inverse entropy h^- obtained by definition as $P^-(0)$ is smaller then or equal to the preimage entropy h_i [6] and actually, in the case of homeomorphisms, they both coincide with the usual topological entropy (definitions and useful properties of h_i are given, for example, in [6, 7]). Another interesting property of P^- gives an alternative way of calculating the inverse pressure, which will be used later in a proof.

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Proposition 1.3 (**[6]**) *Let $f\colon X \to X$ be a continuous surjective map on a compact metric space X, and $\phi \in \mathcal{C}(X, \mathbb{R})$. Let*

$$Q_m^-(\phi, \varepsilon) := \inf\Big\{ \sum_{C \in \Gamma} \exp(S_m^- \phi(C)), \Gamma \subset \mathcal{C}_m, \Gamma \varepsilon - \textit{covering } X \Big\}.$$

Then $P^-(\phi) = \lim_{\varepsilon \to 0} \overline{\lim}_{m \to \infty} \frac{1}{m} \cdot \log Q_m^-(\phi, \varepsilon)$.

In the sequel, we will focus on the case of a holomorphic non-degenerate map $f\colon \mathbb{P}^2 \to \mathbb{P}^2$, where $\mathbb{P}^2$ represents the 2-dimensional complex projective space $\mathbb{P}^2\mathbb{C}$. Any holomorphic map f on $\mathbb{P}^2$ is given as

$$f([z\colon w\colon t]) = [P(z, w, t)\colon Q(z, w, t)\colon R(z, w, t)],$$

with P, Q, R homogeneous polynomials in z, w, t, all having the same degree d. If $d \geq 2$, then f is called non-degenerate; in this case f is non-invertible. We shall assume in the sequel that f is non-degenerate and has Axiom A; let Λ be one of its basic sets of *unstable index* 1, meaning that Df has on Λ both stable and unstable directions. For definitions and discussions of Axiom A for non-invertible maps [3, 8] are good references. An important point to remember is that, since f is not invertible on the invariant set Λ, one has to define hyperbolicity with respect to the *natural extension* of Λ. We recall briefly this notion and also how to define hyperbolicity in this non-invertible case. Denote first

$$\hat{\Lambda} := \{\hat{x} = (x, x_{-1}, \dots) \text{ where } x_{-i} \in \Lambda \text{ and } f(x_{-i-1}) = x_{-i}, i \geq 0, x_0 = x\}$$

and call this set the *natural extension* of Λ with respect to f. Then $\hat{\Lambda}$ is a compact metric space endowed with the metric $d(\hat{x}, \hat{y}) = \sum_{i \geq 0} d(x_{-i}, y_{-i})/2^i$. More generally, we can define a metric d_K on $\hat{\Lambda}$ for any $K > 1$ by setting $d_K(\hat{x}, \hat{y}) = \sum_{i \geq 0} d(x_{-i}, y_{-i})/K^i$. As above, we will not specify the constant K in the notation d_K when $K = 2$. Also, it can be noticed that for all $K > 1$, d_K gives the same topology on $\hat{\Lambda}$, namely the topology induced on the subset $\hat{\Lambda}$ by the product topology on the larger space $\Lambda^{\mathbb{N}}$. We denote by $\pi\colon \hat{\Lambda} \to \Lambda$ the canonical projection $\pi(\hat{x}) = x$ and by $\hat{f}$ the homeomorphism $\hat{f}\colon \hat{\Lambda} \to \hat{\Lambda}$, $\hat{f}(\hat{x}) = (fx, x, x_{-1}, \dots)$. The hyperbolicity of f on Λ means that there exist constants $C > 0$, $\lambda' > 1$, and for every $\hat{x} \in \hat{\Lambda}$, a vector space $E^u_{\hat{x}} \subset T_x\mathbb{P}^2$, and a vector space $E^s_x \subset T_x\mathbb{P}^2$ such that $Df(E^u_{\hat{x}}) \subset E^u_{\widehat{fx}}$, $Df(E^s_x) \subset E^s_{fx}$ and we have the inequalities $\|Df^k_x(v)\| \leq C(\lambda')^{-k}\|v\|$, $\|Df^k_x(w)\| \geq C(\lambda')^k\|w\|$, for every $x \in \Lambda, k \geq 0$ and all vectors $v \in E^s_x, w \in E^u_{\hat{x}}$. In the definition of hyperbolicity on $\hat{\Lambda}$ we assume also that $E^s_x \oplus E^u_{\hat{x}} = T_x\mathbb{P}^2, \forall \hat{x} \in \hat{\Lambda}$ and that E^s_x depends continuously on x, while $E^u_{\hat{x}}$ depends continuously on $\hat{x}$. And E^s_x is called the *stable tangent vector space* (or the *stable space*) at x; $E^u_{\hat{x}}$ is called the *unstable tangent vector space* (or *unstable space*) corresponding to the prehistory $\hat{x}$. As in the diffeomorphism case, it is possible [8] to show that if r is small enough (for example $0 < r < r_0$), there exist stable and unstable local manifolds passing through x:

$$W^s_r(x) := \{y \in \mathbb{P}^2, d(f^i x, f^i y) < r, i \geq 0\},$$

and

$$W^u_r(\hat{x}) := \{y \in \mathbb{P}^2, \exists\, \hat{y} \in \pi^{-1}(y) \text{ with } d(y_{-i}, x_{-i}) < r, i \geq 0\}.$$

If moreover f is holomorphic on $\mathbb{P}^2$, the local (un)stable manifolds on a basic set of unstable index one are analytic disks.

Now, given a point $x \in \Lambda$ and a small fixed number $0 < r < r_0 < \operatorname{diam} \Lambda/2$, let $\delta^s(x) := HD(W^s_r(x) \cap \Lambda)$, where HD stands for the Hausdorff dimension of a set. We shall call $\delta^s(x)$, *the stable dimension at x*. In the sequel we shall suppose also that $\mathcal{C}_f \cap \Lambda = \varnothing$, where $\mathcal{C}_f$ denotes the critical set of f. Hence, one can define the negative function $\phi^s(y) := \log\bigl|Df|_{E^s_y}\bigr|$, $y \in \Lambda$; as a notational remark, E^s_y is a one-dimensional complex space and $\bigl|Df|_{E^s_y}\bigr|$ denotes the norm of Df restricted to this stable space.

We studied the stable dimension in [3, 5, 6]. In [3], the first author showed that $\delta^s(x) \leq t^s_*$, where t^s_* is the unique zero of the pressure function $t \to P(t\phi^s)$ (the topological pressure being calculated with respect to the map $f|_\Lambda$). However in the above inequality, we do not have equality in general. Indeed the gap between $\delta^s(x)$ and t^s_* is influenced by the number of preimages that a point from Λ has in Λ, as was explained in [5], where we obtained a better upper estimate t^s_0:

Theorem 1.4 *In the above setting, assume that the map $f|_\Lambda$ has the property that every point $x \in \Lambda$ has at least $d' \leq d$ preimages in Λ. Then $\delta^s(x) \leq t^s_0$, where t^s_0 is the unique zero of the function $t \to P\bigl(t \log\bigl|Df|_{E^s_y}\bigr| - \log d'\bigr)$ and as a consequence,*

$$\delta^s(x) \leq \frac{h(f|_\Lambda) - \log d'}{\Bigl|\log \sup_{y\in\Lambda}\bigl|Df|_{E^s_y}\bigr|\Bigr|}.$$

Let us focus now on the zeros $t^s_n(\varepsilon)$ of the ε-inverse pressure functions for the iterates $f^n|_\Lambda$. If Λ is a basic set for f, then $f(\Lambda) = \Lambda$, hence $f^n(\Lambda) = \Lambda, \forall n > 0$ integer. Let us denote by $Df_s(y)$ the linear map $Df|_{E^s_y}$; similarly, $Df^n_s(y)$ denotes $Df^n|_{E^s_y}$, $y \in \Lambda$. Since f is conformal on stable manifolds,

$$|Df^n_s(y)| = |Df_s(y)| \cdot |Df_s(fy)| \cdots |Df_s(f^{n-1}y)|, \forall y \in \Lambda.$$

Then $\phi^s_n(y) := \log|Df^n_s(y)|$, $y \in \Lambda$, so ϕ^s_n is a strictly negative function on Λ, which has finite values since $\mathcal{C}_f \cap \Lambda = \varnothing$. From Proposition 1.2(vi) applied to $f^n|_\Lambda \colon \Lambda \to \Lambda$, it follows that the function $t \to P^-_{f^n}(t\phi^s_n, \varepsilon)$ is strictly decreasing; since $P^-_{f^n}(0, \varepsilon) \geq 0$, and $P^-_{f^n}(t\phi^s_n, \varepsilon) < 0$ for $t > 0$ large enough, it follows that this strictly decreasing function has a unique zero, denoted by $t^s_n(\varepsilon)$. The same is true for the function $t \to P^-_{f^n}(t\phi^s_n)$ which has a unique zero t^s_n. When $n = 1$ we denote $t^s_1(\varepsilon)$ by $t^s(\varepsilon)$, and t^s_1 by t^s. We shall prove in the sequel that $t^s_n(\varepsilon) \geq t^s_{np}(\varepsilon)$ and $t^s_n = t^s$, for any positive integers n, p and any $\varepsilon > 0$.

First, we will prove that the stable spaces E^s_y depend Lipschitz-continuously on $y \in \Lambda$. In addition we will show the Lipschitz continuity of $y \to E^s_y$ when y ranges in $W^s_r(x)$ ($x \in \Lambda$), and moreover, that the Lipschitz constant on these stable leaves can be chosen independently of the point $x \in \Lambda$ in the holomorphic case. Remark also that the unstable spaces cannot depend Lipschitz-continuously on their base points since in general they depend on whole prehistories. In [3], one of the authors showed that the unstable spaces $E^u_{\hat{x}}$ depend Hölder-continuously on $\hat{x}$, with respect to a fixed metric d_K on $\hat{\Lambda}$; the respective Hölder exponent depends on the chosen constant $K > 1$. The following theorem was known in the case of conformal diffeomorphisms, but to our knowledge it has never appeared in the case of non-degenerate holomorphic maps on $\mathbb{P}^2$ (which are non-invertible). As it turns out below, the non-invertible case requires its own proof, different from the one given for diffeomorphisms. (For example, in the non-invertible situation we cannot use the inverse iterate f^{-1}, and on the natural extension $\hat{\Lambda}$ we cannot use a differentiable structure.)

Theorem 1.5 *Consider $f\colon \mathbb{P}^2 \to \mathbb{P}^2$ a holomorphic Axiom A map, and let Λ be one of its basic sets of unstable index one such that $\mathcal{C}_f \cap \Lambda = \varnothing$. Then the map $x \to E^s_x$ is Lipschitz continuous as a map from Λ to the bundle $G_1(\Lambda)$ of spaces of complex dimension one in the tangent bundle over Λ,* i.e., *there exists a positive constant Υ such that for all x, y from Λ, $d(E^s_x, E^s_y) \le \Upsilon d(x, y)$. In particular, if $\phi^s(y) := \log |Df|_{E^s_y}|$, $y \in \Lambda$, then ϕ^s is Lipschitz continuous. Moreover, there exist a small $r > 0$ and $\Xi > 0$ such that for any $x \in \Lambda$ and any points $y, z \in W^s_r(x)$, we have $|\phi^s(y) - \phi^s(z)| \le \Xi \cdot d(y, z)$.*

Proof For every $K > 1$, consider the metric d_K on $\hat{\Lambda}$, given by the formula

$$d_K(\hat{x}, \hat{y}) := d(x, y) + \frac{d(x_{-1}, y_{-1})}{K} + \frac{d(x_{-2}, y_{-2})}{K^2} + \cdots .$$

Notice that the topology given by d_K on $\hat{\Lambda}$ is independent of K and is induced by the product topology on a countable product of Λ's. In the sequel we shall use a Pointwise Hölder Section Theorem from [11].

Theorem **(Pointwise Hölder Section Theorem)** *Let $E = X \times Y$ be a vector bundle over a metric space X, where Y is a closed, bounded subset of a Banach space, and let $\pi\colon E \to X$ be the canonical projection. Let $F\colon E \to E$ be a bundle map covering a homeomorphism $h\colon X \to X$,* i.e., *$\pi \circ F = h \circ \pi$. Suppose that F satisfies the following conditions:*

(i) *F contracts the fibers of E in the sense that for all $x \in X$, there exists a constant $0 \le \lambda_x < 1$ such that $d(F(x, y), F(x, z)) \le \lambda_x d(y, z), \forall y, z \in Y$.*
(ii) *There exist constants $L \ge 1$ and $\alpha > 0$ such that for all $x, x' \in X$ and $y \in Y$, $|F(x, y) - F(x', y)| \le L \cdot d(x, x')^\alpha$.*
(iii) *There exists some positive number η such that $\sup_{x\in X} \lambda_x \cdot \mu_x^{-\alpha} =: \rho(\alpha) < 1$ where*

$$\mu_x := \inf\Big\{ \frac{d(hx, hx')}{d(x, x')}, x, x' \in X, x \neq x', d(x, x') < \eta \Big\}.$$

Also, let $\mu := \inf_{x\in X} \mu_x$ and assume that $\mu > 0$.

Then we have the following:

(a) *There exists a unique section $\sigma\colon X \to E$ whose image is invariant under F,* i.e., *$\sigma \circ h(x) = F \circ \sigma(x), x \in X$.*

(b) *σ is Hölder continuous with exponent α,* i.e., $|\sigma(x) - \sigma(x')| \le Hd(x,x')^\alpha$ *for all $x, x' \in X$.*

(c) *Assume that the diameter of Y is bounded by R, then we can bound the Hölder constant H by*

$$H \le \frac{LR}{\mu\eta^\alpha(1 - \sup \lambda_x \mu_x^{-\alpha})}.$$

Let us now return to our setting and see how we can apply this theorem. By definition of hyperbolicity of f, there exists a continuous splitting of the tangent bundle to $\mathbb{P}^2$ over $\hat{\Lambda}$, given by $T_{\hat{\Lambda}}\mathbb{P}^2 = E^s \oplus E^u$, where E^s_x depends continuously on $x \in \Lambda$ and $E^u_{\hat{x}}$ depends continuously on $\hat{x} \in \hat{\Lambda}$. The stable space E^s_x and the stable manifold of size $r > 0$ at x depend only on the forward iterates of x, whereas the unstable space $E^u_{\hat{x}}$ and the unstable manifold $W^u_r(\hat{x})$ depend on the entire prehistory $\hat{x}$ of x. Let us take an arbitrary constant $K > 1$ and consider the metric d_K on $\hat{\Lambda}$. Since continuous maps can be approximated by Lipschitz continuous maps, there exists a splitting $F^s \oplus F^u(K)$ of $T_{\hat{\Lambda}}\mathbb{P}^2$ such that the linear subspaces of complex dimension 1, F^s_x, depend Lipschitz-continuously on $x \in \Lambda$ and the subspaces of dimension 1, $F^u_{\hat{x}}(K)$ depend Lipschitz-continuously on $\hat{x} \in \hat{\Lambda}$; also we assume that F^s_x approximates E^s_x, and $F^u_{\hat{x}}(K)$ approximates $E^u_{\hat{x}}$ uniformly in x, respectively $\hat{x}$. As a remark, the spaces $F^u_{\hat{x}}(K)$ depend in general on K since they must vary Lipschitz-continuously with respect to the metric d_K, whereas the spaces F^s_x are Lipschitz only with respect to the usual Euclidian metric induced on Λ, therefore they do not depend on K. Let us assume that $d(F^s_x, E^s_x) < \varepsilon, d(F^u_{\hat{x}}(K), E^u_{\hat{x}}) < \varepsilon$, for all $\hat{x}$ in $\hat{\Lambda}$, where ε is a small positive number. From the above Lipschitz conditions, there exist positive constants τ and τ_K such that $d(F^s_x, F^s_y) \le \tau d(x,y), \forall x, y \in \Lambda$, and $d(F^u_{\hat{x}}(K), F^u_{\hat{y}}(K)) \le \tau_K d_K(\hat{x}, \hat{y}), \forall \hat{x}, \hat{y} \in \hat{\Lambda}$. In this case, E^s_x can be interpreted as the image of a linear map from F^s_x to $F^u_{\hat{x}}(K)$, for any prehistory $\hat{x}$ of $x \in \Lambda$. Therefore let $\mathcal{L}_{\hat{x}}(K) := L(F^s_x, F^u_{\hat{x}}(K))$ be the space of linear maps from F^s_x to $F^u_{\hat{x}}(K)$. Let $\mathcal{L}(K)$ denote the vector bundle over $\hat{\Lambda}$ given by $\mathcal{L}_{\hat{x}}(K), \hat{x} \in \hat{\Lambda}$, where we consider the metric d_K on $\hat{\Lambda}$. The space X of the Pointwise Hölder Section Theorem will be $\hat{\Lambda}$ endowed with d_K and the homeomorphism h from the statement of the same theorem is the map $\hat{f}^{-1}\colon \hat{\Lambda} \to \hat{\Lambda}$. We will also consider the bundle map $\Psi\colon \mathcal{L}(K) \to \mathcal{L}(K)$ induced by the graph transform associated with the derivative

$$Df^{-1}(\hat{x})\colon F^s_x \oplus F^u_{\hat{x}}(K) \to F^s_{x_{-1}} \oplus F^u_{\hat{f}^{-1}\hat{x}}(K),$$

where $\hat{x} = (x, x_{-1}, \dots) \in \hat{\Lambda}$. The mapping $Df^{-1}(\hat{x})$ represents the derivative at x of the local branch of f^{-1} which takes x into x_{-1}, in case $\hat{x} = (x, x_{-1}, \dots)$ is an arbitrary point of $\hat{\Lambda}$; this derivative does exist because we assumed that the critical set of f does not intersect Λ. In the sequel we shall also use the notation $Df_s^{-1}(\hat{x})$ as being the inverse of the isomorphism $Df_s(x_{-1}) : E^s_{x_{-1}} \to E^s_x$; similarly for the notation $Df_u^{-1}(\hat{x})$. The notion of graph transform used above is explained in [9]. If we assume that

$$Df^{-1}(\hat{x}) = \begin{pmatrix} A_{\hat{x}} & B_{\hat{x}}(K) \\ C_{\hat{x}}(K) & G_{\hat{x}}(K) \end{pmatrix},$$

then we have $A_{\hat{x}}\colon F^s_x \to F^s_{x_{-1}}$, $B_{\hat{x}}(K)\colon F^u_{\hat{x}}(K) \to F^s_{x_{-1}}$, $C_{\hat{x}}(K)\colon F^s_x \to F^u_{\hat{f}^{-1}\hat{x}}(K)$, and $G_{\hat{x}}(K)\colon F^u_{\hat{x}}(K) \to F^u_{\hat{f}^{-1}\hat{x}}(K)$. Let us notice that from the decomposition above, $B_{\hat{x}}(K)$,

$C_{\hat{x}}(K)$, and $G_{\hat{x}}(K)$ depend on K, but $A_{\hat{x}}$ does not, since the bundle F^s is independent of K. From the definition of graph transform,

$$\Psi_{\hat{x}}(g) = (C_{\hat{x}}(K) + G_{\hat{x}}(K)g) \circ (A_{\hat{x}} + B_{\hat{x}}(K)g)^{-1}, \tag{1.1}$$

for any linear map $g \in \mathcal{L}_{\hat{x}}(K)$. So it can be noticed that $\Psi_{\hat{x}}(g) \in \mathcal{L}_{\hat{f}^{-1}\hat{x}}(K)$, for any $\hat{x} \in \hat{\Lambda}$. From construction, $A_{\hat{x}}$ and $G_{\hat{x}}(K)$ approximate $Df_s^{-1}(\hat{x})$, respectively $Df_u^{-1}(\hat{x})$, while $|B_{\hat{x}}(K)| < a_1(\varepsilon)$, $|C_{\hat{x}}(K)| < a_1(\varepsilon)$, where $a_1(\,\cdot\,)$ is a positive continuous function with $a_1(0) = 0$. Hence, if ε is small enough, then the Lipschitz constant of $\Psi_{\hat{x}}$ is smaller than or equal to $\lambda_{\hat{x}}(K)$, where

$$\lambda_{\hat{x}}(K) := |Df_u^{-1}(\hat{x})| \cdot |Df_s(x_{-1})| + a_2(\varepsilon) = \frac{|Df_s(x_{-1})|}{|Df_u(x_{-1})|} + a_2(\varepsilon) < 1, \tag{1.2}$$

and where $a_2(\varepsilon)$ is a positive continuous function in ε, with $a_2(0) = 0$. Let us recall now that the metric on $\hat{\Lambda}$ is d_K which depends on the constant $K > 1$. In the same spirit as in [9], we can also assume that the bundle $E := \mathcal{L}(K)$ is trivial, otherwise we can replace it with $E \oplus E'$, for some complementary bundle E'. This replacement does not depend on the metric d_K, since the metric on E is already induced by the product of the metric d_K on $\hat{\Lambda}$ and the usual Euclidian metric on the spaces of linear maps. We will estimate the local Lipschitz constant $\mu_{\hat{x}}(K)$ of h at $\hat{x} \in \hat{\Lambda}$, where $h = \hat{f}^{-1}$ is our base homeomorphism. Thus, as in the statement of the Pointwise Hölder Section Theorem, let

$$\mu_{\hat{x}}(K) := \inf\Big\{ \frac{d_K(h\hat{x}, h\hat{y})}{d_K(\hat{x}, \hat{y})}, \hat{x} \neq \hat{y}, \hat{x}, \hat{y} \in \hat{\Lambda} \text{ and } d_K(\hat{x}, \hat{y}) < \eta \Big\}$$

for some small $\eta > 0$. Denote also by $\mu(K) := \inf_{\hat{x} \in \hat{\Lambda}} \mu_{\hat{x}}(K)$. Then we have

$$d_K(\hat{x}, \hat{y}) = d(x, y) + \frac{d(x_{-1}, y_{-1})}{K} + \frac{d(x_{-2}, y_{-2})}{K^2} + \cdots = d(x, y) + \frac{1}{K} d(\hat{f}^{-1}\hat{x}, \hat{f}^{-1}\hat{y}).$$

Let us denote by ε_0 a positive constant depending only on f such that f is injective on balls of radius $\varepsilon_0(\inf_\Lambda |Df_s|)^{-1}$ centered on Λ and such that we can apply the Mean Value Inequality on balls of radius $\varepsilon_0(\inf_\Lambda |Df_s|)^{-1}$. Suppose that $0 < \eta < \varepsilon_0$. If $d_K(\hat{x}, \hat{y}) < \eta$, and $d_K(\hat{f}^{-1}\hat{x}, \hat{f}^{-1}\hat{y}) > \eta$, then

$$d_K(\hat{x}, \hat{y}) < (|Df_u(x_{-1})| + \frac{1}{K}) d_K\Big(\hat{f}^{-1}\hat{x}, \hat{f}^{-1}\hat{y} \Big)$$

since $|Df_u(x_{-1})| + \frac{1}{K} > 1$. So, with the assumption that $d_K(\hat{x}, \hat{y}) < \eta$, let us suppose also that $d_K(\hat{f}^{-1}\hat{x}, \hat{f}^{-1}\hat{y}) < \eta$. Hence $d(x_{-1}, y_{-1}) < \eta$ and, from our assumption it follows also that $d(x, y) < \eta$, so, using the Mean Value Inequality, we obtain that

$$\begin{aligned} d_K(\hat{x}, \hat{y}) &\leq \Big(|Df_u(x'_{-1})| + \frac{1}{K} \Big) d_K(\hat{f}^{-1}\hat{x}, \hat{f}^{-1}\hat{y}) \\ &= \Big(|Df_u(x'_{-1})| + \frac{1}{K} \Big) d_K(h\hat{x}, h\hat{y}), \end{aligned} \tag{1.3}$$

where x'_{-1} is some point with $d(x_{-1}, x'_{-1}) < \eta$. This implies that the constant μ_x which appears in the Pointwise Hölder Section Theorem is represented in our situation by $\mu_{\hat{x}}(K)$ and, as we saw in (1.3),

$$\mu_{\hat{x}}(K) \geq \Big(|Df_u(x_{-1})| + \frac{1}{K} + \omega(|Df_u|, \eta) \Big)^{-1}, \tag{1.4}$$

where $\omega(|Df_u|, \eta)$ is the maximum oscillation of $|Df_u|$ on a ball of radius η centered at an arbitrary point of Λ, and we used above that $|Df_u(x'_{-1})| \leq |Df_u(x_{-1})| + \omega(|Df_u|, \eta)$.

Next, we show that $\Psi_{\hat{x}}$ is Lipschitz in $\hat{x}$; recall that we assumed that $\mathcal{L}(K)$ is a trivial bundle, so we can identify all the 1-dimensional complex spaces $\mathcal{L}_{\hat{x}}(K)$ with $\mathbb{C}$, and do this independently of K. We wish to prove that there exists a constant $\Theta_K > 0$ such that

$$|\Psi_{\hat{x}}(g) - \Psi_{\hat{y}}(g)| \leq \Theta_K d_K(\hat{x}, \hat{y}), \forall \hat{x}, \hat{y} \in \hat{\Lambda}, \forall g \in \mathbb{C}, |g| \leq 1. \tag{1.5}$$

From the fact that f is smooth and F^s depends Lipschitz in $x \in \Lambda$, while $F^u_{\hat{x}}(K)$ depends Lipschitz in $\hat{x} \in \hat{\Lambda}$, it follows that $A_{\hat{x}}$ depends Lipschitz in x (with respect to the Euclidian metric induced on Λ) and $B_{\hat{x}}(K)$, $C_{\hat{x}}(K)$, $G_{\hat{x}}(K)$ depend Lipschitz in $\hat{x}$ (with respect to the metric d_K). Recall from (1.1) that

$$\Psi_{\hat{x}}(g) = (C_{\hat{x}}(K) + G_{\hat{x}}(K)g) \cdot (A_{\hat{x}} + B_{\hat{x}}(K)g)^{-1},$$

for any linear map $g \in \mathcal{L}_{\hat{x}}(K)$. But in our case, g, $A_{\hat{x}}$, $B_{\hat{x}}(K)$, $C_{\hat{x}}(K)$, $G_{\hat{x}}(K)$ are just complex numbers. It is enough to show that $\hat{x} \to (A_{\hat{x}} + B_{\hat{x}}(K)g)^{-1}$ is Lipschitz. But since we work with complex numbers, we have

$$|(A_{\hat{x}} + B_{\hat{x}}(K)g)^{-1} - (A_{\hat{y}} + B_{\hat{y}}(K)g)^{-1}| = \Big| \frac{(A_{\hat{y}} - A_{\hat{x}}) + (B_{\hat{y}}(K) - B_{\hat{x}}(K))g}{(A_{\hat{x}} + B_{\hat{x}}(K)g)(A_{\hat{y}} + B_{\hat{y}}(K)g)} \Big|.$$

Now we use the fact that $A_{\hat{x}}$, $B_{\hat{x}}(K)$ depend Lipschitz in $\hat{x}$ and $|B_{\hat{x}}(K)| < a_1(\varepsilon) \ll 1$, $\forall \hat{x} \in \hat{\Lambda}$. Thus, for $|g| \leq 1$ we get that $|A_{\hat{x}} + B_{\hat{x}}(K)g|$ is uniformly (in $\hat{x}$) bounded away from 0, since $|A_{\hat{x}}|$ approximates $|Df_s^{-1}(\hat{x})|$ (and we know that $|Df_s^{-1}(\hat{x})| \geq (\sup_\Lambda |Df_s|)^{-1} > 0$), and $|B_{\hat{x}}(K)|$ is very small in comparison to $|A_{\hat{x}}|$. In conclusion we have obtained the Lipschitz continuity of Ψ, hence inequality (1.5).

Let us check now the condition (iii) of the Pointwise Hölder Section Theorem with $\alpha = 1$. Using the relations in (1.2) and (1.4), we have that

$$\begin{aligned} \rho(1, K) &:= \sup_{\hat{x} \in \hat{\Lambda}} \lambda_{\hat{x}} \cdot \mu_{\hat{x}}(K)^{-1} \\ &\leq \Big(\frac{|Df_s(x_{-1})|}{|Df_u(x_{-1})|} + a_2(\varepsilon) \Big) \cdot \Big(|Df_u(x_{-1})| + \frac{1}{K} + \omega(|Df_u|, \eta) \Big) \\ &= \Big(\frac{|Df_s(x_{-1})|}{|Df_u(x_{-1})|} + a_2(\varepsilon) \Big) \cdot \Big(\frac{1}{K} + \omega(|Df_u|, \eta) \Big) \\ &\quad + \frac{|Df_s(x_{-1})|}{|Df_u(x_{-1})|} \cdot |Df_u(x_{-1})| + a_2(\varepsilon)|Df_u(x_{-1})| \\ &\leq |Df_s(x_{-1})| + M(\varepsilon, \eta, K) < 1, \end{aligned} \tag{1.6}$$

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where $M(\varepsilon, \eta, K)$ is a positive continuous function in ε, η, and K with $M(0, 0, \infty) = 0$. This is why in the last inequality of (1.6) we were able to take $M(\varepsilon, \eta, K) < 1 - \sup_\Lambda |Df_s|$, for ε and η small enough and K large enough. The values of such ε, η, K depend only on f. Therefore, we found that in this case condition (iii) of the Pointwise Hölder Section Theorem is satisfied for $\alpha = 1$.

Now, according to (1.5), it follows that condition (ii) from the statement of the Pointwise Hölder Section Theorem is satisfied as well, so all the conditions of the Pointwise Hölder Section Theorem hold and we get that the unique invariant section σ is Lipschitz. But in our case this unique invariant section σ is just the stable bundle $\sigma(\hat{x}) = E^s_x, \forall \hat{x} \in \hat{\Lambda}$, hence there exists a constant C_K depending on K such that

$$d(E^s_x, E^s_y) \le C_K d_K(\hat{x}, \hat{y}), \forall \hat{x}, \hat{y} \in \hat{\Lambda}. \tag{1.7}$$

Now let $\lambda_s := \inf_{z\in\Lambda} |Df_s(z)|$, and take $\tilde{\varepsilon}_0 := \lambda_s \varepsilon_0$, where the number ε_0 has been introduced earlier; clearly $\tilde{\varepsilon}_0 \neq 0$, since the critical set of f avoids Λ. We want to prove that (1.7) implies that, in fact, $x \to E^s_x$ is Lipschitz.

Case 1: Let us then assume first that $x, y \in \Lambda$ with $d(x, y) \ge \tilde{\varepsilon}_0$. If Δ_0 denotes the diameter of Λ, then

$$\begin{aligned} d_K(\hat{x}, \hat{y}) &\le d(x, y) + \frac{2\Delta_0}{K} \le d(x, y) + \frac{2\Delta_0}{K} \cdot \frac{d(x, y)}{\tilde{\varepsilon}_0} \\ &\le d(x, y)(1 + \frac{2\Delta_0}{K\tilde{\varepsilon}_0}) < d(x, y)(1 + \frac{2\Delta_0}{\tilde{\varepsilon}_0}) \le C' d(x, y), \end{aligned} \tag{1.8}$$

with $C' > 0$ a constant independent of K.

Case 2: Now suppose that $0 < d(x, y) < \tilde{\varepsilon}_0$ for some $x, y \in \Lambda$. We consider here the map f restricted to Λ. We will say that $(x, x_{-1}, \dots, x_{-n})$ are consecutive preimages of x in Λ if $f(x_{-1}) = x, f(x_{-2}) = x_{-1}, \dots, f(x_{-n}) = x_{-n+1}$ and $x_{-j} \in \Lambda, \forall j = 1, \dots, n$. Consider $n = n(x, y)$ to be the largest positive integer such that there exist consecutive preimages of x and of y, $(x, x^*_{-1}, \dots, x^*_{-n})$ and $(y, y^*_{-1}, \dots, y^*_{-n})$ with $d(x^*_{-i}, y^*_{-i}) < \varepsilon_0, i = 1, \dots, n$. Since n is the largest such integer, it follows that for some $x^*_{-n-1} \in f^{-1}(x^*_{-n})$ and $y^*_{-n-1} \in f^{-1}(y^*_{-n})$, with $d(x^*_{-n-1}, y^*_{-n-1}) < \varepsilon_0 \lambda_s^{-1}$, we have

$$\varepsilon_0 < d(x^*_{-n-1}, y^*_{-n-1}) \le \lambda_s^{-1} d(x^*_{-n}, y^*_{-n}). \tag{1.9}$$

We also obtain

$$d(x^*_{-i}, y^*_{-i}) \le \lambda_s^{-i} d(x, y), i = 1, \dots, n. \tag{1.10}$$

From (1.9) and (1.10), we obtain that $d(x^*_{-n-1}, y^*_{-n-1}) \le \lambda_s^{-n-1} d(x, y)$. This implies that for any complete prehistories $\hat{x^*}, \hat{y^*}$ of x, y, which start with the consecutive preimages $(x, x^*_{-1}, \dots, x^*_{-n}), (y, y^*_{-1}, \dots, y^*_{-n})$ considered above, we have

$$\begin{aligned} d_K(\hat{x^*}, \hat{y^*}) &= d(x, y) + \frac{d(x^*_{-1}, y^*_{-1})}{K} + \cdots \\ &\le d(x, y) + \frac{1}{\lambda_s K} d(x, y) + \cdots + \frac{1}{\lambda_s^n K^n} d(x, y) + \frac{2\Delta_0}{K^{n+1}}. \end{aligned} \tag{1.11}$$

Assume that K is fixed such that $K > \lambda_s^{-2}$ and such that $M(\varepsilon, \eta, K) < 1 - \sup_\Lambda |Df_s|$ for some $\varepsilon < 1$ and some $\eta < \varepsilon_0$. Then from (1.9) and (1.10), $\varepsilon_0 < \lambda_s^{-n-1} d(x, y) < K^{n+1} d(x, y)$, which implies that $\frac{1}{K^{n+1}} < \frac{d(x,y)}{\varepsilon_0}$. Introducing this inequality in (1.11), one sees that there exists a positive constant C'' such that for our chosen prehistories $\hat{x^*}, \hat{y^*}$, of x, respectively y,

$$d_K(\hat{x^*}, \hat{y^*}) \le C'' d(x, y). \tag{1.12}$$

By considering now both Case 1, (1.8), and Case 2, (1.12), together with (1.7), we obtain the Lipschitz continuity of the stable spaces with respect to their base points, *i.e.,* there exists a positive constant Υ such that for all x, y from Λ, $d(E^s_x, E^s_y) \le \Upsilon d(x, y)$. This implies immediately that also ϕ^s is Lipschitz on Λ.

Now, we will prove the uniform Lipschitz continuity of the stable distribution and of ϕ^s along the stable leaves in the holomorphic case. We notice that since Λ is compact, one can construct local stable manifolds of uniform size r at all points of Λ if $r > 0$ is small enough. If y is a point in a manifold $W^s_r(x)$, but y is not necessarily in Λ, we shall call stable space at y, denoted by E^s_y, the tangent space at $W^s_r(x)$ at y. We see that the spaces E^s_y vary smoothly when y moves inside $W^s_r(x)$ for x fixed. So the existence of a constant Ξ as in the statement is conditioned only on the boundedness of the "curvature" of these local stable manifolds. Assume then that there exists a sequence $z_n \in \Lambda$ such that the Lipschitz constants L_n of the maps g_n converge to infinity, where $g_n(y) := E^s_y$, $y \in W^s_r(z_n)$. Since Λ is compact, the sequence $(z_n)_n$ has at least one convergent subsequence and without loss of generality we can assume that this subsequence is again $(z_n)_n$ and $z_n \to z$. If x is an arbitrary point in Λ, then $W^s_r(x)$ is an analytic disk which is given as the image of an analytic map h_x from the unit disk Δ to $\mathbb{C}^2$. We denote by h_n the map h_{z_n} for n positive integer. But from the hyperbolicity condition, the analytic maps h_x vary continuously in $x \in \Lambda$, hence also h_n vary continuously in n. The norm on Δ of the second derivative of h_n bounds the Lipschitz constant L_n of the map g_n, for all n. Notice however that since h_n are holomorphic and vary continuously in n, the second derivatives of the maps h_n also vary continuously in n. Therefore, since we assumed $z_n \to z \in \Lambda$, we obtain that L_n are bounded by some finite positive constant L. So the map $y \to E^s_y$ is L-Lipschitz on $W^s_r(x), \forall x \in \Lambda$. Then, due to the smoothness of f, there exists a small $r > 0$ and $\Xi > 0$ such that for any $x \in \Lambda$ and any points $y, z \in W^s_r(x)$, we have $|\phi^s(y) - \phi^s(z)| \le \Xi \cdot d(y, z)$. ■

Proposition 1.6 *Let $f \colon \mathbb{P}^2 \to \mathbb{P}^2$ be holomorphic, with Axiom A and such that $\mathcal{C}_f \cap \Lambda = \varnothing$ for a basic set Λ of unstable index one. Also, let C be a prehistory of a point x in Λ, with respect to f. If $m := n(C)$, $C = (x, x_{-1}, \dots, x_{-m})$ and y is an arbitrary point in $\Lambda(C, \varepsilon)$, with the corresponding prehistory $(y, y_{-1}, \dots, y_{-m})$ ε-shadowed by C, then we have $\frac{1}{C_1} \le \frac{|Df_s^m(y_{-m})|}{|Df_s^m(x_{-m})|} < C_1$, where $C_1 > 1$ is a constant independent of m and C.*

Proof From the fact that $(y, \dots, y_{-m})$ is an m-prehistory of y in Λ, we know in particular that $y_{-m} \in \Lambda$, hence there exists a local stable manifold through y_{-m} of size ε. Let us also take $\hat{x}$ to be any complete prehistory in Λ of x, starting with $(x, x_{-1}, \dots, x_{-m})$. Set $\hat{x}_{-m} := \hat{f}^{-m}(\hat{x})$. In this case $W^u_\varepsilon(\hat{x}_{-m})$ intersects $W^s_\varepsilon(y_{-m})$ in a unique point z. It follows from the local product structure of Λ that z belongs to Λ. From the fact that y belongs to $\Lambda(C,\varepsilon)$ and $(y,\dots,y_{-m})$ is its prehistory ε-shadowed by C, we know that $d(f^i x_{-m}, f^i y_{-m}) < \varepsilon$ for all $i = 0,1,\dots,m$. Also, from the fact that $z \in W^s_\varepsilon(y_{-m})$, it follows that $d(f^i z, f^i y_{-m}) < \varepsilon$ for all $i = 0,1,\dots,m$. From the last two inequalities we get that $d(f^i x_{-m}, f^i z) < 2\varepsilon$ for all $i = 0,1,\dots,m$. But, since $z \in W^u_\varepsilon(\hat{x}_{-m}) \cap W^s_\varepsilon(y_{-m})$, we have that there exist constants $\tilde{c} > 0$ and $\gamma \in (0,1)$ such that for all $i = 0,1,\dots,m$,

$$d(f^i x_{-m}, f^i z) < \tilde{c}\gamma^{m-i} \quad \text{and} \quad d(f^i y_{-m}, f^i z) < \tilde{c}\gamma^i. \tag{1.13}$$

Now from Theorem 1.5, $\phi^s(y)$ depends Lipschitz-continuously on $y \in \Lambda$. This, together with (1.13), implies that there exists a constant $K' > 0$ such that

$$\begin{aligned}
\Big|\sum_{j=0}^m \phi^s(y_{-j}) - \sum_{j=0}^m \phi^s(x_{-j})\Big| &\le \Big|\sum_{j=0}^m \phi^s(y_{-j}) - \sum_{j=0}^m \phi^s(f^{m-j}z)\Big| \\
&\quad + \Big|\sum_{j=0}^m \phi^s(f^{m-j}z) - \sum_{j=0}^m \phi^s(x_{-j})\Big| \\
&\le K'\Big(\sum_{j=0}^m d(y_{-j}, f^{m-j}z) + \sum_{j=0}^m d(f^{m-j}z, x_{-j})\Big) \\
&\le 2K'\tilde{c}\cdot\sum_{j=0}^m \gamma^j < K'',
\end{aligned}$$

where K'' is a constant independent of m and ε. Hence the statement of the proposition follows immediately from the previous inequalities. ■

Proposition 1.7 *Let $f\colon \mathbb{P}^2 \to \mathbb{P}^2$ be holomorphic with Axiom A and such that $\mathcal{C}_f \cap \Lambda = \varnothing$ for a basic set Λ of unstable index one. Denote $\chi_u := \sup_\Lambda |Df_u|$.*

(i) *Then we have that $t^s_n(\varepsilon) \ge t^s_{np}(\varepsilon)$ and that $t^s = t^s_n$, for any positive integers n, p and any $\varepsilon > 0$.*

(ii) *For $\varepsilon < \varepsilon_0$, and ρ an arbitrary number in the interval $(0, \chi_u^{-1})$, let $\rho_n := \varepsilon\cdot\rho^n$, $n > 1$. Then $P^-_{f^n}(t\phi^s_n, \rho_n) = P^-_{f^n}(t\phi^s_n)$, for any t; consequently $t^s_n(\rho_n) = t^s_n = t^s$, $n > 1$.*

Proof (i) First we make the following notations. If m is a positive integer, denote by

$$\mathcal{C}^n_m := \Big\{(y, y^n_{-1}, \dots, y^n_{-m}) \in \Lambda^{m+1}, \\ \text{such that } f^n(y^n_{-i}) = y^n_{-i+1}, i = 1,\dots,m, \text{ and } y_0 = y\Big\}.$$

Also let $\mathcal{C}^n_* := \bigcup_{m\ge 0}\mathcal{C}^n_m$ be the set of prehistories of finite length for f^n in Λ. Now, if n, p, and $\varepsilon > 0$ are fixed, we consider an arbitrary number $t \in (t^s_n(\varepsilon), t^s_n(\varepsilon)+1)$.

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From the definition of $t_n^s(\varepsilon)$, we get that for N large, there exists an ε-covering Γ of Λ, $\Gamma \subset \mathcal{C}_*^n$ with $n(C) \geq N, \forall C \in \Gamma$ and

$$\sum_{C\in\Gamma} \exp\bigl(S^-_{n(C)}(t\phi_n^s(C))\bigr) < \exp\bigl(-(t_n^s(\varepsilon)+1)n(2p-1)\sup_\Lambda|\phi^s|\bigr).$$

For every $C \in \Gamma$, let us divide $n(C)$ by p, and obtain $n(C) = p \cdot m(C) + k(C)$, where $0 \leq k(C) < p$. If $C = (y, y^n_{-1}, \dots, y^n_{-n(C)})$, then denote by C' the $m(C)$-prehistory of y with respect to f^{np} given by $C' = (y, z^{np}_{-1}, \dots, z^{np}_{-m(C)})$, where $z^{np}_{-1} := y^n_{-p}, \dots, z^{np}_{-m(C)} := y^n_{-pm(C)}$. Then it is easy to see that $\Lambda(C,\varepsilon) \subset \Lambda(C',\varepsilon)$, for all $C \in \Gamma$. Denote by Γ' the collection of all the prehistories C' associated by the above procedure with the prehistories C from Γ. We now calculate the consecutive sum

$$\begin{aligned} S^-_{n(C)}\phi_n^s(C) &= \phi_n^s(y) + \dots + \phi_n^s(y^n_{-m(C)p}) + \phi_n^s(y^n_{-m(C)p-1}) + \dots + \phi_n^s(y^n_{-n(C)}) \\ &= \log|Df_s^{n(pm(C)+1)}(y^n_{-m(C)p})| + \log|Df_s^{nk(C)}(y^n_{-n(C)})|. \end{aligned}$$

On the other hand,

$$\begin{aligned} S^-_{m(C)}\phi^s_{np}(C') &= \phi^s_{np}(y) + \dots + \phi^s_{np}(z^{np}_{-m(C)}) \\ &= \phi^s(y^n_{-m(C)p}) \\ &\qquad + \phi^s(fy^n_{-m(C)p}) + \dots + \phi^s(y) + \phi^s(fy) + \dots + \phi^s(f^{np-1}y) \\ &= \log|Df_s^{np(m(C)+1)}(y^n_{-m(C)p})|. \end{aligned}$$

These last two relations show that

$$S^-_{n(C)}\phi_n^s(C) = S^-_{m(C)}\phi^s_{np}(C') + \log|Df_s^n(y)| + \log|Df_s^{nk(C)}(y^n_{-n(C)})| - \log|Df_s^{np}(y)|.$$

Using that $k(C) < p$ and the last equality, we obtain that

$$\begin{aligned} |S^-_{n(C)}\phi_n^s(C) - S^-_{m(C)}\phi^s_{np}(C')| &\leq n(p-1)\cdot\sup_\Lambda|\phi^s| + \bigl|\log|Df_s^{nk(C)}(y^n_{-n(C)})|\bigr| \\ &\leq n(2p-1)\cdot\sup_\Lambda|\phi^s|. \end{aligned}$$

Therefore

$$\begin{aligned} &\inf\Bigl\{\sum_{C'\in\Gamma'} \exp\bigl(S^-_{m(C)}(t\phi^s_{np}(C'))\bigr), \Gamma' \subset \mathcal{C}_*^{np}\varepsilon - \text{ covers } \Lambda\Bigr\} \\ &\qquad \leq \Bigl[\sum_{C\in\Gamma}\exp\bigl(S^-_{n(C)}(t\phi_n^s(C))\bigr)\Bigr]\cdot\exp(tn(2p-1)\sup_\Lambda|\phi^s|) < 1. \end{aligned}$$

The last inequality follows since $t < t_n^s(\varepsilon) + 1$ and from the way we chose Γ in the begining of the proof. But from the definition of P^-_{np}, we obtain then that $t \geq t^s_{np}(\varepsilon)$.

However since t was taken arbitrarily in the finite interval $(t_n^s(\varepsilon), t_n^s(\varepsilon)+1)$, it follows that $t_n^s(\varepsilon) \geq t_{np}^s(\varepsilon)$. The inequality $t^s(\varepsilon) \geq t_n^s(\varepsilon)$ implies that $t^s \geq t_n^s$, $n \geq 1$. We want to prove now the opposite inequality, *i.e.*, $t^s \leq t_n^s$ (actually the same proof shows more generally that $P^-_{f^n}(t\phi_n^s) = nP^-_f(t\phi^s)$). Indeed, let us consider an arbitrary $t > t_n^s$ for a fixed integer n. For a given $\varepsilon > 0$, let $\bar{\varepsilon}_n > 0$ satisfying the following conditions: for any y, z with $d(y,z) < \bar{\varepsilon}_n$ we have $d(f^j y, f^j z) < \varepsilon, 0 \leq j \leq n$, and also $P^-_{f^n}(t\phi_n^s, \bar{\varepsilon}_n) < 0$. Hence for all m large, there exists an $(m, \bar{\varepsilon}_n)$-cover Γ_m^n of Λ (*i.e.*, Γ_m^n is a collection of m-prehistories C' with respect to f^n, so that $\Lambda = \bigcup_{C' \in \Gamma_m^n} \Lambda(C', \bar{\varepsilon}_n)$), satisfying

$$\sum_{C' \in \Gamma_m^n} e^{S_m^-(t\phi_n^s)(C')} < 1.$$

Now, out of every C' we will form a prehistory C with respect to f in the canonical way, *i.e.*, if $C' = (y, y_{-n}, \dots, y_{-nm})$, then

$$C = (y, f^{n-1}y_{-n}, \dots, y_{-n}, \dots, f(y_{-nm}), y_{-nm}).$$

Also, from the condition satisfied by $\bar{\varepsilon}_n$, we see that $\Lambda(C', \bar{\varepsilon}_n) \subset \Lambda(C, \varepsilon)$; so if Γ_{nm} denotes the collection of prehistories C of length nm (with respect to f) obtained as above from the prehistories C' of Γ_m^n, we obtain that Γ_{nm} is an (nm, ε) cover of Λ. Moreover, as found above, $S_{nm}^-(t\phi^s)(C) = S_m^-(t\phi_n^s)(C') + \log|Df_s(y)| - \log|Df_s^n(y)|$. These facts imply that

$$\sum_{C \in \Gamma_{nm}} e^{S_{nm}^-(t\phi^s)(C)} < M_n,$$

where M_n is a constant depending only on n. Therefore if we let $m \to \infty$ (and keep n fixed), we see that $P_f^-(t\phi^s, \varepsilon) \leq 0 \Rightarrow t \geq t^s(\varepsilon)$. But $0 < \varepsilon < \varepsilon_0$ was arbitrary and t was taken arbitrarily larger than t_n^s, hence $t_n^s \geq t^s$. This proves the equality $t^s = t_n^s, n \geq 1$.

(ii) First, from the proof of Proposition 1.6 we know that for all $m \geq 1$ and prehistory $(x, x_{-1}, \dots, x_{-m})$ of x in Λ,

$$\frac{1}{C_1(\varepsilon)} \leq \frac{|Df_s^m(y_{-m})|}{|Df_s^m(x_{-m})|} \leq C_1(\varepsilon),$$

for $(y, y_{-1}, \dots, y_{-m})$ an m-prehistory of y, ε-shadowed by $(x, x_{-1}, \dots, x_{-m})$. The proof of Proposition 1.6 implies also that $C_1(\varepsilon) \leq C_2 \cdot \varepsilon, 0 < \varepsilon < \varepsilon_0$, for some constant $C_2 > 0$. Let us consider now the situation for f^n for some fixed $n \geq 1$. Consider $(x, x_{-n}, \dots, x_{-np})$ a p-prehistory of x in Λ (with respect to f^n), and let $(y, y_{-n}, \dots, y_{-np})$ be another p-prehistory in Λ which is ρ_n-shadowed by $(x, x_{-n}, \dots, x_{-np})$. Then if $d(y_{-np}, x_{-np}) < \rho_n < \varepsilon\rho^n$, we get that $d(f^j(y_{-np}), f^j(x_{-np})) < \varepsilon$, $0 \leq j \leq n$, and similarly we obtain that $d(f^j(y_{-np}), f^j(x_{-np})) < \varepsilon, 0 \leq j \leq np$. Therefore the np-prehistory with respect to f, $(y, y_{-1}, \dots, y_{-np})$ is ε-shadowed by $(x, x_{-1}, \dots, x_{-np})$. So we can apply Proposition 1.6 in this case to obtain similar inequalities for prehistories of f^n:

$$\frac{1}{C_1(\varepsilon)} \leq \frac{|Df_s^{np}(y_{-np})|}{|Df_s^{np}(x_{-np})|} \leq C_1(\varepsilon), \tag{1.14}$$

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for any $p \geq 1$. Next, take C an arbitrary p-prehistory in Λ with respect to f^n for n fixed. If ε' is an arbitrary number in the interval $(0, \rho_n)$, we see that the set $\Lambda(C, \rho_n)$ can be covered with at most $(\frac{\rho_n C_1(\varepsilon)}{\varepsilon'})^4$ sets of the form $\Lambda(C', \varepsilon')$, where C' are p-prehistories with respect to f^n. Thus, recalling the definition of $P^-_{f^n}(t\phi^s_n, \rho_n), P^-_{f^n}(t\phi^s_n, \varepsilon')$ and inequality (1.14), we conclude that $P^-_{f^n}(t\phi^s_n, \rho_n) = P^-_{f^n}(t\phi^s_n, \varepsilon') = P^-_{f^n}(t\phi^s_n)$. The last equality above follows from the fact that $P^-_{f^n}(t\phi^s_n, \varepsilon') \to P^-_{f^n}(t\phi^s_n)$ when $\varepsilon' \to 0$. Hence, recalling also the conclusion of part (i), we get $t^s_n(\rho_n) = t^s_n = t^s, n > 1$. ∎

2 Estimates from Above and Below for the Stable Dimension in the General Holomorphic Case Using the Inverse Pressure of Iterates

Given a map f and a basic set Λ as in Proposition 1.6, define $\lambda_s := \inf_{\omega \in \Lambda} |Df_s(\omega)|$ and $\chi_s := \sup_{\omega \in \Lambda} |Df_s(\omega)|$. Remark that $\lambda_s > 0$ since we assumed that $\Lambda \cap \mathcal{C}_f = \varnothing$. For every positive integer n and small positive number ε, let $t^s_n(\varepsilon)$ (respectively t^s_n) be the unique zero of the function $t \to P^-_{f^n}(t\phi^s_n, \varepsilon)$ (respectively $t \to P^-_{f^n}(t\phi^s_n)$), where $\phi^s_n(y) := \log |Df^n_s(y)|, y \in \Lambda$.

Theorem 2.1 *Let $f\colon \mathbb{P}^2 \to \mathbb{P}^2$ be a holomorphic non-degenerate map with Axiom A and Λ a basic set of f with unstable index one. Assume also that the critical set of f, $\mathcal{C}_f$ does not intersect Λ.*

(i) *Then for every $x \in \Lambda$, we have $\delta^s(x) \leq t^s_n(\rho_n) = t^s$, where $\rho_n > 0$ are small numbers of the form $\varepsilon\rho^n, n \geq 1$, where $\chi_u := \sup_\Lambda |Df_u|$, $\rho > 0$ is an arbitrary number smaller than χ_u^{-1}, and $\varepsilon < \min\{\varepsilon_0, r_0\}$.*

(ii) *For all positive numbers $\varepsilon < \varepsilon_0$, and $\eta > 0$, we get $\delta^s(x) + \eta \geq t^s_n(\varepsilon)$, where $n \geq n(\varepsilon, \eta)$ and $n(\varepsilon, \eta)$ is a positive integer satisfying*

$$n(\varepsilon, \eta) > \frac{4 \log \frac{1}{\varepsilon}}{\eta \cdot \log \chi_s^{-1}}.$$

In particular, if $\eta = \varepsilon$ small enough, we get $t + \varepsilon \geq t^s_n(\varepsilon)$, for $n \geq (\frac{1}{\varepsilon})^{1.1}$.

Proof (i) According to Proposition 1.7, we have $t^s_n(\rho_n) = t^s$. From [6] we have that $\delta^s(x) \leq t^s$. Hence $\delta^s(x) \leq t^s_n(\rho_n), n > 1$.

(ii) We prove now the inequality $\delta^s(x) + \eta \geq t^s_n(\varepsilon)$ for $\varepsilon > 0$ small enough (to be determined next), $\eta > 0$ small, and $n \geq n(\varepsilon, \eta)$.

First let us notice that by definition $\delta^s(x) \leq 2$. Let us take an arbitrary t with $\delta^s(x) < t < 3$. Recall also that ε_0 has been introduced earlier as a positive constant, so that we can apply the mean value inequality for f on balls of radius $\varepsilon_0(\inf_\Lambda |Df_s|)^{-1}$, and also such that f is injective on balls of radius $\varepsilon_0(\inf_\Lambda |Df_s|)^{-1}$ centered on Λ.

Consider now $N_0(\varepsilon)$ to be the smallest cardinality of a covering of Λ with balls of radius ε. Then if $\beta = \overline{\dim}_B(\Lambda)$ denotes the upper box dimension of Λ and $\beta_0 < \beta < \beta_1$, we will have that $(\frac{1}{\varepsilon})^{\beta_0} < N_0(\varepsilon) < (\frac{1}{\varepsilon})^{\beta_1}$, for $\varepsilon > 0$ small enough. With $\varepsilon > 0$

and $\eta > 0$ fixed, consider $n(\varepsilon, \eta)$ to be the smallest positive integer n such that

$$N_0(\varepsilon) \cdot \chi_s^{n\eta} < 1 \tag{2.1}$$

This is satisfied if, for example,

$$n(\varepsilon, \eta) > \frac{4 \log \frac{1}{\varepsilon}}{\eta \cdot \log \frac{1}{\chi_s}}.$$

In the sequel we consider ε with $0 < \varepsilon < \min\{\varepsilon_1/2, r, d(\Lambda, \mathcal{C}_f)/4\}$. We shall prove that for such an ε and $\eta > 0$, the inequality $t + \eta > t_n^s(\varepsilon)$ holds for $n \geq n(\varepsilon, \eta)$.

Define now a constant $0 < \tilde{\alpha} < 1$ which depends only on f and on Λ, such that for all $x' \in \Lambda$ and $0 < r' \ll \operatorname{diam} \Lambda$, we have that $W^s_{r'}(y')$ intersects $W^u_{r'}(\hat{z}')$ for all points $y', z' \in B(x', \tilde{\alpha} r')$ and all prehistories $\hat{z}' \in \hat{\Lambda}$ of z'. The existence of such a constant follows from the transversality of stable and unstable manifolds.

Next let us cover the compact set Λ with a finite number of balls

$$B(y_1, \tilde{\alpha}\varepsilon/4), \dots, B(y_s, \tilde{\alpha}\varepsilon/4)$$

which are centered at points of Λ. Let us choose one such ball and denote its intersection with Λ by Y.

We will show now that there exists a positive integer m such that all local unstable manifolds $W^u_\varepsilon(\hat{y})$ intersect the set $f^{-m}(W)$, for all prehistories $\hat{y} \in \hat{\Lambda}$ of all points $y \in Y$, where we recall that $W := W^s_r(x) \cap \Lambda$.

Indeed, from the transitivity of f on Λ, there exists a positive integer m and a point $z \in Y \cap \Lambda$ such that $f^m(z) \in B(x, \tilde{\alpha}\varepsilon/2) \cap \Lambda$. Take now a complete prehistory $\hat{y} \in \hat{\Lambda}$ of an arbitrary point y from Y. From the fact that Y is contained in a ball of radius $\tilde{\alpha}\varepsilon/4$, we can conclude that $W^s_{\varepsilon/2}(z) \cap W^u_{\varepsilon/2}(\hat{y}) \neq \varnothing$ and denote this intersection (which is a point) by ξ. From the local product structure ξ belongs to Λ. We have also that $f^m(\xi) \in W^s_\varepsilon(f^m z) \cap \Lambda$. Now take $\widehat{f^m\xi}$ to be the prehistory in Λ of $f^m\xi$ given by $(f^m\xi, f^{m-1}\xi, \dots, \xi, \xi_{-1}, \dots)$, where $\hat{\xi} := (\xi, \xi_{-1}, \dots)$ is the prehistory of ξ $\varepsilon/2$-shadowed by $\hat{y}$; such a prehistory of ξ exists since $\xi \in W^u_{\varepsilon/2}(\hat{y})$. So we get that there exists a local unstable manifold $W^u_{\varepsilon/2}(\widehat{f^m\xi})$ which intersects $W^s_{\varepsilon/2}(x)$ in a point ζ; again from the local product structure, $\zeta \in \Lambda$ and since $\zeta \in W^s_{\varepsilon/2}(x)$, we obtain that $\zeta \in W$. If we consider ζ_{-m} the m-th preimage of ζ obtained from the fact that $\zeta \in W^u_{\varepsilon/2}(\widehat{f^m\xi})$, we will have $d(\zeta_{-m}, \xi) < \varepsilon/2$. Combining with the fact that $\hat{\xi}$ corresponds to a prehistory of ξ $\varepsilon/2$-shadowed by $\hat{y}$, it follows that $\zeta_{-m} \in W^u_\varepsilon(\hat{y}) \cap f^{-m}W$. We may denote the point ζ_{-m} also by $\zeta_{-m}(\hat{y})$ when we want to emphasize its dependence on $\hat{y}$.

Therefore, we proved that the set $f^{-m}W$ intersects all unstable manifolds $W^u_\varepsilon(\hat{y})$ for all prehistories $\hat{y} \in \hat{\Lambda}$ of points y from Y.

From the fact that $\zeta \in W^u_{\varepsilon/2}(\widehat{f^m\xi})$, it follows that $d(\zeta_{-m}, \xi) < \varepsilon/2, d(f\zeta_{-m}, f\xi) < \varepsilon/2, \dots, d(\zeta, f^m\xi) < \varepsilon/2$. But $\xi \in \Lambda$ and Λ is f-invariant, hence

$$d(\zeta, \Lambda) < \varepsilon/2, \dots, d(\zeta_{-m}, \Lambda) < \varepsilon/2. \tag{2.2}$$

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Let us denote by J_m the set of these points $\zeta_{-m}(\hat{y})$ obtained for all the prehistories $\hat{y}$ of points $y \in Y$. Relation (2.2) together with the fact that $\zeta \in \Lambda$, imply that $\zeta_{-m}(\hat{y}) \in \Lambda$, therefore $J_m \subset \Lambda$. The relations in (2.2) imply also that f^m is injective on a neighbourhood of J_m, since $\varepsilon < d(\Lambda, \mathcal{C}_f)/4$ and $f^j(J_m) \cap \mathcal{C}_f = \varnothing$, $j = 0, \dots, m$. And from our construction, $f^m(J_m) \subset W$. But from above, f^m is injective on a neighbourhood of J_m and it is bi-Lipschitz on that neighbourhood, hence $HD(J_m) \le HD(W) = \delta^s(x)$. Recall also that $t > \delta^s(x)$, so $t > HD(J_m)$. This means that there exists $0 < \gamma < \varepsilon$, γ small enough, and an open cover of J_m with balls, $\mathcal{U} = (U_i)_{i \in I}$, such that diam $U_i < \gamma$ and

$$\sum_{i \in I} (\operatorname{diam} U_i)^t < \varepsilon^{t+1} \cdot \lambda_s^{4n} \chi_s^n, \tag{2.3}$$

for a fixed n, $n \ge n(\varepsilon)$.

Let us choose now an arbitrary $i \in I$ and assume that $\operatorname{Card}(U_i \cap J_m) > 1$. Let us denote by Y_i the set of points y of Y which have some prehistory $\hat{y}$ with $W^u_\varepsilon(\hat{y}) \cap J_m \cap U_i \neq \varnothing$; denote by F_i the set of prehistories $\hat{y} \in \hat{\Lambda}$ with this property.

For each point $z' \in U_i \cap J_m$, there exists then a point $y \in Y_i$ and a prehistory $\hat{y} \in \hat{\Lambda}$ such that $z' \in W^u_\varepsilon(\hat{y})$, and actually $z' = \zeta_{-m}(\hat{y})$. Therefore z' has a prehistory $\hat{z}'$ given by that procedure, *i.e.,* which is ε-shadowed by $\hat{y}$; this prehistory may also be denoted by $\hat{z}'(\hat{y})$ if we want to emphasize its dependence on $\hat{y}$. Let also $F'_i := \{\hat{z}'(\hat{y}), \hat{y} \in F_i\}$. Let us now take a prehistory $\hat{z}' \in F'_i$. Since ε was assumed sufficiently small, we can define local branches of f^{-1} on balls of radius ε. Let us denote by f_*^{-1} the branch of f^{-1} defined on $B(z', \varepsilon)$ such that $f_*^{-1}(z') = z'_{-1}$. It may happen that the diameter of $f_*^{-1} U_i$ increases. In case diam $f_*^{-1} U_i < \varepsilon$, define afterwards the inverse iterate f_*^{-2} such that $f_*^{-2}(z') = z'_{-2}$, etc. Let us denote by $n_i(\hat{z}')$ the largest integer n' which is a multiple of n and for which diam $f_*^{-k'}(U_i) < \varepsilon, 0 \le k' \le n'$, where $\hat{z}' = \hat{z}'(\hat{y})$ for some $\hat{y} \in F_i \subset \hat{\Lambda}$ as above. We do this for all the points of $U_i \cap J_m$ and denote by n_i the largest integer $n_i(\hat{z}')$ for all $z' \in U_i \cap J_m$ and all prehistories $\hat{z}'$ from F'_i. Obviously, we cannot stretch the open set U_i in backward time forever, while keeping the diameter of its inverse iterates smaller than ε, hence n_i is finite. Also, $n_i, n_i(\hat{z}')$ are multiples of n, so they can be written as $n_i = nm_i, n_i(\hat{z}') = nm_i(\hat{z}')$. In addition, for a point $z' \in U_i \cap J_m$ and a prehistory $\hat{z}' \in F'_i$, we will also define the integer $\bar{n}_i(\hat{z}')$ as the smallest integer (not necessarily a multiple of n) such that diam $f_*^{-\bar{n}_i(\hat{z}')} U_i > \varepsilon$. We remark that the definitions imply the inequalities $n_i(\hat{z}') \le \bar{n}_i(\hat{z}') \le n_i(\hat{z}') + n$, for any point $z' \in J_m \cap U_i$ and any prehistory $\hat{z}' \in F'_i$.

Now we shall cover the set Y_i with sets of type $\Lambda(C', \varepsilon)$, where $C' \in \mathcal{C}^n_*$ (*i.e.,* C' are prehistories with respect to f^n). In order to do this, take an arbitrary $z' \in \frac{1}{2} U_i \cap J_m$ and a prehistory $\hat{z'} = \hat{z}'(\hat{y}) \in F'_i$, which corresponds to some complete (infinite) prehistory $C = \hat{y} \in F_i$. By $\frac{1}{2} U_i$ we understand the ball with the same center as U_i and with half its radius. Then consider the $m_i(\hat{z}')$-prehistory C' of y (prehistory with respect to f^n), coming from the prehistory C, *i.e.,* we have $C' = (y, y_{-n}, \dots, y_{-nm_i(\hat{z}')})$. Recall that $z' \in W^u_{\varepsilon/2}(\hat{y})$. From the definition of $n_i(\hat{z'})$ we see immediately that $U_i \subset \mathbb{P}^2(C', \varepsilon)$, and also $y \in \Lambda(C', \varepsilon)$. Recall that C' is an $m_i(\hat{z}')$-prehistory with respect to f^n. Hence, since $N_0(\varepsilon)$ is the smallest cardinality of a cover of Λ with balls of radius ε, and since $n_i = nm_i$ is the largest integer of the form $n_i(\hat{z}')$, we can cover the set Y_i with at most $N_0(\varepsilon)^{m_i}$ sets of the form $\Lambda(C', \varepsilon)$, where C' are prehistories for f^n of length $n(C')$, with $n(C') \le m_i$. We will denote by Γ_i the set of prehistories C' used for the last covering. So we have $Y_i \subset \bigcup_{C' \in \Gamma_i} \Lambda(C', \varepsilon)$, and $\Gamma_i \subset \mathcal{C}_*^n$, $n(C') \le m_i, \forall C' \in \Gamma_i$. This construction can be done for every $i \in I$, and for each such i, we have $\operatorname{Card} \Gamma_i \le N_0(\varepsilon)^{m_i}$.

But we proved that for all $\hat{y} \in \hat{\Lambda}$, the local unstable manifold $W_\varepsilon^u(\hat{y})$ intersects J_m; on the other hand, $J_m \subset \bigcup_{i \in I} U_i$. In conclusion, $Y \subset \bigcup_{i \in I} Y_i$, hence $Y \subset \bigcup_{i \in I} \bigcup_{C' \in \Gamma_i} \Lambda(C', \varepsilon)$. Using this cover of Y with sets $\Lambda(C', \varepsilon), C' \in \mathcal{C}_*^n$, we will estimate $M_{f^n}^-(0, (t+\eta)\phi_n^s, Y, N, \varepsilon)$ for some large integer N chosen so that $n(C') \ge N, \forall C' \in \bigcup_{i \in I} \Gamma_i$:

$$M_{f^n}^-(0, (t+\eta)\phi_n^s, Y, N, \varepsilon) \le \sum_{i \in I} \sum_{C' \in \Gamma_i} \exp(S_{n(C')}^-(t+\eta)\phi_n^s(C'))$$

Let us investigate now the relation between $\operatorname{diam} U_i$ and $\exp(S_{n(C')}^-(t+\eta)\phi_n^s(C'))$, $C' \in \Gamma_i$. From the definition of $n_i(\hat{z}')$, we know that it represents the largest integer n', multiple of n, such that $\operatorname{diam} f_*^{-k'}(U_i) < \varepsilon, 0 \le k' \le n'$. Also, $\bar{n}_i(\hat{z}')$ represents the smallest integer (not necessarily multiple of n) such that $\operatorname{diam} f_*^{-\bar{n}_i(\hat{z}')} U_i > \varepsilon$, where the inverse branches f_*^{-k} were defined along the prehistory $\hat{z}' = \hat{z}'(C)$.

We consider now what happens to U_i when taking inverse iterates. Let z'' be another point in $\frac{1}{2}U_i \cap \Lambda$, and ζ'' the intersection between $W_r^s(z'')$ and the unstable manifold $W_r^u(\hat{z}')$; from the local product structure $\zeta'' \in \Lambda$. Then since U_i is a ball, we get

$$\operatorname{diam} f^{-\bar{n}_i(\hat{z}')}(W_r^s(z') \cap U_i) = \text{constant} \cdot |Df_s^{\bar{n}_i(\hat{z}')}(z'_{-\bar{n}_i(\hat{z}')})|^{-1},$$
$$\operatorname{diam} f^{-\bar{n}_i(\hat{z}')}(W_r^s(z'') \cap U_i) = \text{constant} \cdot |Df_s^{\bar{n}_i(\hat{z}')}(\zeta''_{-\bar{n}_i(\hat{z}')})|,$$

due to the bounded distortion property from Proposition 1.6. But since $\zeta'' \in W_r^u(\hat{z}')$ and $\hat{\zeta}''$ is the prehistory of ζ'' following $\hat{z}'$, we see that the distance $d(z''_{-j}, \zeta''_{-j})$ decreases exponentially when j increases; thus due to the fact that $|Df_s|(z)$ depends Lipschitz-continuously on z (Theorem 1.5), we get that

$$|Df_s^{\bar{n}_i(\hat{z}')}(\zeta''_{-\bar{n}_i(\hat{z}')}))|, \quad |Df_s^{\bar{n}_i(\hat{z}')}(z'_{-\bar{n}_i(\hat{z}')}|$$

are the same up to a constant independent of z'.

Therefore we will obtain, for every $i \in I$ that

$$\operatorname{diam} U_i > \varepsilon \exp(S_{\bar{n}_i(\hat{z}')}^- \phi^s(C'')) \ge \varepsilon \exp(S_{m_i(\hat{z}')}^- \phi_n^s(C'))\lambda_s^n, \tag{2.4}$$

where we considered first the $\bar{n}_i(\hat{z}')$-prehistory $C'' := (y, y_{-1}, \dots, y_{-\bar{n}_i(\hat{z}')})$, (prehistory with respect to f, induced by the full prehistory $C := \hat{y}$), and then the $m_i(\hat{z}')$-prehistory $C' := (y, y_{-n}, \dots, y_{-nm_i(\hat{z}')})$, (prehistory with respect to f^n, induced by the same complete prehistory C). We used also in (2.4) the fact that $\bar{n}_i(\hat{z}') \le n_i(\hat{z}') + n$.

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Therefore by using (2.4) and the fact that $\operatorname{Card}\Gamma_i \le N_0(\varepsilon)^{m_i}$, we can continue now with the estimate for $M^-_{f^n}(0,(t+\eta)\phi^s_n, Y, N, \varepsilon)$ as follows:

$$
\begin{aligned}
(2.5)\quad & M^-_{f^n}(0,(t+\eta)\phi^s_n, Y, N, \varepsilon) \\
&\le \sum_{i\in I}\sum_{C'\in\Gamma_i} \varepsilon^{-t-\eta}(\operatorname{diam} U_i)^t \cdot \exp(S^-_{m_i(\hat z')}\phi^s_n(C'))^\eta \lambda_s^{-n(t+\eta)} \\
&\le \sum_{i\in I}[N_0(\varepsilon)^{m_i}\cdot \exp(S^-_{m_i(\hat z')}\phi^s_n(C'))^\eta]\varepsilon^{-t-\eta}(\operatorname{diam} U_i)^t\lambda_s^{-n(t+\eta)} \\
&\le \sum_{i\in I}[N_0(\varepsilon)\cdot \chi_s^{n\eta}]^{m_i}\chi_s^{-n\eta}\varepsilon^{-t-\eta}(\operatorname{diam} U_i)^t\lambda_s^{-n(t+\eta)},
\end{aligned}
$$

where we used in the last inequality the definition of $n_i(\hat z')$ and that $|Df_s^{n_i(\hat z')}(z'_{-n_i(\hat z')})|$ is the same as $|Df_s^{n_i(\hat z')}(z'_{-n_i(\hat z')})|$ up to a factor less than χ_s^n for any $z', z' \in U_i\cap J_m$. Thus we may as well use for $\hat z'$ the prehistory with the maximum $n_i(\hat z')$, hence with $n_i(\hat z') = n_i = nm_i$.

In the above sequence of inequalities, we used also that $0<\eta<1$, $0<t<3$. But $n_i = nm_i$, so (2.5) implies that

$$
\begin{aligned}
(2.6)\qquad M^-_{f^n}(0,(t+\eta)\phi^s_n, Y, N, \varepsilon) &\le \varepsilon^{-t-1}\sum_{i\in I}(\operatorname{diam} U_i)^t[N_0(\varepsilon)\chi_s^{\eta n}]^{m_i}\lambda_s^{-4n}\chi_s^{-n} \\
&\le \varepsilon^{-t-1}\lambda_s^{-4n}\chi_s^{-n}\sum_{i\in I}(\operatorname{diam} U_i)^t[N_0(\varepsilon)\chi_s^{\eta n}]^{m_i}.
\end{aligned}
$$

But from (2.1) and since $n\ge n(\varepsilon,\eta)$, we see that $N_0(\varepsilon)\chi_s^{\eta n} < 1$. From the way of choosing the cover $\mathcal{U}$ in (2.3), we have also $\sum_{i\in I}(\operatorname{diam} U_i)^t < \varepsilon^{t+1}\cdot\lambda_s^{4n}\chi_s^n$. In conclusion, the inequality (2.6) becomes

$$
(2.7)\qquad M^-_{f^n}(0,(t+\eta)\phi^s_n, Y, N, \varepsilon) < 1.
$$

Since γ and consequently $\operatorname{diam} U_i$, $i\in I$ can be taken as small as we wish, we see that $n(C')$ can also be made arbitrarily large, for $C'\in\bigcup_{i\in I}\Gamma_i$. Therefore if $\gamma\to 0$, N can be taken arbitrarily large, and (2.7) implies that $M^-_{f^n}(0,(t+\eta)\phi^s_n, Y, \varepsilon) = 0$. Thus one can conclude that $P^-_{f^n}((t+\eta)\phi^s_n, Y, \varepsilon)\le 0$, for $0<\eta<1$ and $n\ge n(\varepsilon,\eta)$. But let us also remember that Y was just the intersection between Λ and one of the balls $B(y_1,\tilde\alpha\varepsilon/4),\dots,B(y_s,\tilde\alpha\varepsilon/4)$ which cover Λ. Therefore by Proposition 1.1(ii), it follows that

$$
P^-_{f^n}((t+\eta)\phi^s_n, \Lambda, \varepsilon)\le 0,\ \text{ for } n\ge n(\varepsilon,\eta).
$$

This implies that $t+\eta\ge t^s_n(\varepsilon)$, for $n\ge n(\varepsilon,\eta)$. Since t was chosen arbitrarily larger than $\delta^s(x)$, we obtain $\delta^s(x)+\eta\ge t^s_n(\varepsilon)$, for $n\ge n(\varepsilon,\eta)$. ■

Corollary 2.2 *In the same setting as in the previous theorem, if x, y are arbitrary points from Λ, then*

$$
|\delta^s(x)-\delta^s(y)| \le \frac{(\overline{\dim}_B\Lambda)\cdot\log\chi_u}{\log\chi_s^{-1}},
$$

where $\chi_u := \sup_{z\in\Lambda}|Df_u(z)|$.

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Proof First, let us notice that $\overline{\dim}_B\Lambda \le 4$ since $\Lambda \subset \mathbb{P}^2$, so even if $\overline{\dim}_B\Lambda$ cannot be calculated explicitly, the statement of the corollary still gives a good estimate of the maximum possible variation of $\delta^s(\cdot)$ on Λ.

Let us take an arbitrary η with $\eta > (\overline{\dim}_B\Lambda)\log\chi_u/\log\chi_s^{-1}$ and an arbitrary t with $t > \delta^s(x)$. Then there exists $\beta_1 > \overline{\dim}_B\Lambda$ such that $\eta > \beta_1\cdot\log\chi_u/\log\chi_s^{-1}$. Now, if $\beta_1 > \overline{\dim}_B\Lambda$, then there will exist a large integer $n_1 = n_1(\beta_1)$ depending on β_1 such that for any $n \ge n_1$, ρ_n is small enough so that $N_0(\rho_n) \le (\frac{1}{\rho_n})^{\beta_1}$, where $N_0(\cdot)$ and ρ_n were introduced in the proof of Theorem 2.1. Hence $N_0(\rho_n)\cdot\chi_s^{n\eta} \le (\varepsilon\rho^n)^{-\beta_1}\chi_s^{n\eta}$. But we assumed $\eta > \beta_1\log\chi_u/\log\chi_s^{-1}$, so there exists n_1 large enough and $\rho \in (0,\chi_u^{-1})$ close to χ_u^{-1}, such that $(\varepsilon\rho^n)^{-\beta_1}\chi_s^{n\eta} < 1$ for $n > n_1$. This then implies that

$$N_0(\rho_n)\cdot\chi_s^{n\eta} < 1. \tag{2.8}$$

Now we can use inequality (2.8) and (2.6) to prove that $M^-_{f^n}(0,(t+\eta)\phi^s_n,Y,\rho_n) < 1$; this implies then that

$$P^-_{f^n}((t+\eta)\phi^s_n,\rho_n) \le 0, \quad \text{for } n > n_1$$

Thus we conclude from above that $t+\eta \ge t^s_n(\rho_n)$. But from Proposition 1.7, $t^s_n(\rho_n) = t^s$, $n \ge 1$. So $t+\eta \ge t^s$. Since t is arbitrarily larger than $\delta^s(x)$ and η is arbitrarily larger than $(\overline{\dim}_B\Lambda)\log\chi_u/\log\chi_s^{-1}$, it follows that

$$\delta^s(x) + \frac{(\overline{\dim}_B\Lambda)\log\chi_u}{\log\chi_s^{-1}} \ge t^s \ge \delta^s(y), \quad y \in \Lambda,$$

where the inequality $t^s \ge \delta^s(y)$ follows from Theorem 2.1. Therefore,

$$|\delta^s(x) - \delta^s(y)| \le (\overline{\dim}_B\Lambda)\cdot\log\chi_u/\log\chi_s^{-1}, \quad \forall x, y \in \Lambda. \qquad \blacksquare$$

3 Independence of $\delta^s(x)$ When the Map f Is Open on Λ

In this section we show that for an Axiom A holomorphic map f on $\mathbb{P}^2$ which, in addition, is also open on the basic set Λ, the stable dimension $\delta^s(x)$ becomes independent of $x \in \Lambda$.

It is easy to prove that for Λ connected, the condition $f|_\Lambda\colon \Lambda \to \Lambda$ open, is equivalent to saying that the cardinality of the set $f^{-1}(x)\cap\Lambda$ is constant when x ranges in Λ.

Fornaess and Sibony [1] have introduced a type of holomorphic maps g on $\mathbb{P}^2$ which are Axiom A and such that the saddle part S_1 of the non-wandering set has a neighbourhood U with the property that $g^{-1}(S_1)\cap U = S_1$ (among other properties). Such maps were called *s-hyperbolic*. Notice that any s-hyperbolic map is in particular open on any basic set Λ of saddle type. Examples of s-hyperbolic maps were given in [1].

In the sequel we will prove that the openness of f on Λ is a sufficient condition in order to guarantee that $\delta^s(x)$ does not depend on $x \in \Lambda$. The proof will use ideas and notations related to the concept of inverse pressure (the sets $\Lambda(C,\varepsilon)$, and their concatenations, for example).

Theorem 3.1 *Consider a holomorphic Axiom A map $f\colon \mathbb{P}^2 \to \mathbb{P}^2$ and a basic set of saddle type Λ which does not intersect the critical set $\mathcal{C}_f$. Moreover assume that $f|_\Lambda\colon \Lambda \to \Lambda$ is open, in particular any point $x \in \Lambda$ has the same number of preimages in Λ (this number being denoted by d'). Then for any $x \in \Lambda$, $\delta^s(x) = t_0^s$, where t_0^s is the unique zero of the pressure function $t \to P(t\phi^s - \log d')$.*

Proof In [5], we proved that $\delta^s(x) \le t_0^s$, so it remains to prove now only the opposite inequality. Denote $W := W_r^s(x) \cap \Lambda$. As in the second part of the proof of Theorem 2.1, we find an integer $m \ge 1$ and a set $J_m \subset f^{-m}W \cap \Lambda$ such that all local unstable manifolds of size $\varepsilon/2$ intersect J_m (for some small fixed $0 < \varepsilon < \varepsilon_0$). Take also $t > \delta^s(x)$ arbitrary. Then there exists a finite open cover $\mathcal{U} = (U_i)_{i\in I}$ of J_m with balls of diameter less than $\gamma \ll 1$, and so that $\sum_{i\in I}(\operatorname{diam} U_i)^t < \frac{1}{2}$. Recall the proof of Theorem 2.1, the definition of F_i', and the set of prehistories in Λ of points from $U_i \cap J_m$. In the sequel, for clarity of notation, we will denote the set $U_i \cap J_m$ by U_i too.

Assume $\hat{z}$ is a prehistory in Λ of a point $z \in U_i$; denote by $n(\hat{z})$ the largest integer such that $\operatorname{diam} f_*^{-k}U_i < \varepsilon/2, 0 \le k \le n(\hat{z})$, where f_*^{-k} is the branch of f^{-k} determined by the prehistory $\hat{z}$. For the prehistory $\hat{z}$, denote by $C(\hat{z})$ the $n(\hat{z})$-prehistory $(z, z_{-1}, \dots, z_{-n(\hat{z})})$ which is obtained by truncating $\hat{z}$.

Now for each $i \in I$, let us fix a point $z_i \in \frac{1}{2}U_i \cap \Lambda$ and then consider the set $\tilde{F}_i$ of all $n(\hat{z}_i)$-prehistories $C(\hat{z}_i)$ obtained as above, for all prehistories in Λ of z_i. Notice that we consider in this case all d' f-preimages in Λ of a given point $z_i \in U_i$.

Denote also by $U_i^* := \bigcup_{C\in\tilde{F}_i} \Lambda(C, \varepsilon)$; then $\Lambda = \bigcup_{i\in I} U_i^*$. For later reference, it is useful to note that for any prehistory $\hat{y} \in \hat{\Lambda}$, there exists $j \in I$ such that $W^u_{\varepsilon/2}(\hat{y}) \cap U_j \neq \varnothing$; but then there exists a certain prehistory $\hat{z}_j$ of z_j such that $W^u_\varepsilon(\hat{y}) \cap \Lambda \subset \Lambda(C(\hat{z}_j), \varepsilon)$ (this follows from the definition of $C(\hat{z}_j)$ and the fact that $f|_\Lambda$ is open). Therefore, all unstable manifolds of prehistories in $\hat{\Lambda}$ (intersected with Λ) are contained in some $\Lambda(C, \varepsilon), C \in \bigcup_{i\in I} \tilde{F}_i$.

For $i \in I$, $C \in \tilde{F}_i$, write C as $(z^C, \dots, z^C_{-n(C)})$ (obviously notationally $z^C = z_i$). Denote also by $G_i := \{n(C), C \in \tilde{F}_i\}$, (recall that $n(C)$ denotes the length of C), and write G_i as $\{n_{i1}, \dots, n_{iq_i}\}$, where $n_{i1} < \cdots < n_{iq_i}$. Now, let N_{ij} be the number of prehistories $C \in \tilde{F}_i$ with $n(C) = n_{ij}, 1 \le j \le q_i, i \in I$.

We will make the connection between the sets $\Lambda(C, \varepsilon)$ (obtained as above in the process of covering Λ, in the definition of inverse pressure P^-), and the Bowen balls needed in the definition of the (forward) pressure. In general by a *Bowen ball* $B_k(z, \varepsilon), z \in \Lambda$, we mean the set $\{y \in \Lambda, d(f^jy, f^jz) < \varepsilon, 0 \le j \le k\}$. Therefore, if $C \in \tilde{F}_i, i \in I$, we have $\Lambda(C, \varepsilon) = f^{n(C)}(B_{n(C)}(z^C_{-n(C)}, \varepsilon))$; for simplicity of notation, denote the Bowen ball $B_{n(C)}(z^C_{-n(C)}, \varepsilon)$ by $B(C), C \in \tilde{F}_i, i \in I$. From the above discussion, we know that $\Lambda = \bigcup_{i\in I} U_i^* = \bigcup_{i\in I}\bigcup_{C\in\tilde{F}_i} f^{n(C)}(B(C))$. However since the integers $n(C)$ are different among themselves, it does not follow directly that the Bowen balls $B(C)$ cover Λ. In order to get a covering of Λ with Bowen balls, we will make a construction using concatenations of sets of type $\Lambda(C, \varepsilon)$; it will be possible then to take the lengths of these concatenations arbitrarily large.

Let in general C and C' be two prehistories of points in Λ, $C = (z, z_{-1}, \dots, z_{-n(C)})$ and $C' = (w, w_{-1}, \dots, w_{-n(C)})$. Assume also that there exists a point $z' \in \Lambda(C, \varepsilon)$, so that $z'_{-n(C)} \in \Lambda(C', \varepsilon)$, where $z'_{-n(C)}$ represents the $n(C)$-preimage of z' which is ε-shadowed by $z_{-n(C)}$. If $z'_{-n(C)} \in \Lambda(C', \varepsilon)$, it follows that it has a prehistory $(z'_{-(n(C)+1)}, \dots, z'_{-(n(C)+n(C'))})$ which is ε-shadowed by C'. So we can form the set $\Lambda(CC', \varepsilon) := \{y \in \Lambda(C, \varepsilon), y_{-n(C)} \in \Lambda(C', \varepsilon)\}$, and from above, if this set is non-empty, then $\Lambda(CC', \varepsilon) \subset \Lambda(C'', 2\varepsilon)$, where C'' is an $(n(C)+n(C'))$-prehistory. This process will be called *concatenation*.

We will use concatenation repeatedly in order to obtain a cover of Λ with sets $\Lambda(C'', 2\varepsilon)$ with $n(C'')$ arbitrarily large. Define now the collection

$$\Gamma_n := \{\bar{C} = C_1 \cdots C_s, C_k \in \tilde{F}_{j_k}, j_k \in I, 1 \le k \le s, n \le n(C_1) + \cdots + n(C_{j_s}) < n+N\},$$

where here $N := \max_{i \in I, C \in \tilde{F}_i} n(C)$. Since $\Lambda = \bigcup_{i \in I} \bigcup_{C \in \tilde{F}_i} \Lambda(C, \varepsilon)$, we also see that $\Lambda = \bigcup_{\bar{C} \in \Gamma_n} \Lambda(\bar{C}, 2\varepsilon)$. If $\bar{C} \in \Gamma_n$ and $\bar{C} = C_1 \cdots C_s$, let $n(\bar{C}) := n(C_1) + \cdots + n(C_s)$. But as noted before, if $\bar{C} \in \Gamma_n$, there exist points $z^{\bar{C}}_{-n(\bar{C})}$ such that $\Lambda(\bar{C}, 2\varepsilon) = f^{n(\bar{C})}(B_{n(\bar{C})}(z^{\bar{C}}_{-n(\bar{C})}, 2\varepsilon))$, and $n \le n(\bar{C}) < n + N$. Therefore $\Lambda = \bigcup_{\bar{C} \in \Gamma_n} f^n(f^{n(\bar{C})-n} B_{n(\bar{C})}(z^{\bar{C}}_{-n(\bar{C})}, 2\varepsilon))$.

Let us recall now the remark made earlier, after the definition of U_i^*. Since any set $W^u_{\varepsilon/2}(\hat{y}) \cap \Lambda$, $\hat{y} \in \hat{\Lambda}$ is contained in $\Lambda(C, \varepsilon)$ for some $C \in \bigcup_{i \in I} \tilde{F}_i$ and since we collected the corresponding $C(\hat{z}_i)$ for all prehistories $\hat{z}_i \in \hat{\Lambda}$ and all $i \in I$, we obtain that any f^n-preimage in Λ of a point from Λ belongs to the union

$$\bigcup_{\bar{C} \in \Gamma_n} f^{n(\bar{C})-n} B_{n(\bar{C})}(z^{\bar{C}}_{-n(\bar{C})}, 2\varepsilon)).$$

So we can conclude that $\Lambda = \bigcup_{\bar{C} \in \Gamma_n} f^{n(\bar{C})-n} B_{n(\bar{C})}(z^{\bar{C}}_{-n(\bar{C})}, 2\varepsilon))$.

On the other hand, notice that $f^{n(\bar{C})-n} B_{n(\bar{C})}(z^{\bar{C}}_{-n(\bar{C})}, 2\varepsilon) \subset B_n(z^{\bar{C}}_{-n}, 2\varepsilon)$.

Denote then $F_n := \{z^{\bar{C}}_{-n}, \bar{C} \in \Gamma_n\}$. From the previous considerations it follows that F_n is an (n, ε)-spanning set for Λ, in the classical (forward) sense. We will use this particular spanning set F_n in order to estimate

$$P_n(t\phi^s - \log d') := \inf\Big\{ \sum_{z \in F} e^{S_n(t\phi^s)(z) - n \log d'},\ F(n, \varepsilon) - \text{ spanning set for } \Lambda \Big\}.$$

Let us remember the construction of the set F_n and the points $z^{\bar{C}}_{-n(\bar{C})}$. If $\bar{C} = C_1 \cdots C_s$, $C_k \in \tilde{F}_{j_k}$, $1 \le k \le s$, then from the proof of Proposition 1.6, we have that there exists a positive constant σ so that

$$|Df_s^{n(C_s)}(z^{\bar{C}}_{-n(\bar{C})})| \le e^{\sigma\varepsilon} \cdot \operatorname{diam} U_{j_s}, \dots, |Df_s^{n(C_1)}(z^{\bar{C}}_{-n(C_1)})| \le e^{\sigma\varepsilon} \cdot \operatorname{diam} U_{j_1},$$

(since $C_1 \in \tilde{F}_{j_1}, \dots, C_s \in \tilde{F}_{j_s}$). Hence since $n \le n(\bar{C}) < n + N$, there will exist a positive constant T_1 independent of n such that

$$|Df_s^{n(\bar{C})}(z^{\bar{C}}_{-n(\bar{C})})| \le T_1 \cdot e^{n\sigma\varepsilon} \cdot (\operatorname{diam} U_{j_1}) \cdots (\operatorname{diam} U_{j_s}).$$

But recall that $|Df_s^{n(\bar{C})}(z^{\bar{C}}_{-n(\bar{C})})| = |Df_s^{n(\bar{C})-n}(z^{\bar{C}}_{-n(\bar{C})})| \cdot |Df_s^n(z^{\bar{C}}_{-n})|$. Thus, for a positive constant T_2 we obtain the inequality:

$$|Df_s^n(z^{\bar{C}}_{-n})| \le T_2 \cdot e^{n\sigma\varepsilon} \cdot (\operatorname{diam} U_{j_1}) \cdots (\operatorname{diam} U_{j_s}), \tag{3.1}$$

for all $\bar{C} \in \Gamma_n$ and all integers $n > 1$.

Now given n, and $j_1, \dots j_s \in I$, we will estimate how many prehistories $\bar{C} = C_1 \cdots C_s$ there are with $C_k \in \tilde{F}_{j_k}$, $1 \le k \le s$ and $\bar{C} \in \Gamma_n$.

For $i \in I$ and $1 \le j \le q_i$, we denoted by N_{ij} the number of prehistories $C \in \tilde{F}_i$ with $n(C) = n_{ij}$, $n_{ij} \in G_i$. Hence for each s, $j_1, \dots, j_s \in I$, and integers $n_{j_k p_k} \in G_{j_k}$, $1 \le k \le s$, satisfying $n \le n_{j_1 p_1} + \cdots + n_{j_s p_s} < n + N$, there exist at most $N_{j_1 p_1} \cdots N_{j_s p_s}$ prehistories of type $\bar{C} = C_1 \cdots C_s$ in Γ_n with $C_k \in \tilde{F}_{j_k}$ and $n(C_k) = n_{j_k p_k}$, $1 \le k \le s$. If $i \in I$, let

$$\Sigma_i := \frac{N_{i1}}{(d')^{n_{i1}}} + \cdots + \frac{N_{iq_i}}{(d')^{n_{iq_i}}}.$$

To start with, let us compare N_{i1} and N_{i2}. Since $n_{i1} < n_{i2}$, the prehistories stopping at n_{i1} cannot be continued to n_{i2}-prehistories; hence using the fact that each point in Λ has at most d' preimages in Λ, it follows that $N_{i2} \le [(d')^{n_{i1}} - N_{i1}] \cdot (d')^{n_{i2} - n_{i1}}$. Similarly one can show that $N_{ij} \le (d')^{n_{ij}} - N_{i1}(d')^{n_{ij} - n_{i1}} - \cdots - N_{i(j-1)}(d')^{n_{ij} - n_{i(j-1)}}$, $2 \le j \le q_i$. This implies that for each $i \in I$, we obtain:

$$\begin{aligned}
\Sigma_i \le& \frac{N_{i1}}{(d')^{n_{i1}}} + \frac{N_{i2}}{(d')^{n_{i2}}} + \cdots + \frac{N_{i(q_i-1)}}{(d')^{n_{i(q_i-1)}}} \\
&\quad + \frac{(d')^{n_{iq_i}} - N_{i1}(d')^{n_{iq_i} - n_{i1}} - \cdots - N_{i(q_i-1)}(d')^{n_{iq_i} - n_{i(q_i-1)}}}{(d')^{n_{iq_i}}} \\
&\le (1 - \frac{N_{i1}}{(d')^{n_{i1}}} - \cdots - \frac{N_{i(q_i-1)}}{(d')^{n_{i(q_i-1)}}}) + \frac{N_{i1}}{(d')^{n_{i1}}} + \cdots + \frac{N_{i(q_i-1)}}{(d')^{n_{i(q_i-1)}}} = 1
\end{aligned}$$

Therefore from the last inequality it follows that $\Sigma_i \le 1, i \in I$ and hence $\Sigma_{j_1} \cdots \Sigma_{j_s} \le 1$, $j_1, \dots, j_s \in I$. This implies then

$$\sum_{1 \le p_1 \le q_{j_1}, \dots, 1 \le p_s \le q_{j_s}} \frac{N_{j_1 p_1} \cdots N_{j_s p_s}}{(d')^{n_{j_1 p_1} + \cdots + n_{j_s p_s}}} \le 1.$$

In particular, if $j_1, \dots, j_s \in I$, we get

$$\sum{}' \frac{N_{j_1 p_1} \cdots N_{j_s p_s}}{(d')^n} \le \Theta, \tag{3.2}$$

where $\Theta > 0$ is a constant independent of n, $j_1, \dots, j_s$ and where the sum $\sum''$ is taken over all integers $n_{j_k p_k} \in G_{j_k}$, $1 \le k \le s$ satisfying $n \le n_{j_1 p_1} + \cdots + n_{j_s p_s} < n + N$.

We will use the above conclusions in order to estimate now $\sum_{z \in F_n} e^{S_n(t\phi^s)(z) - n \log d'}$; first notice that for each $j_1, \dots, j_s \in I$, there exist at most $\sum N_{j_1 p_1} \cdots N_{j_s p_s}$ prehistories $\bar{C} = C_1 \cdots C_s \in \Gamma_n$, with $C_k \in \tilde{F}_{j_k}$, $1 \le k \le s$, where the last sum is taken over all integers $n_{j_k p_k} \in G_{j_k}$, $1 \le k \le s$ satisfying $n \le n_{j_1 p_1} + \cdots + n_{j_s p_s} < n + N$. Then using (3.1) and (3.2), we will obtain:

(3.3)
$$
\begin{aligned}
P_n(t\phi^s - \log d') &\le \sum_{z\in F_n} e^{S_n(t\phi^s)(z) - n\log d'} \\
&\le {\sum}'' \Big(\sum N_{j_1 p_1}\cdots N_{j_s p_s}\Big)\cdot (d')^{-n}\cdot T_2 e^{n\sigma\varepsilon}\cdot (\operatorname{diam} U_{j_1})^t \cdots (\operatorname{diam} U_{j_s})^t \\
&\le \Theta T_2 \cdot e^{n\sigma\varepsilon}\cdot {\sum}''(\operatorname{diam} U_{j_1})^t\cdots(\operatorname{diam} U_{j_s})^t,
\end{aligned}
$$

where the sum $\sum''$ is taken over all integers $s > 0$ and s-uples $j_1, \dots, j_s \in I$ having some prehistories $C_1, \dots, C_s$ in $\tilde F_{j_1}, \dots, \tilde F_{j_s}$ respectively, which satisfy: $C_1 \cdots C_s \in \Gamma_n$. But the cover $(U_i)_{i\in I}$ has been taken such that $\sum_{i\in I}(\operatorname{diam} U_i)^t < \frac{1}{2}$, therefore $\sum_{s>0}(\sum_{i\in I}(\operatorname{diam} U_i)^t)^s < 1$. This implies that

$$\sum_{s>0}\sum_{j_1,\dots,j_s\in I}(\operatorname{diam} U_{j_1})^t\cdots(\operatorname{diam} U_{j_s})^t < 1.$$

Therefore using (3.3) it follows that $P_n(t\phi^s - \log d') < \Theta T_2 \cdot e^{n\sigma\varepsilon}$. The constants Θ, T_2, σ do not depend on n, ε, if $\varepsilon < \varepsilon_1$ is small enough. So we get $P(t\phi^s - \log d') = \overline{\lim}_n \frac{1}{n}\log P_n \le \sigma\varepsilon$, and since $\varepsilon > 0$ is arbitrarily small, we get $P(t\phi^s - \log d') \le 0$. But this means that $t \ge t_0^s$, where t_0^s denotes the unique zero of the function $t \to P(t\phi^s - \log d')$. Now recall that t has been taken arbitrarily larger than $\delta^s(x)$, hence $\delta^s(x) \ge t_0^s$. Recalling that the opposite inequality was proved in [5], we finally get that $\delta^s(x) = t_0^s, x \in \Lambda$. So, in case $f|_\Lambda$ is open, the stable dimension is independent of the point. ■

In particular Theorem 3.1 shows that in the case of s-hyperbolic maps studied in [1], the stable dimension along basic sets of saddle type, is independent of the point .

Finally, notice that the proof of Theorem 3.1 shows more generally that $\delta^s(x) \ge t_0^s$ if each point of Λ has at most d'-preimages in Λ (one may also denote t_0^s by $t_0^s(d')$ when emphasizing its dependence on d'). The number of preimages $d(x)$ that a point x from Λ has in Λ, is not necessarily constant. The above remark and [5, Theorem 1.2] prove the following.

Corollary 3.2 *In the setting of Theorem* 2.1, *if $d' \le d(y) \le d''$, $y \in \Lambda$, then for each $x \in \Lambda$ it follows that $t_0^s(d'') \le \delta^s(x) \le t_0^s(d')$.*

It is important to remark that this Corollary does not require $f|_\Lambda$ to be open; it gives estimates of the stable dimension, for example in the case of quadratic maps from [5].

4 Results in the Real Conformal Case

Most of the results of the previous sections work also in a more general setting, although for historical and example reasons we preferred to give them in the holomorphic case.

Definition 1 Let M be a compact Riemannian manifold of real dimension 4, and $f\colon M \to M$ a $\mathcal{C}^r, r \geq 2$ map, possibly non-invertible. Let also Λ a *basic set of saddle type* for f, *i.e.*, there exists an open neighbourhood V of Λ in M, such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(V)$, $f|_\Lambda\colon \Lambda \to \Lambda$ is transitive and f is hyperbolic on $\hat{\Lambda}$ with both expanding and contracting directions. Suppose also that f is finite-to-one, the dimension of stable tangent spaces on Λ is 2, and f is conformal on its stable manifolds on Λ. We will say that such a map f is *c-hyperbolic* on the basic set Λ (c coming from "conformal").

The notations for the stable dimension $\delta^s(x)$, the zero of the inverse pressure $t^s_n(\varepsilon), t^s_n$, etc., remain the same.

The following theorems are proved in the same way as the previous corresponding theorems in the holomorphic case.

Theorem 4.1 *Consider $f\colon M \to M$ a c-hyperbolic map on the basic set Λ, such that $\mathcal{C}_f \cap \Lambda = \emptyset$. Then the map $x \to E^s_x$ is Lipschitz continuous and in particular, if $\phi^s(y) := \log|Df_s(y)|, y \in \Lambda$, then ϕ^s is Lipschitz continuous on Λ.*

Theorem 4.2 *Let $f\colon M \to M$ be a c-hyperbolic map on a basic set Λ, with $\mathcal{C}_f \cap \Lambda = \emptyset$. Then*

(i) *for every $x \in \Lambda$, we have $\delta^s(x) \leq t^s_n(\rho_n) = t^s$, where ρ_n are numbers of the form $\varepsilon\rho^n, n \geq 1$, with $\chi_u := \sup_\Lambda ||Df_u||$, and $\rho > 0$ is an arbitrary number smaller than χ_u^{-1}, and $\varepsilon < min\{\varepsilon_0, r_0\}$.*

(ii) *for all positive numbers $\varepsilon < \varepsilon_0$, and $\eta > 0$, we obtain $\delta^s(x) + \eta \geq t^s_n(\varepsilon)$, where $n \geq n(\varepsilon, \eta) > 4\log(\varepsilon^{-1})/\eta \log \chi_s^{-1}$. In particular, if $\eta = \varepsilon$ small enough, we get $\delta^s(x) + \varepsilon \geq t^s_n(\varepsilon)$, for $n > \frac{1}{\varepsilon^{1.1}}$.*

Consequently we have the similar corollary.

Corollary 4.3 *In the same setting as in Theorem 4.2, if $x, y \in \Lambda$, then $|\delta^s(x) - \delta^s(y)| \leq (\overline{\dim}_B \Lambda) \cdot \log \chi_u / \log \chi_s^{-1}$.*

Theorem 4.4 *Consider a smooth map $f\colon M \to M$ which is c-hyperbolic on a connected basic set Λ which does not intersect the critical set $\mathcal{C}_f$. Moreover assume that $f|_\Lambda\colon \Lambda \to \Lambda$ is open, in particular any point $x \in \Lambda$ has the same number of preimages in Λ (denote this number by d'). Then for any $x \in \Lambda$, $\delta^s(x) = t^s_0$, where t^s_0 is the unique zero of the pressure function $t \to P(t\phi^s - \log d')$.*

Corollary 4.5 *Let $f\colon M \to M$ be a smooth map, c-hyperbolic on the basic set Λ which does not intersect the critical set $\mathcal{C}_f$; if $d(y)$ denotes the cardinality of $f^{-1}(y) \cap \Lambda$, $y \in \Lambda$ and d', d'' are positive integers such that $d' \leq d(y) \leq d'', \forall y \in \Lambda$, then for each $x \in \Lambda$ it follows that $t^s_0(d'') \leq \delta^s(x) \leq t^s_0(d')$, where $t^s_0(d')$ represents the unique zero of the pressure function $t \to P(t\phi^s - \log d')$.*

This corollary does not require $f|_\Lambda$ to be open.

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References

[1] J. E. Fornaess and N. Sibony, *Hyperbolic maps on* $\mathbb{P}^2$. Math. Ann. **311**(1998), no. 2, 305–333.
[2] A. Manning and H. McCluskey, *Hausdorff dimension for horseshoes.* Ergodic Theory Dynam. Systems **3**(1983), no. 2, 251–260.
[3] E. Mihailescu, *Applications of thermodynamic formalism in complex dynamics on* $\mathbb{P}^2$. Discrete Contin. Dynam. Systems **7**(2001), no. 4, 821–836.
[4] ———, *The set* K^- *for hyperbolic non-invertible maps.* Ergodic Theory Dynam. Systems **22**(2002), no. 3, 873–887.
[5] E. Mihailescu and M. Urbański, *Estimates for the stable dimension for holomorphic maps.* Houston J. Math **31**(2005), 2, 367–389 (electronic).
[6] ———, *Inverse topological pressure with applications to holomorphic dynamics of several complex variables.* Commun. Contemp. Math. **6**(2004), no. 4, 653–679.
[7] Z. Nitecki, *Topological entropy and the preimage structure of maps.* Real Anal. Exchange **29**(2003/04), no. 1, 9–41.
[8] D. Ruelle, *Elements of differentiable dynamics and bifurcation theory.* Academic Press, Boston, MA, 1989.
[9] M. Shub, *Global Stability of Dynamical Systems.* Springer-Verlag, New York, 1987.
[10] A. Verjovsky and H. Wu, *Hausdorff dimension of Julia sets of complex Hénon mappings.* Ergodic Theory Dynam. Systems **16**(1996), no. 4, 849–861.
[11] A. Wilkinson, *Stable ergodicity of the time-one map of a geodesic flow.* Ergodic Theory Dynam. Systems **18**(1998), no. 6, 1545–1587.

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