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ZERO-DIMENSIONAL COMPACTIFICATIONS OF LOCALLY COMPACT SPACES

R. GRANT WOODS

1. Introduction. Let X be a locally compact Hausdorff topological space. A *compactification* of X is a compact Hausdorff space which contains X as a dense subspace. Two compactifications αX and γX of X are *equivalent* if there is a homeomorphism from αX onto γX that fixes X pointwise. We shall identify equivalent compactifications of a given space. If $\mathscr{F}$ is a family of compactifications of X, we can partially order $\mathscr{F}$ by saying that $\alpha X \leqq \gamma X$ if there is a continuous map from γX onto αX that fixes X pointwise. This map is necessarily unique, and will be called the *connecting map* from γX onto αX. The family of all compactifications of X will be denoted by $\mathscr{K}(X)$; it is a complete lattice under the above partial order (see [**8**]). The Boolean algebra of all open-and-closed ("clopen") subsets of a space X will be denoted by $\mathscr{B}(X)$.

If $\alpha X \in \mathscr{K}(X)$, we say that αX is a *zero-dimensional compactification* of X if $\alpha X - X$ has a basis of clopen sets. The family of all zero-dimensional compactifications of X will be denoted by $\mathscr{K}_0(X)$. It is partially ordered (as noted above) and has a largest member, the *Freudenthal compactification* of X (denoted FX). We shall be interested in the family $L(\mathscr{B}(FX - X))$ of all subalgebras of $\mathscr{B}(FX - X)$. When partially ordered by inclusion, $L(\mathscr{B}(FX - X))$ is a complete lattice.

Let $\mathscr{R}(X)$ denote the Boolean algebra of regular closed subsets of X, let $D(X)$ denote the subalgebra of $\mathscr{R}(X)$ consisting of members of $\mathscr{R}(X)$ with compact boundaries, and let $K(X)$ denote the collection of compact members of $\mathscr{R}(X)$. Let $\mathscr{S}(X)$ be the family of all subalgebras $\mathscr{A}$ of $\mathscr{R}(X)$ such that $K(X) \subseteq \mathscr{A} \subseteq D(X)$, and partially order $\mathscr{S}(X)$ by inclusion. Note that each $\mathscr{A}$ in $\mathscr{S}(X)$ is a base for the closed subsets of X.

We prove two main results in this paper. First, we show that if X is a locally compact Hausdorff space, then $\mathscr{K}_0(X)$ is a complete lattice and is lattice-isomorphic to $L(\mathscr{B}(FX - X))$. It follows from this that if X and Y are locally compact Hausdorff spaces, then $\mathscr{K}_0(X)$ and $\mathscr{K}_0(Y)$ are lattice-isomorphic if and only if $FX - X$ and $FY - Y$ are homeomorphic. This result is similar in spirit to results of Magill [**8**] and Thrivikraman [**14**], and should be compared to the theorem of Thrivikraman that if $\alpha_0 X$ and $\gamma_0 Y$ are compactifications of the locally compact Hausdorff spaces X and Y, then $\alpha_0 X - X$

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and $\gamma_0 Y - Y$ are homeomorphic if and only if $\{\alpha X \in \mathscr{K}(X) : \alpha X \leqq \alpha_0 X\}$ and $\{\gamma Y \in \mathscr{K}(Y) : \gamma Y \leqq \gamma_0 Y\}$ are lattice-isomorphic.

Second, we show that if X is a locally compact Hausdorff space, then $\mathscr{S}(X)$ is a complete lattice and is lattice-isomorphic to $\mathscr{K}_0(X)$. This generalizes a result of Magill and Glasenapp [**9**]. In the process of obtaining this result, we obtain information about the relationship between $\mathscr{K}_0(X)$ and $\mathscr{K}_0(E(X))$, where $E(X)$ denotes the projective cover of X.

2. Preliminary results. All topological spaces considered in this paper will be assumed to be locally compact Hausdorff spaces (and hence completely regular). Notation and terminology pertaining to topological concepts may be found in [**4**]; those pertaining to Boolean algebras may be found in [**12**].

A subset A of a topological space X is *regular closed* if it is the closure of its interior. The family $\mathscr{R}(X)$ of all regular closed subsets of X, partially ordered by inclusion, is a complete Boolean algebra under the following operations:

$$\begin{aligned} &\vee_\alpha A_\alpha = \mathrm{cl}_X[\cup_\alpha A_\alpha] \\ &\wedge_\alpha A_\alpha = \mathrm{cl}_X \,\mathrm{int}_X[\cap_\alpha A_\alpha] \qquad ((A_\alpha) \subset \mathscr{R}(X)) \\ &A' = \mathrm{cl}_X(X - A) \end{aligned}$$

(here A' denotes the Boolean-algebraic complement of the element A of $\mathscr{R}(X)$).

If S is a subset of a Boolean algebra B, we denote by $\langle S\rangle$ the subalgebra of B generated by S; this, by definition, is the intersection of all subalgebras of B that contain S.

A topological space X is called *zero-dimensional* if $\mathscr{B}(X)$ is a base for the open (and closed) subsets of X. In the next few paragraphs we outline a portion of the well-known "Stone duality theory" of the relationship between Boolean algebras and compact zero-dimensional Hausdorff spaces. A treatment of this topic may be found in [**12**].

If U is a Boolean algebra, let $S(U)$ denote the family of all ultrafilters on U. For each $b \in U$, put $\lambda(b) = \{\alpha \in S(U) : b \in \alpha\}$. Then $S(U)$, topologized by letting $\{\lambda(b) : b \in U\}$ be a base for the open sets of $S(U)$, is a compact zero-dimensional Hausdorff space called the *Stone space* of U. The map $b \mapsto \lambda(b)$ is a Boolean algebra isomorphism from U onto $\mathscr{B}(S(U))$. If V is a subalgebra of U, and $\alpha \in S(U)$, then $\alpha \cap V \in S(V)$, and each ultrafilter on V is of this form. The function k from $S(U)$ to $S(V)$ defined by

$$k(\alpha) = \alpha \cap V \qquad (\alpha \in S(U))$$

is a continuous map from $S(U)$ onto $S(V)$. We call k the *dual map* of the inclusion map from V into U.

2.1 LEMMA. *If V is a subalgebra of the Boolean algebra U, then $\{k^{\leftarrow}[B]: B \in \mathscr{B}(S(V))\} = \{\lambda(v): v \in V\}$. (here λ is the isomorphism from U onto $\mathscr{B}(S(U))$.*

Proof. It follows from the above remarks that if $B \in \mathscr{B}(S(V))$, then there

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exists $v \in V$ such that $B = \{\alpha \in S(V): v \in \alpha\}$. Hence from the definition of k, it follows that

$$\begin{aligned} k^{\leftarrow}[B] &= \{\gamma \in S(U): v \in \gamma \cap V\} \\ &= \{\gamma \in S(U): v \in \gamma\} \\ &= \lambda(v). \end{aligned}$$

Conversely, if $v \in V$, then $\lambda(v) = k^{\leftarrow}[\{\alpha \in S(V): v \in \alpha\}] \in \{k^{\leftarrow}[B]: B \in \mathscr{B}(S(V))]$, and the lemma follows.

If K is a compact zero-dimensional space, then the function $h: K \rightarrow S(\mathscr{B}(K))$ defined by

$$h(x) = \{B \in \mathscr{B}(K) : x \in B\} \qquad (x \in K)$$

is a homeomorphism from K onto $S(\mathscr{B}(K))$. With this we end our discussion on Stone duality theory.

Let X be a locally compact zero-dimensional Hausdorff space, and let $\mathscr{S}_0(X)$ be the family of all subalgebras of $\mathscr{B}(X)$ that contain $\langle \mathscr{B}(X) \cap K(X) \rangle$ (note that $\langle \mathscr{B}(X) \cap K(X) \rangle$ is the smallest subalgebra of $\mathscr{B}(X)$ that is a base for the open sets of X; see [**9**, Lemma 2.4]). If $\mathscr{A} \in \mathscr{S}_0(X)$ and $\alpha \in S(\mathscr{A})$, we call α a *fixed* ultrafilter on $\mathscr{A}$ if $\bigcap \alpha \neq \emptyset$. If $x \in X$, put $\alpha(x) = \{A \in \mathscr{A} : x \in A\}$. Then $\{\alpha(x) : x \in X\}$ is the set of all fixed ultrafilters on $\mathscr{A}$. Let $\Gamma(\mathscr{A})$ be an index set for $S(\mathscr{A})$, with the convention that for each $x \in X$, the index of $\alpha(x)$ is the point x. Thus $S(\mathscr{A}) = \{\alpha(p) : p \in \Gamma(\mathscr{A})\}$. Topologize $\Gamma(\mathscr{A})$ by letting $\{\{p \in \Gamma(\mathscr{A}) : A \in \alpha(p)\} : A \in \mathscr{A}\}$ be a base for the open sets of $\Gamma(\mathscr{A})$. Then the map $p \mapsto \alpha(p)$ is a homeomorphism $h_{\mathscr{A}}$ from $\Gamma(\mathscr{A})$ onto $S(\mathscr{A})$, and X is a dense subspace of $\Gamma(\mathscr{A})$ whose subspace topology coincides with the original topology on X. Thus $\Gamma(\mathscr{A}) \in \mathscr{K}_0(X)$. This construction is discussed by Magill and Glasenapp in [**9**, 3.6].

In Theorem 3.6 of [**9**], Magill and Glasenapp prove the following result.

2.2 THEOREM. *Let X be a locally compact zero-dimensional space. Partially order $\mathscr{S}_0(X)$ by inclusion. Then $\mathscr{A} \rightarrow \Gamma(\mathscr{A})$ is an order isomorphism from $\mathscr{S}_0(X)$ onto $\{\alpha X \in \mathscr{K}(X) : \alpha X$ is zero-dimensional$\}$ and $\mathscr{S}_0(X)$ is a complete lattice.*

2.3 LEMMA. *Let X be a locally compact zero-dimensional space and let $\alpha X \in \mathscr{K}(X)$. Then αX is zero-dimensional if and only if $\alpha X \in \mathscr{K}_0(X)$.*

Proof. Obviously if αX is zero-dimensional then $\alpha X \in \mathscr{K}_0(X)$. Conversely, if $\alpha X - X$ is zero-dimensional and if C is connected component of αX containing more than one point, then C cannot lie entirely in $\alpha X - X$. Let p and q be distinct points of C, and let $p \in X$. As X is locally compact and zero-dimensional, there is a compact member of $\mathscr{B}(X)$ such that $p \in B$ and $q \notin B$. Now $B \in \mathscr{B}(\alpha X)$, so $\{C \cap B, C - B\}$ is a disconnection of C. The lemma follows.

It is worth noting that the local compactness of X is essential in the above; the one-point compactification of the space of real numbers is a compactification $\alpha\mathbf{Q}$ of the zero-dimensional space $\mathbf{Q}$ of rational numbers, and $\alpha\mathbf{Q} - \mathbf{Q}$ is zero-dimensional while $\alpha\mathbf{Q}$ is connected.

The proof of Theorem 3.6 of [**9**] can be modified, with the aid of 2.3 but with essentially no other changes, to a proof of the following generalization of 2.2.

2.4 THEOREM. *Let X be a locally compact, zero-dimensional space, and let $\mathscr{A}_0 \in \mathscr{S}_0(X)$. Then the restriction of Γ to $\{\mathscr{A} \in \mathscr{S}_0(X) : \mathscr{A} \subseteq \mathscr{A}_0\}$ is a lattice-isomorphism onto $\{\alpha X \in \mathscr{K}_0(X) : \alpha X \leqq \Gamma(\mathscr{A}_0)\}$.*

A perfect map k from a space X onto a space Y is called *irreducible* if proper closed subsets of X are mapped onto proper closed subsets of Y by k. We need the following result which, we believe, has the status of a "folk theorem". We include a proof for the sake of completeness.

2.5 THEOREM. *Let k be a perfect irreducible map from X onto Y. Then $A \mapsto k[A]$ is a Boolean algebra isomorphism from $\mathscr{R}(X)$ onto $\mathscr{R}(Y)$, and $A = \mathrm{cl}_X k^{\leftarrow}[\mathrm{int}_Y k[A]]$ for each $A \in \mathscr{R}(X)$.*

Proof. In Lemma 2.1 of [**5**] it is shown that if V is open in X, then $k[V] \subseteq \mathrm{cl}_Y[Y - k[X - V]]$. Hence if $A \in \mathscr{R}(X)$, then $k[\mathrm{int}_X A] \subseteq \mathrm{cl}_Y[Y - k[X - \mathrm{int}_X A]]$. As k is a closed map, this implies that

$$k[A] = k[\mathrm{cl}_X \mathrm{int}_X A] = \mathrm{cl}_Y k[\mathrm{int}_X A] \subseteq \mathrm{cl}_Y[Y - k[X - \mathrm{int}_X A]].$$

But $Y - k[X - \mathrm{int}_X A]$ is open in Y and contained in $k[A]$, which is closed. Hence $k[A] = \mathrm{cl}_Y[Y - k[X - \mathrm{int}_X A]] \in \mathscr{R}(Y)$.

As $k[A \cup B] = k[A] \cup k[B]$, k preserves finite suprema. If $A \in \mathscr{R}(X)$, then

$$k[A'] = k[\mathrm{cl}_X[X - A]] = \mathrm{cl}_Y k[X - A] \subseteq \mathrm{cl}_Y[Y - k[A]] \text{ (using } [\mathbf{5}, 2.1]) = (k[A])'.$$

But $k[A] \cup k[A'] = Y$, so $\mathrm{cl}_Y[Y - k[A]] \subseteq k[A']$. Thus $k[A'] = (k[A])'$, and our correspondence preserves Boolean-algebraic complements. Hence it is a Boolean algebra homomorphism. It is onto, for if $B \in \mathscr{R}(Y)$, then $\mathrm{cl}_X k^{\leftarrow}[\mathrm{int}_Y B] \in \mathscr{R}(X)$ and is mapped to B by k. Finally, the correspondence is one-to-one, for $k[A] = \emptyset$ if and only if $A = \emptyset$. Hence our correspondence is a Boolean algebra isomorphism.

Recall that a Hausdorff space is called *extremally disconnected* if its open sets have open closures. If X is extremally disconnected, then X is zero-dimensional and $\mathscr{B}(X) = \mathscr{R}(X)$. Associated with each completely regular Hausdorff space X there is an extremally disconnected space $E(X)$, and a perfect irreducible map $k_X : E(X) \to X$; the points of $E(X)$ are the ultrafilters of $\mathscr{R}(X)$ that converge to points of X, and $E(X)$ is given the subspace topology that it inherits from $S(\mathscr{R}(X))$. If $\alpha \in E(X)$, $k_X(\alpha)$ is defined to be $\bigcap\{A : A \in \alpha\}$. $E(X)$ is called the *projective cover* of X, and the above description of it characterizes it up to homeomorphism. See [**5**; **13**], and [**15**, §1] for further details.

Denote

$$\{\mathrm{cl}_{E(X)}\, k_X{}^{\leftarrow}[\mathrm{int}_X A] : A \in D(X)\} \text{ by } \tau(D(X)).$$

Our approach in proving the second of our main results (Theorem 3.7) is as follows. If X is a locally compact Hausdorff space, we show that $\mathscr{S}(X)$ is lattice-isomorphic with $\{\mathscr{A} \in \mathscr{S}_0(E(X)) : \mathscr{A} \subseteq \tau(D(X))\}$. Then using Theorem 2.4, we show that $\mathscr{S}(X)$ is lattice-isomorphic to $\{\alpha\, E(X) \in \mathscr{K}_0(E(X)) : \alpha\, E(X) \leqq \Gamma\, (\tau(D(X)))\}$. The final step is to show that $FX - X$ and $\Gamma(\tau(D(X))) - E(X)$ are homeomorphic.

We shall need the following result about Boolean algebras, proved by Sachs [**11**] and Filippov [**2**]. I am grateful to Professor G. Grätzer for calling it to my attention.

2.6 THEOREM. *Let B_1 and B_2 be two Boolean algebras, and let $L(B_i)$ denote the lattice of subalgebras of B_i $(i = 1, 2)$. Then B_1 and B_2 are isomorphic as Boolean algebras if and only if $L(B_1)$ and $L(B_2)$ are lattice-isomorphic.*

Finally, if X is a locally compact Hausdorff space we describe how to construct a member $\mathscr{A}X$ of $\mathscr{K}_0(X)$ from a member $\mathscr{A}$ of $\mathscr{S}(X)$. The construction is essentially that discussed by Fan and Gottesman in [**1**], and by Njåstad in [**10**, §3]. Proofs of the following assertions can be found therein. Our terminology differs slightly from that of Njåstad, but can readily be seen to be equivalent to his.

If $\mathscr{A} \in \mathscr{S}(X)$, call a subfamily α of $\mathscr{A}$ a *maximal binding family* if α has the finite intersection property (F.I.P.) and is not contained in any larger subfamily of $\mathscr{A}$ with F.I.P. If $x \in X$ then $\{A \in \mathscr{A} : x \in A\} = \alpha(x)$ is a maximal binding family, and $\bigcap \alpha(x) = \{x\}$, as $\mathscr{A}$ forms a base for the closed subsets of X. Each maximal binding family with non-empty intersection is an $\alpha(x)$ for some $x \in X$. Let $\mathscr{A}X$ be the collection of all maximal binding families of $\mathscr{A}$. If $A \in \mathscr{A}$, let $A^* = \{\alpha \in \mathscr{A}X : A \in \alpha\}$. If $A, B \in \mathscr{A}$ then $A \cup B \in \mathscr{A}$ and $(A \cup B)^* = A^* \cup B^*$. Hence $\{A^* : A \in \mathscr{A}\}$ forms a base for the closed sets of a topology on $\mathscr{A}X$. Thus topologized, $\mathscr{A}X$ becomes a compact Hausdorff space and $\{\alpha(x) : x \in X\}$ becomes a dense subspace of $\mathscr{A}X$ that is homeomorphic to X. We identify X and $\{\alpha(x) : x \in X\}$; then $\mathscr{A}X \in \mathscr{K}(X)$, and it follows from 2.7(c) below that $\mathscr{A}X - X$ is zero-dimensional. Thus $\mathscr{A}X \in \mathscr{K}_0(X)$. If $\mathscr{A} = D(X)$, then $\mathscr{A}X$ is just FX; the above construction is similar to that used by Freudenthal in [**3**] in his original construction of FX.

We state below some elementary facts about $\mathscr{A}X$. Proofs of (a) and (b) may be found (at least implicitly) in [**10**, §3]. The symbol "$\mathrm{bd}_Y A$" denotes the topological boundary in Y of the subset A of the space Y.

2.7 PROPOSITION. *Let $A \in \mathscr{A}$ and let $\mathscr{A} \in \mathscr{S}(X)$. Then:*

(a) $\mathrm{cl}_{\mathscr{A}X}\, A = A^*$. *Thus $A^* \in \mathscr{R}(\mathscr{A}X)$.*

(b) $\mathrm{int}_{\mathscr{A}X} A^* = \mathscr{A}X - (A')^*$. *Thus $\{\mathrm{int}_{\mathscr{A}X}\, A^* : A \in \mathscr{A}\}$ is a base for the open sets of $\mathscr{A}X$.*

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(c) $\mathrm{bd}_{\mathscr{A}X} A^* \subset X$. *Thus* $\mathscr{A}X - X$ *is zero dimensional.*

Proof of (c): Let $\alpha \in \mathrm{bd}_{\mathscr{A}X} A^*$. By (b), $\alpha \in (A')^*$ and so A and A' both belong to α. But $A \cap A' = \mathrm{bd}_X A$, which is compact as $\mathscr{A} \in \mathscr{S}(X)$. Hence if $\bigcap \alpha = \emptyset$, there exist $B_1, \ldots, B_n \in \alpha$ such that $A \cap A' \cap [\bigcap_{i=1}^n B_i] = \emptyset$. This violates the fact that α has F.I.P. Hence $\bigcap \alpha \neq \emptyset$, and so $\alpha \in X$ (in our identification of X with members of $\mathscr{A}X$ with non-empty intersection). Thus $\mathrm{bd}_{\mathscr{A}X} A^* \subset X$. Hence $\{A^* - X : A \in \mathscr{A}\}$ is a collection of clopen subsets of $\mathscr{A}X - X$ that form a base for the closed sets of $\mathscr{A}X - X$, and $\mathscr{A}X - X$ is zero-dimensional.

The final result of this section is new.

2.8 THEOREM. *Let X be a locally compact space and let $\mathscr{A} \in \mathscr{S}(X)$. Define a function g from $S(\mathscr{A})$ to $\mathscr{A}X$ as follows:*

$$g(\alpha) = \bigcap \{A^* : A \in \alpha\} \qquad (\alpha \in S(\mathscr{A})).$$

Then:

(a) *g is a well-defined continuous function from $S(\mathscr{A})$ onto $\mathscr{A}X$.*

(b) $g^{\leftarrow}[X] = \{\alpha \in S(\mathscr{A}) : \bigcap \alpha \neq \emptyset\}$.

(c) *The restriction of g to $S(\mathscr{A}) - g^{\leftarrow}[X]$ is a homeomorphism from $S(\mathscr{A}) - g^{\leftarrow}[X]$ onto $\mathscr{A}X - X$.*

Proof. (a) Let $\alpha \in S(\mathscr{A})$. To show that g is well-defined, we must show that $\bigcap \{A^* : A \in \alpha\}$ contains precisely one point of $\mathscr{A}X$. If $\bigcap \{A^* : A \in \alpha\} = \emptyset$, by the compactness of $\mathscr{A}X$ there exist $A_1, \ldots, A_n \in \alpha$ such that $\bigcap_{i=1}^n A_i^* = \emptyset$. By [**10**, Lemma 6], it follows that $\bigcap_{i=1}^n A_i = \emptyset$, which contradicts the fact that $\bigwedge_{i=1}^n A_i \in \alpha$ and $\bigwedge_{i=1}^n A_i \subseteq \bigcap_{i=1}^n A_i$. Hence $g(\alpha) \neq \emptyset$. If p and q are distinct points of $g(\alpha)$ by 2.7 (b) there exist $A, B \in \mathscr{A}$ such that $p \in \mathrm{int}_{\mathscr{A}X} A^*$, $q \in \mathrm{int}_{\mathscr{A}X} B^*$, and $A^* \cap B^* = \emptyset$. As α is an ultrafilter on $\mathscr{A}$, either $A \in \alpha$ or $A' \in \alpha$. If $A \in \alpha$, then $g(\alpha) \subset A^*$ so $q \notin g(\alpha)$, which is a contradiction. Hence $A' \in \alpha$. But $p \notin (A')^*$ (2.7(b)) and so $p \notin g(\alpha)$. It follows that $g(\alpha)$ contains precisely one point of $\mathscr{A}X$, and so g is well-defined.

It follows from 2.7(b) that if $p \in \mathscr{A}X$, then $\bigcap \{A : A \in \mathscr{A}$ and $p \in \mathrm{int}_{\mathscr{A}X} A^*\} = \{p\}$. But $\{A \in \mathscr{A} : p \in \mathrm{int}_{\mathscr{A}X} A^*\}$ is a filter on $\mathscr{A}$, and so is contained in some ultrafilter α on $\mathscr{A}$. Obviously $g(\alpha) = p$, and so g maps $S(\mathscr{A})$ onto $\mathscr{A}X$.

If $\alpha \in S(\mathscr{A})$ and V is open in $\mathscr{A}X$ with $g(\alpha) \in V$, then by 2.7(b) and the regularity of $\mathscr{A}X$, there exists $A \in \mathscr{A}$ such that $g(\alpha) \in \mathrm{int}_{\mathscr{A}X} A^* \subseteq A^* \subseteq V$. Thus $g(\alpha) \notin (A')^*$ so $A' \notin \alpha$. Thus $A \in \alpha$, so $\alpha \in \lambda(A)$, where λ is the canonical isomorphism from $\mathscr{A}$ onto $\mathscr{B}\mathscr{Z}(S(\mathscr{A}))$. If $\gamma \in \lambda(A)$, then $A \in \gamma$ so $g(\gamma) \in A^*$. Hence $g[\lambda(A)] \subset V$ and g is continuous at α. As α was arbitrary in $S(\mathscr{A})$, g is continuous.

(b) If $\bigcap \alpha \neq \emptyset$, then $\bigcap \{A^* \in \mathscr{A} : A \in \alpha\} = \bigcap \alpha \in X$. Conversely, if $g(\alpha) \in X$, then $X \cap \bigcap \{A^* : A \in \alpha\} \neq \emptyset$. As $X \cap A^* = A$, this means that $\bigcap \alpha \neq \emptyset$.

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(c) Evidently the restriction of g to $S(\mathscr{A}) - g^{\leftarrow}[X]$ is a perfect map from $S(\mathscr{A}) - g^{\leftarrow}[X]$ onto $\mathscr{A}X - X$, and hence is closed. Thus if g can be shown to be one-to-one on $S(\mathscr{A}) - g^{\leftarrow}[X]$, our assertion will follow. Let α and γ be distinct members of $S(\mathscr{A}) - g^{\leftarrow}[X]$. There exists $A \in \mathscr{A}$ such that $A \in \alpha$ and $A' \in \gamma$. If $g(\alpha) = g(\gamma)$, then

$$g(\alpha) \in A^* \cap (A')^* = A^* \cap \mathrm{cl}_{\mathscr{A}X}(\mathscr{A}X - A^*) = \mathrm{bd}_{\mathscr{A}X} A^*.$$

By 2.7(c), $\mathrm{bd}_{\mathscr{A}X} A^* \subseteq X$, and $g(\alpha) \in \mathscr{A}X - X$, so we have a contradiction. Hence g is one-to-one on $S(\mathscr{A}) - g^{\leftarrow}[X]$.

We remark that the subspace $g^{\leftarrow}[X]$ in the above theorem is homeomorphic to $E(X)$. A justification of this remark appears in the proof of 3.6.

3. The main results. Our first goal is to prove Theorems 3.3 and 3.4. We need several lemmas. Throughout what follows, X is assumed to be a locally compact Hausdorff space. Let $\alpha_0 X$ be a fixed member of $\mathscr{K}_0(X)$, and let $\overline{\mathscr{K}}_0(X) = \{\alpha X \in \mathscr{K}_0(X) : \alpha X \leqq \alpha_0 X\}$.

3.1 LEMMA. *Let $\alpha X \in \overline{\mathscr{K}}_0(X)$, let k_α be the connecting map from $\alpha_0 X$ onto αX, and set $\sigma(\alpha X) = \{k_\alpha^{\leftarrow}[B] : B \in \mathscr{B}(\alpha X - X)\}$. Then $\sigma(\alpha X) \in L(\mathscr{B}(\alpha_0 X - X))$* (see 2.6 for notation).

Proof. As $\alpha X \leqq \alpha_0 X$, the map k_α certainly exists: Furthermore, $k_\alpha^{\leftarrow}[\alpha X - X] = \alpha_0 X - X$. As k_α is continuous, if $B \in \mathscr{B}(\alpha X - X)$ then $k_\alpha^{\leftarrow}[B] \in \mathscr{B}(\alpha_0 X - X)$. Obviously $\sigma(\alpha X)$ is closed under finite unions and intersections as $\mathscr{B}(\alpha X - X)$ is. Hence $\sigma(\alpha X) \in L(\mathscr{B}(\alpha_0 X - B))$.

3.2 LEMMA. *If $\mathscr{E} \in L(\mathscr{B}(\alpha_0 X - X))$, then $\mathscr{E} = \sigma(\alpha X)$ for some $\alpha X \in \overline{\mathscr{K}}_0(X)$.*

Proof. As $\mathscr{E} \in L(\mathscr{B}(\alpha_0 X - X))$, $\mathscr{E}$ is a subalgebra of $\mathscr{B}(\alpha_0 X - X)$. Let f denote the dual map from $S(\mathscr{B}(\alpha_0 X - X))$ onto $S(\mathscr{E})$. As $\alpha_0 X - X$ is compact and zero-dimensional, $S(\mathscr{B}(\alpha_0 X - X))$ is homeomorphic to $\alpha_0 X - X$, and so we can regard f as being a continuous map from $\alpha_0 X - X$ onto $S(\mathscr{E})$. Let αX be the free union of X and $S(\mathscr{E})$. Let k_α map $\alpha_0 X$ onto αX as follows: $k_\alpha | X = 1_X$ and $k_\alpha | \alpha_0 X - X = f$. Give αX the quotient topology that k_α induces. Then αX becomes a compact Hausdorff space, and the topology that X inherits from αX is identical to the original topology on X. As X is dense in αX, evidently $\alpha X \in \overline{\mathscr{K}}_0(X)$ (see [**8**, Theorem 13], or [**7**, Theorem 20, Chapter 5] for details), and $\alpha X - X = S(\mathscr{E})$. It follows from lemma 2.1 that $\sigma(\alpha X) = \mathscr{E}$, and we are done.

3.3 THEOREM. *The correspondence $\alpha X \mapsto \sigma(\alpha X)$ is an order isomorphism from $\overline{\mathscr{K}}_0(X)$ onto $L(\mathscr{B}(\alpha_0 X - X))$. Hence $\overline{\mathscr{K}}_0(X)$ is a complete lattice and is lattice-isomorphic to $L(\mathscr{B}(\alpha_0 X - X))$.*

Proof. If $\alpha X \leqq \gamma X$, with $\alpha X, \gamma X \in \overline{\mathscr{K}}_0(X)$, let k_α and k_γ be the connecting maps from $\alpha_0 X$ onto αX and γX respectively. Let g be the connecting map from γX onto αX. As $g \circ k_\gamma$ and k_α agree on X, evidently $g \circ k_\gamma = k_\alpha$. If $B \in \sigma(\alpha X)$, then $B = k_\alpha^{\leftarrow}[C]$ for some $C \in \mathscr{B}(\alpha X - X)$; hence $B = k_\gamma^{\leftarrow}[g^{\leftarrow}[C]]$. But $g^{\leftarrow}[C] \in \mathscr{B}(\gamma X - X)$, so $B \in \sigma(\gamma X)$. Thus $\alpha X \leqq \gamma X$ implies $\sigma(\alpha X) \subseteq \sigma(\gamma X)$. Conversely, if αX and γX are in $\overline{\mathscr{K}}_0(X)$, and $\sigma(\alpha X) \subseteq \sigma(\gamma X)$, let $k : S(\sigma(\gamma X)) \to S(\sigma(\alpha X))$ be the dual map. $S(\sigma(\gamma X))$ and $\gamma X - X$ are homeomorphic, as $\gamma X - X$ is compact and zerodimensional; so are $S(\sigma(\alpha X))$ and $\alpha X - X$. Thus there is a continuous map k from $\gamma X - X$ onto $\alpha X - X$. A repetition of the argument used to prove 3.2 allows us to conclude that $\alpha X \leqq \gamma X$.

The above remarks show that σ is a one-to-one order-preserving map from $\overline{\mathscr{K}}_0(X)$ into $L(\mathscr{B}(\alpha_0 X - X))$. Lemma 3.2 tells us that σ sends $\overline{\mathscr{K}}_0(X)$ onto $L(\mathscr{B}(\alpha_0 X - X))$; thus $L(\mathscr{B}(\alpha_0 X - X))$ and $\overline{\mathscr{K}}_0(X)$ are order-isomorphic. As $L(\mathscr{B}(\alpha_0 X - X))$ is a complete lattice, so is $\overline{\mathscr{K}}_0(X)$.

3.4 THEOREM. *Let X and Y be two locally compact Hausdorff spaces. Then $\mathscr{K}_0(X)$ and $\mathscr{K}_0(Y)$ are lattice-isomorphic if and only if $FX - X$ and $FY - Y$ are homeomorphic.*

Proof. By Theorem 3.3, (with $\alpha_0 X = FX$, $\alpha_0 Y = FY$) $\mathscr{K}_0(X)$ and $\mathscr{K}_0(Y)$ are lattice-isomorphic if and only if $L(\mathscr{B}(FX - X))$ and $L(\mathscr{B}(FY - Y))$ are lattice-isomorphic. By Theorem 2.6, $L(\mathscr{B}(FX - X))$ are lattice-isomorphic if and only if $\mathscr{B}(FX - X)$ and $\mathscr{B}(FY - Y)$ are isomorphic as Boolean algebras. As $FX - X$ and $FY - Y$ are compact and zero-dimensional, this occurs if and only if $FX - X$ and $FY - Y$ are homeomorphic.

Next we show that $\mathscr{S}(X)$ is order-isomorphic with $\mathscr{K}_0(X)$.

3.5 LEMMA. *For each $\mathscr{A} \in \mathscr{S}(X)$, put*

$$\tau(\mathscr{A}) = \{\mathrm{cl}_{E(X)}\, k_X^{\leftarrow}[\mathrm{int}_X A] : A \in \mathscr{A}\}.$$

Then τ is an order-isomorphism from $\mathscr{S}(X)$ onto $\{\mathscr{E} \in \mathscr{S}_0(E(X)) : \mathscr{E} \subset \tau(D(X))\}$. Thus $\mathscr{S}(X)$ is a complete lattice.

Proof. As k_X is a perfect irreducible map from $E(X)$ onto X, by Theorem 2.5 it is evident that the correspondence $\mathscr{E} \mapsto \{k_X[E] : E \in \mathscr{E}\}$ is a lattice-isomorphism from the lattice of subalgebras of $\mathscr{R}(E(X))$ $(=\mathscr{B}(E(X)))$ onto the lattice of subalgebras of $\mathscr{R}(X)$, and that τ is the restriction of the inverse of this correspondence to $\mathscr{S}(X)$. But

$$\{\tau(\mathscr{A}) : \mathscr{A} \in \mathscr{S}(X)\} = \{\tau(\mathscr{A}); K(X) \subseteq \mathscr{A} \subseteq D(X) \text{ and } \mathscr{A} \in \mathscr{S}(X)\} = \{\mathscr{E} \in \mathscr{S}_0(E(X)) : \tau(\langle K(X)\rangle) \subseteq \mathscr{E} \subseteq \tau(D(X))\}.$$

Evidently $\langle K(X)\rangle = \{A \in \mathscr{R}(X)$: either A or $\mathrm{cl}_X(X - A)$ is compact$\}$ and it follows quickly that $\tau(\langle K(X)\rangle) = \{B \in \mathscr{B}(E(X))$: either B or $E(X) - B$ is compact$\}$. Hence $\tau(\langle K(X)\rangle)$ is the smallest member of $\mathscr{S}_0(E(X))$ (see [**9**, Lemma 2.10 and Corollary 3.5]). Thus the lemma holds.

3.6 THEOREM. *Let $\mathscr{A} \in \mathscr{S}(X)$ and define a map f from $\Gamma(\tau(\mathscr{A}))$ to $\mathscr{A}X$*

as follows:

$$f(p) = \bigcap \{(k_X[F])^*;\ F \in \alpha(p)\} \qquad (p \in \Gamma(\tau(\mathscr{A}))).$$

Then f is a well-defined continuous map from $\Gamma(\tau(\mathscr{A}))$ onto $\mathscr{A}X$ whose restriction to $E(X)$ is k_X and whose restriction to $\Gamma(\tau(\mathscr{A})) - E(X)$ is a homeomorphism onto $\mathscr{A}X - X$.

Proof. From the definition of τ it is evident that the map $C \mapsto k_X[C]$ is a Boolean algebra isomorphism from $\tau(\mathscr{A})$ onto $\mathscr{A}$. It follows that the map $\bar{k} : S(\tau(\mathscr{A})) \to S(\mathscr{A})$ defined by $\bar{k}(\alpha) = \{k_X[C] : C \in \alpha\}$ $(\alpha \in S(\tau(\mathscr{A})))$ is a homeomorphism from $S(\tau(\mathscr{A}))$ onto $S(\mathscr{A})$. The map $h_{\tau(\mathscr{A})} : \Gamma(\tau(\mathscr{A})) \to S(\tau(\mathscr{A}))$ defined by $h_{\tau(\mathscr{A})}(p) = \alpha(p)$ is, as we have seen, a homeomorphism from $\Gamma(\tau(\mathscr{A}))$ onto $S(\tau(\mathscr{A}))$. Hence $f = g \circ \bar{k} \circ h_{\tau(\mathscr{A})}$ is a continuous map from $\Gamma(\tau(\mathscr{A}))$ onto $\mathscr{A}X$, where g is the map defined in 2.8.

Let $p \in E(X)$; then $h_{\tau(\mathscr{A})}(p) = \{C \in \tau(\mathscr{A}) : p \in C\}$ (see the discussion preceding 2.2).
Thus

$$\begin{aligned} \bar{k} \circ h_{\tau(\mathscr{A})}(p) &= \{k[C] : C \in \tau(\mathscr{A}) \text{ and } p \in C\} \\ &= \{k[C] : p \in C \text{ and } C = \mathrm{cl}_{E(X)}\, k_X^{\leftarrow}[\mathrm{int}_X A] \text{ for some } A \in \mathscr{A}\} \\ &= \{A \in \mathscr{A} : A \in p\}. \end{aligned}$$

It is straightforward to check that this last subfamily of $\mathscr{A}$ is an ultrafilter α on $\mathscr{A}$ and hence a point of $S(\mathscr{A})$. As $\mathscr{A}$ is a base for the closed subsets of X, and as $\bigcap \alpha \neq \emptyset$, it follows that

$$g(\alpha) = \bigcap \{A \in \mathscr{A} : A \in p\} = \bigcap \{G \in \mathscr{R}(X) : G \in p\} = k_X(p).$$

Hence $f(p) = k_X(p)$, so f agrees with k_X on $E(X)$.

Since $\bar{k} \circ h_{\tau(\mathscr{A})}$ is a homeomorphism from $\Gamma(\tau(\mathscr{A}))$ onto $S(\mathscr{A})$, and since g restricted to $S(\mathscr{A}) - g^{\leftarrow}[X]$ is a homeomorphism from $S(\mathscr{A}) - g^{\leftarrow}[X]$ onto $\mathscr{A}X - X$ (2.8), to complete the proof it suffices to show that if $p \in \Gamma(\tau(\mathscr{A})) - E(X)$, then $\bar{k} \circ h_{\tau(\mathscr{A})}(p) \notin g^{\leftarrow}[X]$. If $p \in \Gamma(\tau(\mathscr{A})) - E(X)$, then $h_{\tau(\mathscr{A})}(p)$ is an ultrafilter on $\tau(\mathscr{A})$ with empty intersection, and $\bar{k} \circ h_{\tau(\mathscr{A})}(\alpha) = \{k_X[A] : A \in h_{\tau(\mathscr{A})}(p)\}$. Note that $k_X[A]$ is a regular closed subset of X for each $A \in h_{\tau(\mathscr{A})}(p)$. If $q \in X$ and $q \in \bigcap \{k_X[A] : A \in h_{\tau(\mathscr{A})}(p)\}$, then there exists an ultrafilter δ on $\mathscr{R}(X)$ such that $\bigcap \delta = \{q\}$ and $\{k_X[A] : A \in h_{\tau(\mathscr{A})}(p)\} \subseteq \delta$ (note that $\{k_X[A] : A \in h_{\tau(\mathscr{A})}(p)\}$, together with the filter $\{F \in \mathscr{R}(X) : q \in \mathrm{int}_X F\}$, form a filter base on $\mathscr{R}(X)$ which will be contained in some ultrafilter; this is our δ). Evidently $\delta \in A$ for each $A \in h_{\tau(\mathscr{A})}(p)$, and so $h_{\tau(\mathscr{A})}(p)$ does not have an empty intersection, in contradiction to our assumptions. Hence $\bigcap \{k_X[A] : A \in h_{\tau(\mathscr{A})}(p)\} = \emptyset$, and so by 2.8(b) $\bar{k} \circ h_{\tau(\mathscr{A})}(p) \notin g^{\leftarrow}[X]$. The theorem follows.

3.7 THEOREM. (a) *$FX - X$ and $\Gamma(\tau(D(X))) - E(X)$ are homeomorphic.* (b) *$\mathscr{S}(X)$ is lattice-isomorphic to $\mathscr{K}_0(X)$.*

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Proof. (a) As remarked earlier, the compactification $\mathscr{A}X$, when $\mathscr{A} = D(X)$, is FX. Part (a) now follows from 3.6

(b) By 3.5, $\mathscr{S}(X)$ is lattice-isomorphic to $\{\mathscr{E} \in \mathscr{S}_0(E(X)) : \mathscr{E} \subseteq \tau(D(X))\}$. As $E(X)$ is zero-dimensional, by 2.5 this latter lattice is lattice-isomorphic to

$$\{\alpha E(X) \in \mathscr{K}_0(E(X)) : \alpha X \leqq \Gamma(\tau(D(X)))\}.$$

It follows from 3.3 that this lattice is lattice-isomorphic to $L(\mathscr{B}(\Gamma(\tau(D(X))) - E(X)))$, which by part (a) and 2.6 is lattice-isomorphic to $L(\mathscr{B}(FX - X))$. By 3.3 this lattice is lattice-isomorphic to $\mathscr{K}_0(X)$, and the theorem follows.

We conclude this paper with three remarks. First, the methods of proof used above can be adapted, with little change, to prove the following more general result: if X is locally compact and $\mathscr{A}_0 \in \mathscr{S}(X)$, then $\{\mathscr{A} \in \mathscr{S}(X) : \mathscr{A} \subseteq \mathscr{A}_0\}$ is lattice-isomorphic to $\{\alpha X \in \mathscr{K}_0(X) : \alpha X \leqq \mathscr{A}_0 X\}$. Second, those lattices which are lattice-isomorphic to the lattice of subalgebras of some Boolean algebra have been characterized in [**6**]. Since, given a Boolean algebra B, we can find a locally compact space X such that $FX - X$ is homeomorphic to $S(B)$ (see [**4**, 9*K*]), it follows that we have a lattice-theoretic characterization of those lattices that are lattice-isomorphic to $\mathscr{K}_0(X)$ for some locally compact space X. Third, it follows from 2.5 that

$$\mathscr{A} \mapsto \{\mathrm{cl}_{E(X)}\, k_X^{\leftarrow}(\mathrm{int}_X A) : A \in \mathscr{A}\}$$

is a lattice isomorphism from the lattice $\mathscr{P}(X)$ of subalgebras of $\mathscr{R}(X)$ that contain $\langle K(X)\rangle$ onto $\mathscr{S}_0(E(X))$. Hence we obtain the following result (βS denotes the Stone-Čech compactification of the space S).

3.8 THEOREM. *Let X be a locally compact Hausdorff space.*

(a) *$\mathscr{P}(X)$ is lattice-isomorphic to $L(\mathscr{B}(\beta E(X) - E(X)))$.*

(b) *If X and Y are locally compact Hausdorff spaces, then $\mathscr{P}(X)$ and $\mathscr{P}(Y)$ are lattice-isomorphic if and only if $\beta E(X) - E(X)$ and $\beta E(Y) - E(Y)$ are homeomorphic.*

(c) *Assume the continuum hypothesis ($\aleph_1 = 2^{\aleph_0}$). If X is locally compact, realcompact and non-compact, and if $\mathscr{R}(X)$ has cardinality $2^{\aleph_0}$, then $\mathscr{P}(X)$ is lattice-isomorphic to $\mathscr{P}(\mathbf{N})$, where $\mathbf{N}$ is the countably infinite discrete space.*

Proof. (a) This follows from the above remark, and Theorems 2.2 and 3.3.

(b) This follows from (a) and Theorems 3.3 and 3.4.

(c) In Theorem 3.2 of [**16**], it is proved that if X is as described in (c), and if the continuum hypothesis is assumed, then $\beta E(X) - E(X)$ is homeomorphic to $\beta\mathbf{N} - \mathbf{N}$. As $\mathbf{N} = E(\mathbf{N})$, our result now follows from (b).

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University of Manitoba,
Winnipeg, Manitoba