ASSOCIATIVITY OF THE TENSOR PRODUCT OF SEMILATTICES

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(Received 19th March 1984)

The tensor product of semilattices has been studied in [2], [3] and [5]. A survey of this work is given in [4]. Although a number of problems were settled completely in these papers, the question of the associativity of the tensor product was only partially answered. In the present paper we give a complete solution to this problem.

For terminology and basic results of lattice theory and universal algebra, consult Birkhoff [1] and Grätzer [6], [7]. The join and meet of elements $a_1, \ldots, a_n$ of a lattice are denoted by $\sum_{i=1}^{n} a_i$ and $\prod_{i=1}^{n} a_i$ respectively. All semilattices considered are join-semilattices. The reader is referred to [2] for definitions and results concerning the tensor product $A \otimes B$ of semilattices $A$ and $B$. In fact, much of [2] is concerned with the special situation in which $A$ and $B$ are distributive lattices, and $A \otimes B$ is obtained by considering $A$ and $B$ as join-semilattices.

We will need the following results from the earlier papers [2, Theorem 2.5; 3, Theorem 1].

**Theorem 1.** Let $A$ and $B$ be the distributive lattices and let $a, a_i \in A$ and $b, b_i \in B$ for $i=1, \ldots, n$. Let $n$ be the set $\{1, \ldots, n\}$. Then $a \otimes b \leq \sum_{i=1}^{n} (a_i \otimes b_i)$ in $A \otimes B$ if and only if there exist non-empty subsets $S_1, \ldots, S_m$ of $n$ such that $a \leq \sum_{i=1}^{m} \prod_{i \in S_j} a_i$ and $b \leq \prod_{i=1}^{m} \sum_{i \in S_j} b_i$.

**Theorem 2.** Let $A$ and $B$ be semilattices and let $a, a_i \in A$ and $b, b_i \in B$ for $i=1, \ldots, n$. Then $a \otimes b \leq \sum_{i=1}^{n} (a_i \otimes b_i)$ in $A \otimes B$ if and only if there exists an $n$-ary lattice polynomial $p$ such that $a \in p((a_1), \ldots, (a_n))$ and $b \in p^*((b_1), \ldots, (b_n))$.

Here $(x)$ denotes the principal ideal generated by $x$ and $p^*$ is the polynomial obtained by interchanging the lattice operations in $p$.

Now the partial result on associativity of the tensor product obtained earlier is the following [2, Theorem 5.1].

**Theorem 3.** Let $A$, $B$ and $C$ be finite distributive lattices. Then $(A \otimes B) \otimes C$ is isomorphic with $A \otimes (B \otimes C)$.

We shall extend this result first to arbitrary distributive lattices and then to arbitrary semilattices.

**Theorem 4.** Let $A$, $B$ and $C$ be distributive lattices. Then $(A \otimes B) \otimes C$ is isomorphic with $A \otimes (B \otimes C)$.
Proof. Every element of \((A \otimes B) \otimes C\) can be written in the form \(\sum_{i=1}^{n} [(a_i \otimes b_i) \otimes c_i]\), where \(a_i \in A\), \(b_i \in B\) and \(c_i \in C\) for \(i = 1, \ldots, n\). Let \(\varphi : (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C)\) be defined by 
\(\varphi(\sum_{i=1}^{n} [(a_i \otimes b_i) \otimes c_i]) = \sum_{i=1}^{n} [a_i \otimes (b_i \otimes c_i)]\). We prove that \(\varphi\) is an isomorphism by showing that for all \(a_i, e_j \in A\), \(b_i, f_j \in B\), \(c_i, g_j \in C\), \(i = 1, \ldots, n, j = 1, \ldots, m\), we have 
\(\sum_{i=1}^{n} [(a_i \otimes b_i) \otimes c_i] \leq \sum_{i=1}^{n} [(e_j \otimes f_j) \otimes g_j]\) if and only if \(\sum_{i=1}^{n} [a_i \otimes (b_i \otimes c_i)] \leq \sum_{j=1}^{m} [e_j \otimes (f_j \otimes g_j)]\). Clearly it suffices to prove that \((a \otimes b) \otimes c \leq \sum_{i=1}^{n} [(a_i \otimes b_i) \otimes c_i]\) if and only if \(a \otimes (b \otimes c) \leq \sum_{i=1}^{n} [a_i \otimes (b_i \otimes c_i)]\). In view of the symmetry of this assertion, it is enough to prove it in one direction.

Suppose that \((a \otimes b) \otimes c \leq \sum_{i=1}^{n} [(a_i \otimes b_i) \otimes c_i]\). Then by Theorem 1, there are non-empty subsets \(S_1, \ldots, S_m\) of \(n\) such that

\[
a \otimes b \leq \prod_{j=1}^{m} \left( \prod_{i \in S_j} (a_i \otimes b_i) \right) = \prod_{j=1}^{m} \left( \prod_{i \in S_j} a_i \right) \otimes \left( \prod_{i \in S_j} b_i \right)
\]

and

\[
c \leq \prod_{j=1}^{m} \sum_{i \in S_j} c_i. \tag{1}
\]

Again using Theorem 1, we have that there exist non-empty subsets \(T_1, \ldots, T_p\) of \(m\) such that

\[
a \leq \prod_{k=1}^{p} \left( \prod_{j \in T_k} a_i \right) \tag{2}
\]

and

\[
b \leq \prod_{k=1}^{p} \sum_{j \in T_k} \left( \prod_{i \in S_j} b_i \right). \tag{3}
\]

Now for \(k = 1, \ldots, p\), let \(U_k = \{i \in S_j : j \in T_k\}\). Then \(U_1, \ldots, U_p\) are non-empty subsets of \(n\). Then by (2) we have

\[
a \leq \prod_{k=1}^{p} \left( \prod_{i \in U_k} a_i \right). \tag{4}
\]

Using (1) and (3) we have that for \(k = 1, \ldots, p\), \(b \leq \sum_{j \in T_k} \prod_{i \in S_j} b_i\) and \(c \leq \prod_{j \in T_k} \sum_{i \in S_j} c_i\). Then it follows by Theorem 1 that for \(k = 1, \ldots, p\), we have \(b \otimes c \leq \sum_{i \in U_k} (b_i \otimes c_i)\). Hence \(b \otimes c \leq \prod_{k=1}^{p} \sum_{i \in U_k} (b_i \otimes c_i)\). Applying Theorem 1 to the preceding result and (4), we obtain \(a \otimes (b \otimes c) \leq \sum_{i=1}^{n} [a_i \otimes (b_i \otimes c_i)]\).

Theorem 5. Let \(A\), \(B\) and \(C\) be semilattices. Then \((A \otimes B) \otimes C\) is isomorphic with \(A \otimes (B \otimes C)\).

Proof. The initial remarks made in the proof of Theorem 4 remain valid in this case and we define the map \(\varphi\) in the same way as before. Again it suffices to prove that if \((a \otimes b) \otimes c \leq \sum_{i=1}^{n} [(a_i \otimes b_i) \otimes c_i]\) then \(a \otimes (b \otimes c) \leq \sum_{i=1}^{n} [a_i \otimes (b_i \otimes c_i)]\). Now it follows
by Theorem 2 that this assertion is equivalent to the statement: if there is an $n$-ary polynomial $p$ such that $a \otimes b \in p((a_1 \otimes b_1), \ldots, (a_n \otimes b_n))$ and $c \in p^*((c_1), \ldots, (c_n))$, then there is an $n$-ary polynomial $t$ such that $a \in t((a_1), \ldots, (a_n))$ and $b \otimes c \in t^*((b_1 \otimes c_1), \ldots, (b_n \otimes c_n))$.

We will prove this statement by induction on the complexity of the polynomials involved. It is clearly true for polynomials of length 1. Now assume that the statement holds for the $n$-ary polynomials $p$ and $q$. Let $a \otimes b \in (p + q)((a_1 \otimes b_1), \ldots, (a_n \otimes b_n))$ and $c \in p^*q^*((c_1), \ldots, (c_n))$. Then $a \otimes b \leq \sum_{i=1}^k (x_i \otimes y_i) + \sum_{k+1}^m (x_i \otimes y_i)$ where $x_i \in A$ and $y_i \in B$ for $i = 1, \ldots, m$ and

$$
\sum_{i=1}^k (x_i \otimes y_i) \in p((a_1 \otimes b_1), \ldots, (a_n \otimes b_n))
$$

and

$$
\sum_{k+1}^m (x_i \otimes y_i) \in q((a_1 \otimes b_1), \ldots, (a_n \otimes b_n)).
$$

Thus $a \otimes b \leq \sum_{i=1}^n (x_i \otimes y_i)$, where for each $i$ either $x_i \otimes y_i \in p((a_1 \otimes b_1), \ldots, (a_n \otimes b_n))$ or $x_i \otimes y_i \in q((a_1 \otimes b_1), \ldots, (a_n \otimes b_n))$. Since $c \in p^*q^*((c_1), \ldots, (c_n))$ it follows that for all $i$ there is an $n$-ary polynomial $s$ (where $s$ is either $p$ or $q$) such that

$$
x_i \otimes y_i \in s((a_1 \otimes b_1), \ldots, (a_n \otimes b_n))
$$

and

$$
c \in s^*((c_1), \ldots, (c_n)).
$$

Now by the inductive hypothesis, for each $i$ there is an $n$-ary polynomial $u_i$ such that

$$
x_i \in u_i((a_1), \ldots, (a_n))
$$

and

$$
y_i \otimes c \in u_i^*((b_1 \otimes c_1), \ldots, (b_n \otimes c_n)).
$$

Since $a \otimes b \leq \sum_{i=1}^n (x_i \otimes y_i)$ it follows by Theorem 2 that there is an $n$-ary polynomial $r$ such that $a \in r((x_1), \ldots, (x_n))$ and $b \in r^*((y_1), \ldots, (y_n))$. Let $t$ be the $n$-ary polynomial $r(u_1, \ldots, u_n)$. Since $x_i \in u_i((a_1), \ldots, (a_n))$ for all $i$, it is easy to see that $a \in t((a_1), \ldots, (a_n))$. Also since $b \in r^*((y_1), \ldots, (y_n))$ it is readily verified that $b \otimes c \in r^*((y_1 \otimes c), \ldots, (y_n \otimes c))$.

It follows that

$$
b \otimes c \in r^*(u_1^*, \ldots, u_n^*)((b_1 \otimes c_1), \ldots, (b_n \otimes c_n))
$$

$$
= t^*((b_1 \otimes c_1), \ldots, (b_n \otimes c_n)).
$$

Thus the statement holds for the polynomial $p + q$. 

Finally, assume the statement holds for $p$ and $q$ and suppose that

$$a \otimes b \in (pq)(((a_1 \otimes b_1), \ldots, (a_n \otimes b_n)))$$

and

$$c \in (p^* + q^*)((c_1), \ldots, (c_n)).$$

Then

$$a \otimes b \in p(((a_1 \otimes b_1), \ldots, (a_n \otimes b_n))),$$

$$a \otimes b \in q(((a_1 \otimes b_1), \ldots, (a_n \otimes b_n))),$$

and there exist $x \in A$ and $y \in B$ such that $c \leq x + y$ where $x \in p^*((c_1), \ldots, (c_n))$ and $y \in q^*((c_1), \ldots, (c_n))$. Now by the inductive hypothesis there exist polynomials $r$ and $s$ such that

$$a \in r((a_1), \ldots, (a_n)), b \otimes x \in r^*((b_1 \otimes c_1), \ldots, (b_n \otimes c_n)),$$

and

$$a \in s((a_1), \ldots, (a_n)), b \otimes y \in s^*((b_1 \otimes c_1), \ldots, (b_n \otimes c_n)).$$

Let $t$ be the polynomial $rs$. Then $a \in t((a_1), \ldots, (a_n))$ and

$$b \otimes c \leq b \otimes (x + y) \in (r^* + s^*)((b_1 \otimes c_1), \ldots, (b_n \otimes c_n))$$

so that

$$b \otimes c \in t^*((b_1 \otimes c_1), \ldots, (b_n \otimes c_n)).$$

Thus the statement holds for the polynomial $pq$.

This completes the induction and the Theorem is now established.

REFERENCES

5. G. Fraser and A. Bell, The word problem in the tensor product of distributive semilattices, Semigroup Forum 30 (1984), 117–120.

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