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A NOTE ON PERIODIC SOLUTIONS OF SOME NONAUTONOMOUS DIFFERENTIAL EQUATIONS

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We prove the existence of nontrivial periodic solutions of some nonlinear ordinary differential equations with time-dependent coefficients using variational methods.

0. Introduction and statement of the results

In this work we study the existence, under suitable conditions, of nontrivial T-periodic solutions of the following nonlinear equations:

$$\ddot{x} - \alpha(t)x - \beta(t)x^2 + \gamma(t)x^3 = 0 \tag{0.1}$$

$$\ddot{x} + \alpha(t)x - \beta(t)x^2 + \gamma(t)x^3 = 0 \tag{0.2}$$

where α, β and γ are measurable T-periodic functions such that if we denote by a , A , c and C the infimum and supremum of α and γ , respectively, and $B = \|\beta\|_{L^\infty}$, then

$$0 < a \leqslant \alpha(t) \leqslant A < \infty \ , \quad 0 < c \leqslant \gamma(t) \leqslant C < \infty \quad \text{and} \quad B < \infty \ . \tag{0.3}$$

The study of these equations was suggested by a paper of Cronin [4] which deals with an equation related with the biomathematical model of the aneurysm of the circle of Willis introduced by Austin [2]. In fact equation (0.2) is the homogeneous analogue of the equation studied in [4] for the case in which α , β and γ are positive constants. Also in [4] there is a forcing term of the type $k\cdot\cos(\omega t)$.

Solutions of (0.1) and (0.2) will always be considered in the sense

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of C^1-functions, x , such that $\ddot{x}$ exists almost everywhere and $x(t) = x(t+T)$.

For the sake of simplification of computation we shall deal with the case of period 2π . The results, however, apply to every period T with obvious modifications.

Abstract results concerning the existence of nontrivial periodic solutions of nonlinear problems have recently appeared in the literature (see [3], [5] and [7]). However they do not apply to equations (0.1) or (0.2). We also observe that, if α , β and γ are constants, (0.1) and (0.2) can be studied using the phase plane.

We prove the following theorems:

THEOREM 0.1. *Let* α , β *and* γ *be measurable* 2π*-periodic functions that satisfy (0.3). Then equation (0.1) has a nontrivial* 2π*-periodic solution.*

THEOREM 0.2. *Let* α , β *and* γ *be measurable* 2π*-periodic functions that satisfy (0.3) and such that*

(0.4) $$m^2 < a \leqslant A < (m+1)^2 \quad \textit{for some integer} \quad m \geqslant 0 ,$$

and

(0.5) $$B^2 \leqslant \frac{9}{2} \delta c , \quad \textit{where} \quad \delta = a - m^2 .$$

Then equation (0.2) has a nontrivial 2π*-periodic solution.*

This work is divided into two parts in which we prove theorems 0.1 and 0.2, respectively. Those proofs have an analogous structure and we use similar notations in both. The arguments and computations of section 1 are, however, much simpler than those of section 2. We consider the interval $[0,2\pi]$ and first we solve the projections of (0.1) and (0.2) onto spaces of finite dimension using in the case of (0.1) a "Mountain Pass Lemma" ([1]) and in (0.2) a generalization of that lemma ([6]). Then, after adequate estimates, we pass to the limit. We observe that working with the finite dimensional approach the boundary conditions are well defined (see (1.2) and (2.2)) and the variational principles that we use are simpler.

NOTATIONS. Throughout this paper we use the standard spaces $L^p(0,2\pi)$, $C^p(\overline{0,2\pi})$, and the Sobolev spaces $H^p(0,2\pi) = W^{p,2}(0,2\pi)$ which we denote

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simply by L^p, C^p and H^p. We use the symbols $|\cdot|_p$ and $\|\cdot\|_p$ to denote the usual norms of L^p and H^p, respectively.

1. Equation (0.1)

Consider the interval $[0,2\pi]$ and the equation

$$\ddot{u} - \alpha(t)u - \beta(t)u^2 + \gamma(t)u^3 = 0 , \tag{1.1}$$

with the periodic conditions

$$u(0) = u(2\pi) , \quad \dot{u}(0) = \dot{u}(2\pi) , \tag{1.2}$$

where α, β and γ are measurable 2π-periodic functions that satisfy (0.3).

Proof of Theorem 0.1. The proof is divided in three steps:

Step 1: Approximate solution in finite dimension

For each positive integer N consider the finite dimensional space

$$Y_N = \{ \sum_{k=-N}^{N} c_k e^{ikt} : c_k \in \mathbb{C} \text{ and the sum is real}\}$$

and the functional $J_N : Y_N \to \mathbb{R}$ defined by

$$J_N(u) = \int_0^{2\pi} (\tfrac{1}{2}|\dot{u}|^2 + \tfrac{1}{2}\alpha(t)u^2 + \tfrac{1}{3}\beta(t)u^3 - \tfrac{1}{4}\gamma(t)u^4)dt .$$

It is easy to see that every critical point of J_N is a solution of the equation

$$\ddot{x} - \alpha(t)x = P_N(\beta(t)x^2 - \gamma(t)x^3) \tag{1.3}$$

where P_N is the orthogonal projection of $L^2(0,2\pi)$ onto Y_N. So we are going to look for critical points of J_N. The tool we use is the "Mountain Pass Lemma" contained in [1].

By Hölder's inequality and (0.3) we have for $u \in Y_N$

$$J_N(u) \geq \frac{a}{2} |u|_2^2 - \frac{B}{3} |u|_3^3 - \frac{C}{4} |u|_4^4 \geq |u|_2^2 (a_1 - a_2|u|_2 - a_3|u|_2^2) ,$$

(where a_1, a_2, a_3 are positive constants) which implies that there exist $r, \rho > 0$ such that $J(u) > 0$ if $0 < |u|_2 \leq r$ and $J(u) \geq \rho > 0$ if

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$|u|_2 = r$. On the other hand, denoting by $\|\cdot\|_\alpha$ the norm $\|u\|_\alpha = (\int_0^{2\pi}(|\dot{u}|^2 + \alpha(t)u^2)dt)^{\frac{1}{2}}$, since Y_N has finite dimension, we derive

$$J_N(u) \leqslant \frac{1}{2}\|u\|_\alpha^2 + \frac{B}{3}|u|_3^3 - \frac{c}{4}|u|_4^4 \leqslant |u|_2^2(a_4 + a_5|u|_2 - a_6|u|_2^2) ,$$

(where a_4 , a_5 and a_6 are positive constants) which implies that

(1.4) $$J_N(u) < 0 \quad \text{for} \quad |u|_2 \quad \text{large enough} ,$$

and therefore there is $w_N \in Y_N$ such that

(1.5) $$|w_N|_2 > r \quad \text{and} \quad J_N(w_N) = 0 .$$

In particular there is w_N in Y_N such that

(1.5) $$|w_N|_2 > r \quad \text{and} \quad J_N(w_N) = 0 ,$$

and obviously we can take $w_N = w \in Y_1$ since $J_{N|Y_1} = J_1$. This fact will be useful in step 2. Inequality (1.4) and the fact that Y_N has finite dimension show that the Palais-Smale condition is trivially satisfied in $[0,+\infty)$. Then by the "Mountain Pass Lemma" ([1]) J_N has a nonzero critical point, u_N , and a corresponding critical value b_N characterized by:

(1.6) $$b_N = \inf_{g\in\Gamma} \max_{y\in[0,1]} J_N(g(y)) > 0 ,$$

where

$$\Gamma_N = \{g \in C([0,1], Y_N) : g(0) = 0, g(1) = w\} .$$

Step 2: Estimates for u_N

Now we are going to prove that there are positive constants independent of N , A_1 and A_2 , such that

(1.7) $$\|u_N\|_2 \leqslant A_1 ,$$

(1.8) $$|u_N|_\infty \geqslant A_2 > 0 .$$

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Let b_N be the critical value that corresponds to the critical point of J_N, u_N. From its variational characterization (1.6) and from the fact that, for $t \in [0,1]$, $g(t) = t\,w \in \Gamma_N$ we derive

(1.9) $$0 < b_N = \inf_{g\in\Gamma_N} \max_{t\in[0,1]} J_N(g(t)) \leq \max_{t\in[0,1]} J_N(t\,w) = J_1(\bar{w})$$

where $\bar{w}$ is an element of Y_1 in which $\max_{t\in[0,1]} J_1(t\,w)$ is achieved.

If we denote by $\langle \cdot,\cdot \rangle$ the duality bracket between Y_N and its dual, by (1.9) and by the fact that u_N is a critical point of J_N we have

$$0 < b_N = J(u_N) - \frac{1}{3}\langle J'(u_N), u_N\rangle$$

$$= \frac{1}{6}\int_0^{2\pi}(|\dot{u}_N|^2 + \alpha u_N^2) + \frac{1}{12}\int_0^{2\pi}\gamma u_N^4 \leq J_1(\bar{w}) ,$$

which implies the existence of a constant B_1 (independent of N) such that

(1.10) $$\|u_N\|_\alpha < B_1$$

and, since $\|\cdot\|_\alpha$ is equivalent to the standard H^1-norm, by the Sobolev imbedding theorem, together with (0.3) and (1.3) it follows easily that

$$|\ddot{u}_N|_2 \leq |\ddot{u}_N - \alpha u_N|_2 + |\alpha u_N|_2 \leq |P_N(\beta u_N^2 - \gamma u_N^3)|_2 + B_2 \leq B_3 ,$$

where B_2 and B_3 are positive constants independent of N. Hence (1.7) holds.

As for (1.8), we observe that, since u_N is a critical point of J_N in Y_N, we have

$$\langle J'(u_N), u_N\rangle = \int_0^{2\pi}(|\dot{u}_N|^2 + \alpha u_N^2 + \beta u_N^3 - \gamma u_N^4) = 0 .$$

Therefore by (0.3)

$$0 < a|u_N|_2^2 \leq B|u_N|_3^3 + C|u_N|_4^4 \leq (B|u_N|_\infty + C|u_N|_\infty^2)\,|u_N|_2^2 ,$$

which implies (1.8) since $u_N \not\equiv 0$.

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Step 3: Passing to the limit

By (1.7) and imbedding theorems, (u_N) has a subsequence (which we still denote (u_N)) such that

$$u_N \rightharpoonup u \quad \text{in} \quad H^2$$

and

$$u_N \to u \quad \text{in} \quad C^1 .$$

Estimate (1.8) ensures that u is nonzero. Since

$$P_N(\beta u_N^2 - \gamma u_N^3) \to \beta u^2 - \gamma u^3 \quad \text{in} \quad L^2 ,$$

(1.3) shows that u satisfies the equation

$$\ddot{u} - \alpha u - \beta u^2 + \gamma u^3 = 0 \tag{1.9}$$

in the L^2 sense. The periodicity of u_N and $\dot{u}_N$ and the C^1-convergence obviously imply

$$u(0) = u(2\pi) \, , \quad \dot{u}(0) = \dot{u}(2\pi) \, . \tag{1.10}$$

(1.9) and (1.10) show that u can be extended to $(-\infty,+\infty)$ as a C^1-function, with period 2π, solving equation (0.1). This ends the proof.

REMARK 1. We observe that if α, β and γ are C^p-functions, solutions of equation (0.1) are classical solutions, more precisely, solutions of class C^{p+2}.

2. Equation (0.2)

Consider the interval $[0,2\pi]$ and the equation

$$\ddot{u} + \alpha(t)u - \beta(t)u^2 + \gamma(t)u^3 = 0 \tag{2.1}$$

with the periodic conditions

$$u(0) = u(2\pi) \, , \quad \dot{u}(0) = \dot{u}(2\pi) \tag{2.2}$$

where α, β and γ are functions that satisfy (0.3).

In the proof of Theorem (0.2) we use the following result, which follows easily from a theorem of Rabinowitz [6], and which is a generalization of the "Mountain Pass Lemma"

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THEOREM 2.1. *Let* H *be a finite dimensional Hilbert space and* H_1, H_2 *subspaces of* H *such that* $H_2 = H_1^{\perp}$. *Suppose that* $J \in C^1(H,\mathbb{R})$ *and* $J(x) \leqslant 0$ *for every* $x \in H_1$. *If there are constants* $r_1 > r_2 > 0$ *such that* $J(x) > 0$ *in* $(B_{r_2} \setminus \{0\}) \cap H_2)$ *and* $J(x) \leqslant 0$ *in* $H \setminus B_{r_1}$, *then* J *has a critical point in* $\{x \in H : J(x) > 0\}$ *and a corresponding critical value characterized by*

$$c = \inf_{h\in\Gamma} \max_{x\in H_3\cap\bar{B}_{r_1}} J(h(x)) > 0$$

where, choosing $y \in H_2$, $H_3 = H_1 \oplus \operatorname{span}\{y\}$ *and*

$$\Gamma = \{h \in C(H_3 \cap \bar{B}_{r_1}, H) : h(x) = x \ \text{ if } \ J(x) \leqslant 0\} .$$

Proof of Theorem 0.2. As in Theorem 0.1 we divide the proof into three steps:

Step 1: Approximate solution in finite dimension

For each positive integer N such that $N^2 > A$ consider the finite dimensional space defined in the first step of the proof of Theorem 0.1, Y_N , and the functional $J_N : Y_N \to \mathbb{R}$ defined by

$$J_N(u) = \int_0^{2\pi} \left(\frac{|\dot{u}|^2}{2} - \frac{\alpha(t)}{2} u^2 + \frac{\beta(t)}{3} u^3 - \frac{\gamma(t)}{4} u^4\right) dt .$$

Analogously to (1.3) every critical point of J_N is a solution of the equation

$$\ddot{x} + \alpha(t)x = P_N(\beta(t)x^2 - \gamma(t)x^3) , \tag{2.3}$$

and therefore we are going to look for critical points of J_N .

Let m be as in the statement of Theorem (0.2) and consider the following subspaces of Y_N :

$$Y_N^+ = \{ \sum_{k=-N}^{N} c_k e^{ikt} \in Y_N : k^2 \geqslant (m+1)^2 \} ,$$

and

$$Y_N^- = \{ \sum_{k=-m}^{m} c_k e^{ikt} \in Y_N \} .$$

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It is clear that $Y_N = Y_N^+ \oplus Y_N^-$.

We claim that:

(2.4) $$J_N(u) \leqslant 0 \text{ for every } u \in Y_N^-$$

and

(2.5) there are constants $r_1 > r_2 > 0$ such that $J_N(u) > 0$

in $Y_N^+ \cap (B_{r_2} \setminus \{0\})$ and $J_N(u) \leqslant 0$ in $Y_N \setminus B_{r_1}$.

In fact if $u \in Y_N^-$ we have

$$J_N(u) \leqslant \pi \sum_{k=-m}^{m} |c_k|^2 (k^2-a) + \int_0^{2\pi} (\frac{\beta}{3} u^3 - \frac{\gamma}{4} u^4)$$

$$\leqslant \pi(m^2-a) \sum_{k=-m}^{m} |c_k|^2 + \int_0^{2\pi} (\frac{B}{3} |u|^3 - \frac{c}{4} u^4)$$

$$= \int_0^{2\pi} u^2 (\frac{m^2-a}{2} + \frac{B}{3} |u| - \frac{c}{4} u^2)$$

which, by condition (0.5), implies (2.4). As for (2.5), take $u \in Y_N^+$.

Computing $J_N(u)$ we have

$$J_N(u) \geqslant \pi \sum_{\substack{k=-N \\ k^2 \geqslant (m+1)^2}}^{N} |c_k|^2 (k^2-A) + \int_0^{2\pi} (\frac{\beta}{3} u^3 - \frac{\gamma}{4} u^4)$$

$$\geqslant \frac{(m+1)^2-A}{2} |u|_2^2 - \frac{B}{3} |u|_3^3 - \frac{C}{4} |u|_4^4$$

$$\geqslant |u|_2^2 (q_1-q_2 |u|_2 - q_3 |u|_2^2) .$$

This inequality implies the first statement of (2.5). As for the second one it is obtained as in the proof of Theorem 0.1 (see (1.4)). Hence, by Theorem 2.1, J_N has a critical point in Y_N and corresponding critical value c_N characterized by

(2.6) $$c_N = \inf_{h \in \Gamma_N} \max_{u \in (Y_N^- \oplus \operatorname{span}\{y\}) \cap \bar{B}_{r_1}} J_N(h(u)) > 0 ,$$

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where $y \in Y_N^+$ and

$$\Gamma_N = \{h \in C(Y_N^- \oplus \text{span}\{y\}) \cap \bar{B}_{r_1}, Y_N) : h(u) = u \text{ if } J_N(u) \leqslant 0\}$$

Step 2: Estimates for U_N

As in the proof of Theorem 0.1 we are going to prove that there are constants independent of N, A_1 and A_2, such that

$$\|u_N\|_2 \leqslant A_1 \tag{2.7}$$

and

$$|u_N|_\infty \geqslant A_2 > 0 . \tag{2.8}$$

Let c_N be the critical value that corresponds to the critical point of J_N, u_N. We claim that

$$c_N \leqslant B_1 \tag{2.9}$$

and

$$|u_N|_4 \leqslant B_2 , \tag{2.10}$$

where B_1 and B_2 are positive constants independent of N. In fact, using the characterization of c_N (2.6) and the fact that $h(u) = u \in \Gamma_N$, we have

$$\begin{aligned} 0 < c_N &= \inf_{h \in \Gamma_N} \max_{u \in (Y_N^- \oplus \text{span}\{y\}) \cap \bar{B}_{r_1}} J_N(h(u)) \\ &\leqslant \max_{u \in (Y_N^- \oplus \text{span}\{y\}) \cap \bar{B}_{r_1}} J_N(u) \\ &\leqslant \max_{u \in Y_N^- \oplus \text{span}\{y\}} J_N(u) . \end{aligned}$$

We observe that (2.5) guarantees the existence of $\max_{u \in Y_N^- \oplus \text{span}\{y\}} J_N(u)$.

Suppose that the maximum is achieved at $\bar{u}_N$. Take $y = \cos jt$ where j is an integer such that $N^2 \geqslant j^2 \geqslant (m+1)^2$. Then $\bar{u}_n = \sum_{k=-m}^{m} c_k e^{ikt} + \lambda \cos jt$

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and we have

$$0 \leq c_N \leq J_N(\bar{u}_N) \leq \sum_{k=-m}^{m} (k^2-a)|c_k|^2\pi + \frac{\pi}{2}\lambda^2(j^2-a) + \int_0^{2\pi} (\frac{\beta}{3}\bar{u}_N^3 - \frac{\gamma}{4}\bar{u}_N^4)$$

$$\leq \pi\lambda^2 + \frac{B}{3}\int_0^{2\pi}|\bar{u}_N|^3 - \frac{c}{4}\int_0^{2\pi}|\bar{u}_N|^4$$

$$\leq |\bar{u}_N|_2^2 + \frac{B}{3}|\bar{u}_N|_3^3 - \frac{c}{4}|\bar{u}_N|_4^4 .$$

Then

$$\frac{c}{4}|\bar{u}_N|_4^4 \leq |\bar{u}_N|_2^2 + \frac{B}{3}|\bar{u}_N|_3^3 ,$$

which implies the existence of a constant B_3 , independent of N , such that

$$|\bar{u}_N|_4 \leq B_3 . \tag{2.11}$$

Let $\bar{c}_N = J_N(\bar{u}_N)$. If we get a bound for $\bar{c}_N$, (2.9) is proved since $c_N \leq \bar{c}_N$. Using the fact that $\bar{u}_N$ is a critical point of J_N in $Y_N^- \oplus \text{span}\{y\}$ and Hölder's inequality, we get

$$c_N \leq \bar{c}_N = \bar{c}_N - \frac{1}{2}\langle J_N'(\bar{u}_N), \bar{u}_N\rangle$$

$$= \int_0^{2\pi} ((\frac{\beta}{3} - \frac{\beta}{2})\bar{u}_N^3 + (\frac{\gamma}{2} - \frac{\gamma}{4})\bar{u}_N^4)$$

$$\leq \frac{B}{6}|\bar{u}_N|_3^3 + \frac{c}{4}|\bar{u}_N|_4^4 \leq B_1$$

because of (2.11). Then (2.9) is proved. The fact that u_N is a critical point of J_N in Y_N and (2.9) imply

$$B_1 \geq c_N = J_N(u_N) - \frac{1}{2}\langle J_N'(u_N), u_N\rangle$$

$$= \int_0^{2\pi} (-\frac{\beta}{6}u_N^3 + \frac{\gamma}{4}u_N^4)$$

$$\geq -\frac{B}{6}|u_N|_3^3 + \frac{c}{4}|u_N|_4^4 .$$

Therefore

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$$\frac{c}{4} |u_N|_4^4 \leq B_1 + \frac{B}{6} |u_N|_3^3 ,$$

which implies (2.10).

Now using (2.10), a Galiardo-Nirenberg interpolation and the Sobolev imbedding theorem we are going to prove (2.7). Throughout this proof we denote by k several positive constants independent of N. By (2.3), (2.10) and Hölder's inequality we have

$$\begin{aligned} |\ddot{u}_N|_2 &\leq |\ddot{u}_N + \alpha u_N|_2 + |\alpha\, u_N|_2 \\ &\leq |P_N(\beta\, u_N^2 - \gamma\, u_N^3)|_2 + K \\ &\leq B|u_N^2|_2 + C|u_N^3|_2 + k \\ &\leq k|u_N|_\infty\, |u_N|_4^2 + k \\ &\leq k|u_N|_\infty + k \end{aligned}$$

then

$$|\ddot{u}_N|_2 \leq k + k\, |u_N|_\infty . \tag{2.12}$$

Write u_N in the form $u_N = a_N + \tilde{u}_N$ where $\int_0^{2\pi} \tilde{u}_N = 0$ and a_N is a constant. Since $\tilde{u}_N$ vanishes for some $t_o \in [0,2\pi]$ we can apply the Gagliardo-Nirenberg interpolation to $\tilde{u}_N$:

$$|\tilde{u}_N|_\infty \leq M|\tilde{u}_N|_4^{\frac{2}{3}}\, |\dot{\tilde{u}}_N|_2^{\frac{1}{3}} . \tag{2.13}$$

By (2.10) and Hölder's inequality

$$|a_N| \leq \frac{1}{2\pi}\, |u_N|_1 \leq k . \tag{2.14}$$

By (2.13), (2.14) and (2.10) we have

$$\begin{aligned} |u_N|_\infty &\leq k + |\tilde{u}_N|_\infty \\ &\leq k + M\, |\tilde{u}_N|_4^{\frac{2}{3}}\, |\dot{\tilde{u}}_N|_2^{\frac{1}{3}} \\ &\leq k + M\, (k+|u_N|_4)^{\frac{2}{3}}\, |\dot{u}_N|_2^{\frac{1}{3}} \\ &\leq k + k\, |\dot{u}_N|_2^{\frac{1}{3}} . \end{aligned} \tag{2.15}$$

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By (2.12) and (2.15) we get

$$|\ddot{u}_N|_2 \leqslant k + k\,|\dot{u}_N|_2^{\frac{1}{3}} . \tag{2.16}$$

Since $|\dot{u}_N|_2 \leqslant |\ddot{u}_N|_2$, from (2.16), (2.10) and Hölder's inequality we conclude that (2.7) holds.

As for (2.8), we observe that from the variational characterization of the eigenvalues and by (0.4) it follows that $\lambda = 0$ is not an eigenvalue of the linear problem

$$\begin{cases} \ddot{u} + \alpha(t)u = \lambda u , \\ u(o) - u(2\pi) = \dot{u}(o) - \dot{u}(2\pi) = 0 . \end{cases}$$

From this fact, it is easy to see, arguing by contradiction, that there is $k > 0$ such that

$$|\ddot{u} + \alpha u|_2 \geqslant k|u|_2$$

for every $u \in H^2$ that satisfies the periodic conditions (2.2). Then, using (2.3),

$$\begin{aligned} k|u_N|_2 &\leqslant |P_N(\beta\; u_N^2 - \gamma\; u_N^3)|_2 \\ &\leqslant (B\;|u_N|_\infty + C\;|u_N|_\infty^2)\;|u_N|_2 \end{aligned}$$

which implies (2.8), since $u_N \not\equiv 0$.

Step 3: Passing to the limit

This is analogous to step 3 of the proof of Theorem 0.1.

This ends the proof.

REMARK 2.1. As in the case of remark 1.1, the regularity of the solution thus obtained depends on the regularity of the coefficient functions α , β and γ .

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