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AN ACTION OF THE KLEIN FOUR-GROUP ON THE IRRATIONAL ROTATION C^*-ALGEBRA

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Explicit automorphisms of the irrational rotation algebra are constructed which are associated with the two 2×2 diagonal integer matrices of determinant -1. The fixed point algebra of the product of these two automorphisms is shown to be isomorphic to the fixed point algebra of the flip.

1. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ action

Let θ be an irrational number with $0 < \theta < 1$ and let the irrational rotation algebra A_θ be the universal C^*-algebra generated by two unitaries U, V satisfying $VU = e^{2\pi i\theta}UV$. This algebra has been the subject of extensive study in recent years. In particular it has been shown, as a consequence of the remarkable results in [**8**] and [**9**], that there exists a surjective homomorphism from $\operatorname{Aut}(A_\theta)$, the group of $*$-automorphisms of A_θ, onto $\operatorname{Aut}(K_1(A_\theta))$, which can be identified with $GL_2(\mathbb{Z})$ via the identification of $K_1(A_\theta)$ with $\mathbb{Z}^2$. It was shown in [**6**] and [**17**] how to construct a partial lifting of the map $\operatorname{Aut}(A_\theta) \to GL_2(\mathbb{Z})$ via an isomorphism from $SL_2(\mathbb{Z})$ into $\operatorname{Aut}(A_\theta)$. The purpose of the present paper is to describe explicit automorphisms of A_θ arising from the matrices $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and to investigate the associated automorphism arising from the matrix $-I$. The construction is given in terms of the inductive limit description of A_θ developed in [**9**, Section 5] which is now described.

Let the continued fraction expansion of θ be $[m_0, m_1, m_2, \ldots]$, with associated partial quotients $p_0/q_0, p_1/q_1, p_2/q_2, \ldots$, and let the natural numbers a_n, b_n, c_n, d_n be defined by

$$\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \begin{pmatrix} m_{4n} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} m_{4n-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} m_{4n-2} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} m_{4n-3} & 1 \\ 1 & 0 \end{pmatrix}.$$

Furthermore, for each $k \in \mathbb{N}$, let $k \times k$ matrices R_k and S_k with entries in $C(S^1)$ be defined by

$$R_k = \begin{pmatrix} 0 & \mathrm{id} \\ I_{k-1} & 0 \end{pmatrix} \quad \text{and} \quad S_k = \begin{pmatrix} 0 & 1 \\ I_{k-1} & 0 \end{pmatrix}.$$

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The construction given in [**9**] gives A_θ as an inductive limit of a sequence $A_1 \to A_2 \to A_3 \to \ldots$ where $A_n = M_{q_{4n}}(C(S^1)) \oplus M_{q_{4n-1}}(C(S^1))$ and where the connecting maps $\theta_n : A_n \to A_{n+1}$ are specified, for constant A, B, by

$$\text{(i)}\quad \theta_n(\mathrm{id}I_{q_{4n}}, 0) = \left(\begin{pmatrix} I_{q_{4n}} \otimes R_{a_{n+1}} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} I_{q_{4n}} \otimes S_{c_{n+1}} & 0 \\ 0 & 0 \end{pmatrix} \right)$$

$$\text{(ii)}\quad \theta_n(0, \mathrm{id}I_{q_{4n-1}}) = \left(\begin{pmatrix} 0 & 0 \\ 0 & I_{q_{4n-1}} \otimes S_{b_{n+1}} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & I_{q_{4n-1}} \otimes R_{d_{n+1}} \end{pmatrix} \right)$$

$$\text{(iii)}\quad \theta_n(A, B) = \left(\begin{pmatrix} A \otimes I_{a_{n+1}} & 0 \\ 0 & B \otimes I_{b_{n+1}} \end{pmatrix}, \begin{pmatrix} A \otimes I_{c_{n+1}} & 0 \\ 0 & B \otimes I_{d_{n+1}} \end{pmatrix} \right).$$

Here, for an $\ell \times \ell$ matrix M, $I_k \otimes M$ denotes the $k\ell \times k\ell$ matrix with k copies of M down the main diagonal and $M \otimes I_k$ denotes the $k\ell \times k\ell$ matrix consisting of $k \times k$ blocks $m_{ij}I_k$ in the obvious way. As the notation indicates, this description corresponds to an isomorphism between $M_{k\ell}$ and the tensor product $M_k \otimes M_\ell$.

The connecting maps θ_n were chosen to give the following prescribed effect on $K_0(A_n)$ and $K_1(A_n)$ when these are both identified with $\mathbb{Z}^2$: on $K_0(A_n)$, θ_n corresponds to the matrix $\begin{pmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix}$, whereas on $K_1(A_n)$ θ_{n*} corresponds to the identity. It will be important in the sequel to note that the identification of $K_1(A_n)$ with $\mathbb{Z}^2$ can be given by associating $(1,0)$ with $([u_{q_{4n}}], [I])$ and $(0,1)$ with $([I], [u_{q_{4n-1}}])$, where, for each $k \in \mathbb{N}$,

$$u_k = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & I_{k-1} \end{pmatrix}.$$

The inductive limit of the system $A_1 \to A_2 \to \ldots$ is A_θ because of the main results of [**8**] and [**9**], which show that it is sufficient to demonstrate that the inductive limit has the same K-theory as A_θ.

It may be of interest to note the following alternative way of describing the maps θ_n, which will however not be used in the sequel.

Proposition 1.1. *Let*

$$\theta_n : M_{q_{4n}}(C(S^1)) \oplus M_{q_{4n-1}}(C(S^1)) \to M_{q_{4n+4}}(C(S^1)) \oplus M_{q_{4n+3}}(C(S^1))$$

be defined by (i), (ii), (iii) above. Then $\theta_n((f_{ij}), (g_{ij}))$ is equal to

$$\left(\begin{pmatrix} (f_{ij}(R_{a_{n+1}})) & 0 \\ 0 & (g_{ij}(S_{b_{n+1}})) \end{pmatrix}, \begin{pmatrix} (f_{ij}(S_{c_{n+1}})) & 0 \\ 0 & (g_{ij}(R_{d_{n+1}})) \end{pmatrix} \right),$$

where, for each i, j, f_{ij} and g_{ij} are elements of $C(S^1)$ and $f_{ij}(R_{a_{n+1}}), f_{ij}(S_{c_{n+1}})$, $g_{ij}(S_{b_{n+1}})$, $g_{ij}(R_{d_{n+1}})$ use the usual functional calculus for a unitary element of a C^*-algebra.

Proof: It is easily seen that the formula given above gives the appropriate answers (i), (ii), (iii) and defines a $*$-homomorphism. ∎

The $\mathbb{Z}_2 \times \mathbb{Z}_2$ action on A_θ will be defined in terms of certain inductively defined matrices V_n, V'_n, W_n, W'_n. To define these, first let Q_k be the $k \times k$ matrix

$$Q_k = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$$

and let $W_1 = I_{q_4}, V_1 = Q_{q_3}, W'_1 = Q_{q_4}$ and $V'_1 = I_{q_3}$. Then, for each $n \geqslant 1$, let

$$W_{n+1} = \begin{pmatrix} W_n \otimes I_{a_{n+1}} & \\ & V_n \otimes Q_{b_{n+1}} \end{pmatrix}$$
$$V_{n+1} = \begin{pmatrix} W_n \otimes I_{c_{n+1}} & \\ & V_n \otimes Q_{d_{n+1}} \end{pmatrix}$$
$$W'_{n+1} = \begin{pmatrix} W'_n \otimes Q_{a_{n+1}} & \\ & V'_n \otimes I_{b_{n+1}} \end{pmatrix}$$
$$V'_{n+1} = \begin{pmatrix} W'_n \otimes Q_{c_{n+1}} & \\ & V'_n \otimes I_{d_{n+1}} \end{pmatrix}$$

and, for each $f \in C(S^1, M_k)$, where $k \in \mathbb{N}$, let $(Rf)(t) = f(\bar{t})$. It will be shown that an action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on A_θ is defined by the formulae

$$\alpha_n = (AdW_n, (AdV_n) \circ R)$$
$$\alpha'_n = ((AdW'_n) \circ R, AdV'_n)$$

where W_n, W'_n, V_n, V'_n are regarded as constant matrix valued functions on S^1.

Proposition 1.2.

(i) *α_n, α'_n are automorphisms of A_n.*

(ii) *$\alpha_n^2 = \mathrm{id}$, $\alpha_n'^2 = \mathrm{id}$ and $\alpha_n\alpha'_n = \alpha'_n\alpha_n$, so that α_n, α'_n define a $\mathbb{Z}_2 \times \mathbb{Z}_2$ action on A_n.*

(iii) *$\alpha_{n*} = (id, -\mathrm{id})$ and $\alpha'_{n*} = (-\mathrm{id}, \mathrm{id})$ on $\mathbb{Z}^2 \cong K_1(A_n)$.*

Proof:

(i) This is clear from the definitions.

(ii) It follows inductively that, for all $n \in \mathbb{N}$, $W_n^2 = I$, $W_n'^2 = I$, $V_n^2 = I$, $V_n'^2 = I$, $W_nW'_n = W'_nW_n$ and $V_nV'_n = V'_nV_n$, from which the result follows.

(iii) This follows from $R(u) = u^*$, where ([I], [u]) and ([u], [I]) are the generators of $K_1(A_n)$ described earlier.

∎

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Theorem 1.1. *There exists an isomorphism β from $\mathbb{Z}_2 \times \mathbb{Z}_2$ to $Aut(A_\theta)$ with $\beta(1,0)_* = (\mathrm{id}, -\mathrm{id})$ and $\beta(0,1)_* = (-\mathrm{id}, \mathrm{id})$ on $\mathbb{Z}^2 \cong K_1(A_\theta)$.*

Proof: The result will follow from Proposition 1.2 and the fact that $\theta_{n*} = \mathrm{id}$ on $K_1(A_\theta)$ if it can be shown that $\alpha_{n+1}\theta_n = \theta_n\alpha_n$ and $\alpha'_{n+1}\theta_n = \theta_n\alpha'_n$ for each $n \in \mathbb{N}$. However

$$\begin{aligned}
\alpha_{n+1}\theta_n(\mathrm{id}\, I_{q_{4n}}, 0) &= \left(\begin{pmatrix} W_nIW_n^* \otimes R_{a_{n+1}} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} W_nIW_n^* \otimes S_{c_{n+1}} & 0 \\ 0 & 0 \end{pmatrix}\right) \\
&= \theta_n(\mathrm{id}\, I_{q_{4n}}, 0) \\
&= \theta_n\alpha_n(\mathrm{id}\, I_{q_{4n}}, 0), \\
\alpha_{n+1}\theta_n\big(0, \mathrm{id}\, I_{q_{4n-1}}\big) &= \left(\begin{pmatrix} 0 & 0 \\ 0 & I \otimes Q_{b_{n+1}} S_{b_{n+1}} Q_{b_{n+1}} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & I \otimes Q_{d_{n+1}} \overline{R}_{d_{n+1}} Q_{d_{n+1}} \end{pmatrix}\right) \\
&= \left(\begin{pmatrix} 0 & 0 \\ 0 & I \otimes S^*_{b_{n+1}} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & I \otimes R^*_{d_{n+1}} \end{pmatrix}\right) \\
&= \theta_n\Big(0, \big(\mathrm{id}\, I_{q_{4n-1}}\big)^*\Big) \\
&= \theta_n\alpha_n\big(0, \mathrm{id}\, I_{q_{4n-1}}\big)
\end{aligned}$$

and

$$\begin{aligned}
\alpha_{n+1}\theta_n(A, B) &= \Bigg(\begin{pmatrix} W_nAW_n^* \otimes I_{a_{n+1}} & 0 \\ 0 & V_nBV_n^* \otimes I_{b_{n+1}} \end{pmatrix}, \\
&\qquad\qquad \begin{pmatrix} W_nAW_n^* \otimes I_{c_{n+1}} & 0 \\ 0 & V_nBV_n^* \otimes I_{d_{n+1}} \end{pmatrix}\Bigg) \\
&= \theta_n(W_nAW_n^*, V_nBV_n^*) \\
&= \theta_n\alpha_n(A, B).
\end{aligned}$$

Hence $\alpha_{n+1}\theta_n = \theta_n\alpha_n$. The proof for $\alpha'_{n+1}\theta_n = \theta_n\alpha'_n$ is exactly similar. ∎

2. A fixed point algebra

It is natural to ask if the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action and the $SL_2(\mathbb{Z})$ action on A_θ can be combined to give a $GL_2(\mathbb{Z})$ action. If this were to be possible then $\beta(1,1)$ would be conjugate to the flip automorphism, studied in [**3**, **4**, **5**], which is the automorphism of A_θ corresponding to $-I \in SL_2(\mathbb{Z})$ under the correspondence established in [**6**] and [**17**]. Consequently the fixed point algebra of $\beta(1,1)$ would be isomorphic to the fixed point algebra of the flip. We shall now show this to be the case, which leaves open the possibility of a $GL_2(\mathbb{Z})$ action on A_θ.

To facilitate the statement of the results let

$$E_n = (I_{q_{4n}} + W_nW'_n)/2 \quad \text{and} \quad F_n = \big(I_{q_{4n-1}} + V_nV'_n\big)/2.$$

Then E_n and F_n are the projections associated with the symmetries W_nW_n' and V_nV_n'. It will also be convenient to abbreviate $\alpha\alpha' = \beta(1,1)$ to β and to use β_n for its restriction to A_n.

Proposition 2.1.

(i) *The fixed point algebra B_n of β_n is $B_n^1 \oplus B_n^2$ where B_n^1 is isomorphic to*

$$\{f \in C([0,1], M_{q_{4n}}) : f(0), f(1) \in \{E_n\}'\}$$

and B_n^2 is isomorphic to

$$\{f \in C\big([0,1], M_{q_{4n-1}}\big) : f(0), f(1) \in \{F_n\}'\}.$$

(ii) $K_1(B_n) = \{0\}$.

(iii) *The ordered group $K_0(B_n)$ is isomorphic to $\mathbb{Z}^6$ with positive cone $\{(n_1, n_2, n_3, n_4, n_5, n_6) : n_i \geqslant 0$ for $1 \leqslant i \leqslant 6,\ n_1 + n_2 \geqslant n_3,\ n_4 + n_5 \geqslant n_6\}$.*

Proof: (i) Recall that $\beta_n = (Ad(W_nW_n') \circ R,\ Ad(V_nV_n') \circ R)$ from which it follows that $B_n^1 = \{f \in C\big(S^1, M_{q_{4n}}\big) : f(t) = W_nW_n'f(\bar{t})W_nW_n'$ for each $t \in S^1\}$. Each element of B_n^1 is determined by its restriction to $\{t \in S^1 : \operatorname{Im} t \geqslant 0\}$ and is therefore specified by the function $g : s \to f\big(e^{i\pi s}\big)$ on $[0,1]$ which satisfies $g(0), g(1) \in \{W_nW_n'\}' = \{E_n\}'$. The result for B_n^2 is exactly similar.

(ii) By [**10**, Lemma 2.1], $K_1\big(B_n^1\big)$ is isomorphic to $K_0(M_{q_{4n}})/\{i_*(a) - i_*(b) : a, b \in K_0(\{E_n\}')\}$, where i_* results from the inclusion map from $\{E_n\}'$ into $M_{q_{4n}}$. It is readily seen that $K_0(\{E_n\}')$ is isomorphic to $\mathbb{Z}^2$ with $i_* : \mathbb{Z}^2 \to \mathbb{Z}$ given by $i_*(n_1, n_2) = n_1 + n_2$. Hence $K_1\big(B_n^1\big)$ is isomorphic to $\mathbb{Z}/\mathbb{Z}$. Similarly $K_1\big(B_n^2\big)$ is also zero, as therefore is $K_1(B_n)$.

(iii) By [**10**, Lemma 2.1], $K_0\big(B_n^1\big)$ is isomorphic to $\{(a,b) : a, b \in K_0(\{E_n\}'),\ i_*(a) = i_*(b)\}$ which is isomorphic to $\{(n_1, n_2, m_1, m_2) : n_1 + n_2 = m_1 + m_2\}$, with positive cone given by the usual positive cone on $\mathbb{Z}^4$. Hence, restricting to the first three coordinates, $K_0\big(B_n^1\big)$ is isomorphic to $\mathbb{Z}^3$ with positive cone $\{(n_1, n_2, n_3) : n_1, n_2, n_3 \geqslant 0,\ n_3 \leqslant n_1 + n_2\}$. The result for $K_0\big(B_n^2\big)$ is exactly similar, from which the result for $K_0(B_n)$ follows. ∎

The next step is to determine the 6×6 matrices corresponding to the maps $\theta_{n*} : K_0(B_n) \to K_0(B_{n+1})$ and to check that they are non-singular. In order to do this, an explicit expression for E_n and F_n will be obtained. The expression will involve matrices of the form $(I_k + Q_k)/2$, where Q_k was defined earlier: it will be convenient to denote such a matrix $(I_k + Q_k)/2$ by P_k.

Lemma 2.1. *For each $n \in \mathbb{N}$,*

(i) $W_n W_n' = \operatorname{diag}\Big(Q_{k_n(1)}, Q_{k_n(2)}, \dots, Q_{k_n(2^{n-1})}\Big)$

(ii) $V_n V_n' = \operatorname{diag}\Big(Q_{\ell_n(1)}, Q_{\ell_n(2)}, \dots, Q_{\ell_n(2^{n-1})}\Big)$

(iii) $E_n = \operatorname{diag}\Big(P_{k_n(1)}, P_{k_n(2)}, \dots, P_{k_n(2^{n-1})}\Big)$

(iv) $F_n = \operatorname{diag}\Big(P_{\ell_n(1)}, P_{\ell_n(2)}, \dots, P_{\ell_n(2^{n-1})}\Big)$,

where the integers $k_n(i)$ and $\ell_n(i)$ are defined inductively by $k_1(1) = q_4, \ell_1(1) = q_3$ and, for each $n \in \mathbb{N}$ and each $1 \leqslant i \leqslant 2^{n-1}$, $k_{n+1}(i) = a_{n+1}k_n(i)$, $k_{n+1}(2^{n-1}+i) = b_{n+1}\ell_n(i)$, $\ell_{n+1}(i) = c_{n+1}k_n(i)$ and $\ell_{n+1}(2^{n-1}+i) = d_{n+1}\ell_n(i)$.

Proof: These are all straightforward consequences of the definitions, using the observation that, for any matrix (x_{ij}), the matrix $(x_{ij}) \otimes Q_k$ consists of blocks $x_{ij}Q_k$. ▯

Lemma 2.1 can now be used to compute the images of the generators of $\mathbb{Z}^6 \cong K_0(B_n)$ under θ_{n*}.

Lemma 2,2. *Let $K_0(B_n)$ and $K_0(B_{n+1})$ be identified with $\mathbb{Z}^6$ as in Proposition 2.1 and let $\alpha_{n+1} = [a_{n+1}/2]$, $\beta_{n+1} = [b_{n+1}/2]$, $\gamma_{n+1} = [c_{n+1}/2]$ and $\delta_{n+1} = [d_{n+1}/2]$, where $[x]$ denotes the integral part of x. Then*

$$\theta_{\theta*}(1,0,1,0,0,0) = (a_{n+1} - \alpha_{n+1}, \alpha_{n+1}, a_{n+1} - \alpha_{n+1}, c_{n+1} - \gamma_{n+1}, \gamma_{n+1}, c_{n+1} - \gamma_{n+1})$$
$$\theta_{n*}(0,1,0,0,0,0) = (\alpha_{n+1}, a_{n+1} - \alpha_{n+1}, \alpha_{n+1}, \gamma_{n+1}, c_{n+1} - \gamma_{n+1}, \gamma_{n+1})$$
$$\theta_{n*}(0,0,0,1,0,1) = (b_{n+1} - \beta_{n+1}, \beta_{n+1}, b_{n+1} - \beta_{n+1}, d_{n+1} - \delta_{n+1}, \delta_{n+1}, d_{n+1} - \delta_{n+1})$$
$$\theta_{n*}(0,0,0,0,1,0) = (\beta_{n+1}, b_{n+1} - \beta_{n+1}, \beta_{n+1}, \delta_{n+1}, d_{n+1} - \delta_{n+1}, \delta_{n+1}).$$

Proof: Define $e_{k_n(1)} \leqslant P_{k_n(1)}$ to be the $k_n(1) \times k_n(1)$ matrix whose sole non-zero entries are $1/2$ in each of the four corners and let $e = \operatorname{diag}\Big(e_{k_n(1)}, 0_{k_n(2)}, \dots, 0_{k_n(2^{n-1})}\Big)$, so that e is a minimal projection in $E_n M_{q_{4n}} E_n$ (regarded also as a constant element of B_n^1). Then

$$\theta_n(e,0) = \left(\begin{pmatrix} e \otimes I_{a_{n+1}} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} e \otimes I_{c_{n+1}} & 0 \\ 0 & 0 \end{pmatrix}\right).$$

Furthermore $P_{k_{n+1}(1)}\big(e \otimes I_{a_{n+1}}\big) = \big(I_{k_n(1)} \otimes I_{a_{n+1}} + Q_{k_n(1)} \otimes Q_{a_{n+1}}\big)/2\ \big(e \otimes I_{a_{n+1}}\big) = e \otimes P_{a_{n+1}} = \big(e \otimes I_{a_{n+1}}\big)P_{k_{n+1}(1)}$ and, similarly, $P_{\ell_{n+1}(1)}\big(e \otimes I_{c_{n+1}}\big) = e \otimes P_{c_{n+1}} = \big(e \otimes I_{c_{n+1}}\big)P_{\ell_{n+1}(1)}$. Hence, using Lemma 2.1,

$$\begin{aligned}(E_{n+1}, 0)\theta_n(e,0) &= \big(\operatorname{diag}\ (e \otimes P_{a_{n+1}}, 0), 0\big) \\ &= \theta_n(e,0)(E_{n+1}, 0)\end{aligned}$$

and

$$\begin{aligned}(0, F_{n+1})\theta_n(e,0) &= \big(0, \operatorname{diag}(e \otimes P_{c_{n+1}}, 0)\big) \\ &= \theta_n(e,0)(0, F_{n+1}).\end{aligned}$$

It follows that $\theta_{n*}(1,0,1,0,0,0) = (\tau_1, \tau_2, \tau_1, \sigma_1, \sigma_2, \sigma_1)$ where $\tau_1 = Tr\big(e \otimes P_{a_{n+1}}\big)$, $\tau_2 = Tr\big(e \otimes I_{a_{n+1}}\big) - \tau_1 = a_{n+1} - \tau_1$, $\sigma_1 = Tr\big(e \otimes P_{c_{n+1}}\big)$ and $\sigma_2 = Tr\big(e \otimes I_{c_{n+1}}\big) - \sigma_1 = c_{n+1} - \sigma_1$. However $Tr\big(e \otimes P_{a_{n+1}}\big) = Tr\big(P_{a_{n+1}}\big) = [(a_{n+1}+1)/2] = a_{n+1} - [a_{n+1}/2]$ and, similarly, $Tr\big(e \otimes P_{c_{n+1}}\big) = c_{n+1} - [c_{n+1}/2]$. Hence $\theta_{n*}(1,0,1,0,0,0) = (a_{n+1} - \alpha_{n+1}, \alpha_{n+1}, a_{n+1} - \alpha_{n+1}, c_{n+1} - \gamma_{n+1}, \gamma_{n+1}, c_{n+1} - \gamma_{n+1})$. The image of $(0,0,0,1,0,1)$ is computed in exactly the same way using $\theta_n(0,e)$ where e is a minimal projection in $F_n M_{q_{4n-1}} F_n$ obtained from $e_{\ell_n(1)} \leqslant P_{\ell_n(1)}$. The other two images are then computed in a similar manner but based on minimal projections

$$e_{k_n(1)} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \leqslant 1 - P_{k_n(1)}$$

and

$$e_{\ell_n(1)} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \leqslant 1 - P_{\ell_n(1)}.$$

∎

In order to find the images of other generators, such as $(1,0,0,0,0,0)$ and $(0,0,0,1,0,0)$, it is necessary to consider non-constant projections and to find their image under θ_n. This is done in the following lemma.

LEMMA 2.3. *Maintaining the notation of Lemma 2.2,*

$$\begin{aligned}\theta_{n*}&(1,0,0,0,0,0)\\ &= (\alpha_{n+1}+1, a_{n+1} - \alpha_{n+1} - 1, \alpha_{n+1}, \gamma_{n+1}+1, c_{n+1} - \gamma_{n+1} - 1, \gamma_{n+1}+1)\end{aligned}$$

and

$$\begin{aligned}\theta_{n*}&(0,0,0,1,0,0)\\ &= (\beta_{n+1}+1, b_{n+1} - \beta_{n+1} - 1, \beta_{n+1}+1, \delta_{n+1}+1, d_{n+1} - \delta_{n+1} - 1, \delta_{n+1}).\end{aligned}$$

PROOF: As in the proof of Lemma 2.2, construct a minimal projection e with $e(0)$ in $E_n M_{q_{4n}} E_n$ and $e(1)$ in $(1-E_n) M_{q_{4n}} (1-E_n)$, starting with a minimal $k_n(1) \times k_n(1)$ projection $e_{k_n(1)}$ defined by

$$e_{k_n(1)}(s) = \frac{1}{2}\begin{pmatrix} 1 & e^{\pi i s} \\ e^{-\pi i s} & 1 \end{pmatrix}$$

which satisfies $e_{k_n(1)}(0) \leqslant P_{k_n(1)}$ and $e_{k_n(1)}(1) \leqslant 1 - P_{k_n(1)}$. Then, using Lemma 2.1, $e(0) \leqslant E_n$ and $e(1) \leqslant 1 - E_n$, so $[e] \in K_0(B_n^1)$ corresponds to $(1,0,0) \in \mathbb{Z}^3$. In order to calculate $\theta_n(e,0)$ it is convenient to regard e as an element of $C(S^1, M_{q_{4n}})$ by identifying $e_{k_n(1)}$ above with the matrix valued function defined on S^1 by

$$e_{k_n(1)}(t) = \frac{1}{2}\begin{pmatrix} 1 & t \\ \bar{t} & 1 \end{pmatrix}.$$

Note that $Q_{k_n(1)} e_{k_n(1)}(t) Q_{k_n(1)} = e_{k_n(1)}(\bar{t})$ so, using Lemma 2.1, $W_n W_n' e(t) W_n W_n' = e(\bar{t})$, as required for e to belong to the fixed point algebra B_n^1 of β_n restricted to A_n^1.

The image of $\theta_n(e,0)$ can be found by writing $e_{k_n(1)}$ as

$$e_{k_n(1)} = \frac{1}{2}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \frac{1}{2}\,\mathrm{id}\begin{pmatrix} & 1 \\ & \end{pmatrix} + \frac{1}{2}\,\mathrm{id}^*\begin{pmatrix} & \\ 1 & \end{pmatrix}.$$

from which it follows that $\theta_n(e,0) = (\mathrm{diag}(f,0), \mathrm{diag}(g,0))$ where f is the $k_{n+1}(1) \times k_{n+1}(1)$ matrix and g is the $\ell_{n+1}(1) \times \ell_{n+1}(1)$ matrix given by

$$\begin{aligned} f &= \frac{1}{2}\begin{pmatrix} I_{a_{n+1}} & \\ & I_{a_{n+1}} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} & 1 \\ & \end{pmatrix} \otimes R_{a_{n+1}} + \frac{1}{2}\begin{pmatrix} & \\ 1 & \end{pmatrix} \otimes R_{a_{n+1}}^* \\ &= \frac{1}{2}\begin{pmatrix} I_{a_{n+1}} & R_{a_{n+1}} \\ R_{a_{n+1}}^* & I_{a_{n+1}} \end{pmatrix} \end{aligned}$$

and, similarly,

$$g = \frac{1}{2}\begin{pmatrix} I_{c_{n+1}} & S_{c_{n+1}} \\ S_{c_{n+1}}^* & I_{c_{n+1}} \end{pmatrix}.$$

To calculate $(E_{n+1}, 0)\ \theta_n(e,0)(1)$ observe that

$$\begin{aligned} P_{k_{n+1}(1)} f(1) &= \frac{1}{4}\begin{pmatrix} I_{a_{n+1}} & Q_{a_{n+1}} \\ Q_{a_{n+1}} & I_{a_{n+1}} \end{pmatrix}\begin{pmatrix} I_{a_{n+1}} & R_{a_{n+1}}(1) \\ R_{a_{n+1}}^*(1) & I_{a_{n+1}} \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} I_{a_{n+1}} + Q_{a_{n+1}} R_{a_{n+1}}^*(1) & R_{a_{n+1}}(1) + Q_{a_{n+1}} \\ R_{a_{n+1}}^*(1) + Q_{a_{n+1}} & I_{a_{n+1}} + Q_{a_{n+1}} R_{a_{n+1}}(1) \end{pmatrix} \end{aligned}$$

From $Q_{a_{n+1}} R_{a_{n+1}}^*(t) = R_{a_{n+1}}(\bar{t}) Q_{a_{n+1}}$ it follows that $P_{k_{n+1}(1)} f(1) = f(1) P_{k_{n+1}(1)}$ and therefore that $(E_{n+1}, 0)\theta_n(e,0)(1) = \theta_n(e,0)(1)(E_{n+1}, 0)$. The trace is given by $\mathrm{Tr}(E_{n+1}\mathrm{diag}(f(1),0)) = \mathrm{Tr}(P_{k_{n+1}(1)} f(1)) = [a_{n+1}/2] + 1$ (as can be seen by a separate analysis of the cases a_{n+1} odd and a_{n+1} even). Similarly $(E_{n+1}, 0)\theta_n(e,0)(-1) =$

$\theta_n(e,0)(-1)(E_{n+1},0)$ with $\mathrm{Tr}(E_{n+1}\mathrm{diag}(f(-1),0)) = \mathrm{Tr}\big(P_{k_{n+1}(1)}f(-1)\big) = [a_{n+1}/2]$. Thus the first three components of $\theta_{n*}(1,0,0,0,0,0)$ are $\alpha_{n+1}+1, a_{n+1}-\alpha_{n+1}-1, \alpha_{n+1}$. The second three components are calculated similarly considering $(0,F_{n+1})\theta_n(e,0)$ via $P_{\ell_{n+1}(1)}g$ where g is constant. The formula for $\theta_{n*}(0,0,0,1,0,0)$ then follows by an exactly similar argument. $\square$

Proposition 2.2. *Let A_n be the 6×6 matrix corresponding to θ_{n*} : $K_0(B_n)\to K_0(B_{n+1})$. Then $\det(A_n)=\pm 1$.*

Proof: To ease the complexity of the notation omit subscripts so, by Lemmas 2.2 and 2.3,

$$A_n = \begin{pmatrix} \alpha+1 & \alpha & a-2\alpha-1 & \beta+1 & \beta & b-2\beta-1 \\ a-\alpha-1 & a-\alpha & 2\alpha+1-a & b-\beta-1 & b-\beta & 2\beta+1-b \\ \alpha & \alpha & a-2\alpha & \beta+1 & \beta & b-2\beta-1 \\ \gamma+1 & \gamma & c-2\gamma-1 & \delta+1 & \delta & d-2\delta-1 \\ c-\gamma-1 & c-\gamma & 2\gamma+1-c & d-\delta-1 & d-\delta & 2\delta+1-d \\ \gamma+1 & \gamma & c-2\gamma-1 & \delta & \delta & d-2\delta \end{pmatrix}.$$

Denoting the ith column by C_i and the ith row by R_i and performing the following sequence of row and column operations:

$$C_1' = C_1 - C_2,\ C_4' = C_4 - C_5,\ C_3' = C_3 + C_2,\ C_6' = C_6 + C_5,$$
$$R_2' = R_2 + R_1,\ R_5' = R_4 + R_5,\ R_1' = R_1 - R_3,\ R_4' = R_4 - R_6$$
$$C_3' = C_3 + C_2 + C_1,\ C_6' = C_6 + C_5 + C_4$$

yields

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 2a & 0 & b & 2b \\ 0 & \alpha & a & 1 & \beta & b \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & c & 2c & 0 & d & 2d \\ 1 & \gamma & c & 0 & \delta & d \end{pmatrix}.$$

Using the fact that $ad-bc=1$, the determinant of this matrix reduces to $(a-2\alpha)$ $(d-2\delta)+(c-2\gamma)(2\beta-b)$. Thus when either a or d is even (so that both b and c are odd) $\det(A_n)=0-1=-1$ and when either b or c is even (so that both a and d are odd) $\det(A_n)=1-0=1$. Since $ad-bc=1$ these are the only possible cases. $\square$

Proposition 2.3. *Let B be the fixed point algebra of A_θ under the automorphism β. Then $K_0(B)$ is isomorphic to $\mathbb{Z}^6$ and $K_1(B)$ is isomorphic to $\{0\}$.*

Proof: B is the inductive limit of the sequence $B_1 \xrightarrow{\theta_1} B_2 \xrightarrow{\theta_2} \dots$, so the result follows immediately from Propositions 2.1 and 2.2. $\square$

Proposition 2.3 is the basis for showing that B is isomorphic to the fixed point algebra B_θ of the flip. Another key ingredient is the observation that B is simple with a unique tracial state.

PROPOSITION 2.4. *Let B be the fixed point subalgebra of A_θ under the automorphism β. Then*

(a) *B is simple;*
(b) *B has a unique tracial state (obtained by restricting that on A_θ).*

PROOF: (a) The action of β on $K_1(A_\theta)$ is given by $-I$ whereas any unitarily implemented automorphism acts trivially on $K_1(A_\theta)$. Hence by a standard result, described in [**12,** Theorem 8.10.12], the fixed point algebra B of β is simple.

(b) It is shown in [**3,** Remark 4.7] that it is sufficient to show that the crossed product $M_\theta \times_{\overline{\beta}} \mathbb{Z}_2$ of the weak closure M_θ of A_θ in the trace representation by the unique extension $\overline{\beta}$ of β to M_θ is a factor. Equivalently, it is sufficient to show that the weak closure of B in the trace representation is a factor, that is, that the unique trace on A_θ restricts to a factor state on B. Thus the tracial state space of B will now be investigated.

The commutative diagram

$$\begin{array}{ccccc} B_1 & \longrightarrow & B_2 & \longrightarrow & \dots B \\ \downarrow & & \downarrow & & \downarrow \\ A_1 & \longrightarrow & A_2 & \longrightarrow & \dots A_\theta \end{array}$$

where the vertical maps are inclusions, gives rise to a commutative diagram of tracial state spaces

$$\begin{array}{ccccc} T(B_1) & \longleftarrow & T(B_2) & \longleftarrow & \dots T(B) \\ \psi_1 \uparrow & & \psi_2 \uparrow & & \uparrow \psi_\infty \\ T(A_1) & \longleftarrow & T(A_2) & \longleftarrow & \dots T(A_\theta) \end{array}$$

Recall that $A_n = M_{q_{4n}}(C(S^1)) \oplus M_{q_{4n-1}}(C(S^1))$ from which it follows that $T(A_n)$ has a direct sum decomposition $T^1_{A_n} \oplus T^2_{A_n}$. For each probability measure μ on S^1 there exists a trace τ^q_μ on $M_q(C(S^1))$ defined by $\tau^q_\mu(f) = \int \mathrm{Tr}(f) d\mu$, where Tr is the normalised trace on M_q, and $T^i_{A_n} = \{\tau^q_\mu : \mu \in M^+_1(S^1)\}$ for the appropriate choice of q that is, q_{4n} for $T^1_{A_n}$ or q_{4n-1} for $T^2_{A_n}$. Similarly $T(B_n)$ has a direct sum decomposition $T^1_{B_n} \oplus T^2_{B_n}$, where $T^1_{B_n}$ is the tracial state space of $\{f \in C([0,1], M_q) : f(0), f(1) \in \{E_n\}'\}$. Using $M^+_1(0,1)$ to denote the set of probability measures μ on $[0,1]$ for which $\mu(\{0\}) = \mu(\{1\}) = 0$ and τ_μ to denote the associated trace in $T^1_{B_n}$, the trace space $T^1_{B_n}$ is the convex hull of the face $F^1_n = \{\tau_\mu : \mu \in M^+_1(0,1)\}$ and four other traces

$\tau_{0,E}, \tau_{0,1-E}, \tau_{1,E}, \tau_{1,1-E}$. The first of these is defined by $\tau_{0,E}(f) = \mathrm{Tr}(E_n f(0))$, where Tr is the normalised trace on $E_n M_{q_{4n}} E_n$, and the others are defined similarly.

Let K_n^1 be $\{\tau_\mu : \mu \in M_1^+[0,1]\}$, which is equal to the convex hull of F_n^1, $\lambda\tau_{0,E} + (1-\lambda)\tau_{0,1-E}$ and $\lambda\tau_{1,E} + (1-\lambda)\tau_{1,1-E}$, where $\lambda = \dim(E_n)/(q_{4n} - \dim(E_n))$, and let K_n^2 be defined similarly. The image of $T(A_n)$ under the mapping ψ_n is $K_n^1 \oplus K_n^2$ and it follows from the uniqueness of the trace on A_θ that there is therefore a unique element in the limit K of the system

$$K_1^1 \oplus K_1^2 \leftarrow K_2^1 \oplus K_2^2 \leftarrow \dots .$$

This unique element τ_λ corresponds to the restriction to B of the unique trace on A_θ. It belongs to the face F of $T(B)$ determined by the system of faces

$$F_1^1 \oplus F_1^2 \leftarrow F_2^1 \oplus F_2^2 \leftarrow \dots .$$

and therefore this face F, which is contained in K, contains a unique element. Hence τ_λ is an extreme point of $T(B)$ and so, by [**7**, 6.8.6 and 6.8.7], is a factor trace, as required. □

PROPOSITION 2.5. *B is a simple AF algebra.*

PROOF: By Proposition 2.4 B is a simple unital C^*-algeba with a unique tracial state and, by construction, B is an inductive limit of the algebras $B_n = B_n^1 \oplus B_n^2$. Hence (as observed in [**16**, Theorem 3.0]) B is covered by the classification results of Su, obtained in [**13**] and [**14**], from which the result follows. □

Propositions 2.4 and 2.5 can now be used to determine the order structure on $K_0(B)$ and hence to compare it with the known order structure on $K_0(B_\theta)$.

PROPOSITION 2.6. *There is an order isomorphism from the group $K_0(B)$ onto $K_0(B_\theta)$.*

PROOF: The ordered group $K_0(B_\theta)$ is isomorphic to the ordered group $K_0(C_\theta)$, where C_θ is the crossed product of A_θ by the flip. However $K_0(C_\theta)$ has been computed by Kumjian [**11**] to be equal to $\mathbb{Z}^6$. Its order structure has been investigated in [**15**] where it is shown (in the course of the proof of [**15**, Theorem 4.1]) that there are generators for $\mathbb{Z}^6$ such that the coordinate 6-tuple $(n_1, n_2, n_3, n_4, n_5, n_6)$ is positive if and only if it is zero or $2n_1 + n_2 + n_3 + n_4 + n_5 + \theta n_6 > 0$. Let A be the invertible 6×6 matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & -1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and note that if $Ax = y$ then $2y_1 + y_2 + y_3 + y_4 + y_5 + \theta y_6 = \theta x_1 + \theta x_2 + x_4 + x_5$, so that $K_0(B_\theta)$ is isomorphic to $\mathbb{Z}^6$ with positive cone consisting of 0 and $\{(n_1, n_2, n_3, n_4, n_5, n_6) : \theta n_1 + \theta n_2 + n_4 + n_5 > 0\}$. However, using [**2**, Propositions 2.4 and 2.5 and Theorem 3.1], the ordering on $K_0(B)$ is the strict ordering determined by the unique trace τ.

Let $p_n : K_0(B_n) \to K_0(A_n)$ be the map arising from the inclusion $B_n \to A_n$ and let p_∞ be the corresponding map from $K_0(B)$ to $K_0(A)$. Then $\tau_*(n_1, \ldots, n_6) = \tau_* p_\infty(n_1, \ldots, n_6)$, where τ is used both for the unique trace on A_θ and its restriction to B. However it is known that $\tau_*(m_1, m_2) = \theta m_1 + m_2$ for $(m_1, m_2) \in \mathbb{Z}^2 \cong K_0(A_\theta)$ and so the proof will be complete if it is shown that $p_\infty(n_1, n_2, n_3, n_4, n_5, n_6) = (n_1 + n_2, n_4 + n_5)$. It is clear from a consideration of the trace of constant projections that, for each $n \in \mathbb{N}$, $p_n(n_1, n_2, n_3, n_4, n_5, n_6) = (n_1 + n_2, n_4 + n_5)$. So consider the following commutative diagram

$$\begin{tikzcd}
\mathbb{Z}^6 \arrow[r, "\mathrm{id}"] & \mathbb{Z}^6 \arrow[r, "\mathrm{id}"] & \cdots \\
\mathbb{Z}^6 \arrow[u, "\theta_{0*}^{-1}"] \arrow[r, "\theta_{1*}"] \arrow[d, "P_1"'] & \mathbb{Z}^6 \arrow[u, "\theta_{0*}^{-1}\theta_{1*}^{-1}"'] \arrow[r, "\theta_{2*}"] \arrow[d, "P_2"'] & \cdots \\
\mathbb{Z}^2 \arrow[r, "\theta_{1*}"] \arrow[d, "\theta_{0*}^{-1}"'] & \mathbb{Z}^2 \arrow[r, "\theta_{2*}"] \arrow[d, "\theta_{0*}^{-1}\theta_{1*}^{-1}"] & \cdots \\
\mathbb{Z}^2 \arrow[r, "\mathrm{id}"] & \mathbb{Z}^2 \arrow[r, "\mathrm{id}"] & \cdots
\end{tikzcd}$$

where θ_{0*} is defined on $\mathbb{Z}^6$ by a matrix A_0 as in Proposition 2.5 with $a_0 = q_4$, $b_0 = p_4$, $c_0 = q_3$, $d_0 = p_3$ and is defined on $\mathbb{Z}^2$ by the 2×2 matrix with entries a_0, b_0, c_0, d_0. It follows that $p_\infty(n_1, n_2, n_3, n_4, n_5, n_6) = \theta_{0*}^{-1} p_1 \theta_{0*}(n_1, n_2, n_3, n_4, n_5, n_6) = (n_1 + n_2, n_4 + n_5)$, as required. ∎

The major result of this section can now be obtained:

THEOREM 2.1. *Let β be the isomorphim from $\mathbb{Z}_2 \times \mathbb{Z}_2$ to $Aut(A_\theta)$ described in Theorem 1.1. Then the fixed point algebra B of $\beta(1,1)$ is isomorphic to the fixed point algebra B_θ of the flip determined by $U \mapsto U^*$ and $V \mapsto V^*$.*

PROOF: By [5] B_θ is an AF algebra so, by Proposition 2.5 and the classification theorem for AF algebras, it suffices to show that the order isomorphism of Proposition 2.6 preserves the scales. From the commutative diagram in the proof of Proposition 2.6, the order unit of $K_0(B)$ corresponding to the identity of B is $\theta_{0*}^{-1}(d_4, q_4 - d_4, d_4, d_3, q_3 - d_3, d_3) = (0, 0, 0, 0, 1, 0)$ where $d_4 = [q_4/2]$ and $d_3 = [q_3/2]$. If A is the 6×6 matrix used in the proof of Proposition 2.6 then $A^{-1}(0, 0, 0, 0, 1, 0)^{tr} =$

$(0,0,0,0,1,0)^{tr}$ so, by [**1**, Theorem 7.4.3], it suffices to show that $\tau_*^B(0,0,0,0,1,0) = 1$, where τ^B is the normalised trace on B_θ. However it is shown in the course of the proof of [**15**, Theorem 4.1] that $\tau_*^C(0,0,0,0,1,0) = 1/2$, where τ^C is the normalised trace on the crossed product $C_\theta = A_\theta \times \mathbb{Z}_2$ of A_θ by the flip. However when B_θ is embedded in C_θ in the usual way, described for example in [**4**, Section 4], $\tau^B = 2\tau^C$. Hence $\tau_*^B(0,0,0,0,1,0) = 1$, as required. ∎

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