TRANSCENDENTAL ELEMENTS IN CONTINUOUS RINGS

In memory of MAURICE AUDIN

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In **(2)**, John von Neumann introduced the concept of a continuous ring $\mathfrak{R}$ as a generalization to the infinite limiting case of the total matric algebras over a division ring. Von Neumann sketched a theory of arithmetic for such continuous rings $\mathfrak{R}$ and asserted:

($\bigstar$) *every continuous ring* $\mathfrak{R}$ *contains purely transcendental elements* c.

This means: *for every polynomial* $p(t) = t^m + z_1 t^{m-1} + \ldots + z_m$ $(m \geqslant 1)$ *which has all coefficients* z_i *in the centre of* $\mathfrak{R}$, *the element* $p(c)$ *has a reciprocal in* $\mathfrak{R}$, *that is,* $(p(c))^{-1}$ *exists such that* $p(c) \cdot (p(c))^{-1} = (p(c))^{-1} \cdot p(c) = 1$.

A manuscript found in von Neumann's files after his death (see **(1)**) gives detailed proofs for all statements in **(2)** with one exception: no indication of proof is given for ($\bigstar$). In the present note we give a proof of ($\bigstar$).

Continuous rings were characterized by von Neumann in **(2)** as those irreducible associative regular rings which possess a unity element and are complete, continuous rank rings.

We recall some definitions of von Neumann **(**see **3**, Part II, chapters XVII, XVIII**)**: an associative ring $\mathfrak{R}$ is called *regular* if for each a in $\mathfrak{R}$, the equation $axa = a$ has at least one solution x in $\mathfrak{R}$. A regular ring $\mathfrak{R}$ is called a *rank* ring if a real-valued function $R(a)$ is defined for all a in $\mathfrak{R}$ with the properties:

Always $0 \leqslant R(a) \leqslant 1$.
$R(a) = 0$ *if and only if* $a = 0$.
$R(1) = 1$.
$R(ab) \leqslant R(a)$, $R(ab) \leqslant R(b)$.
For $e^2 = e, f^2 = f, ef = fe = 0$, *always* $R(e+f) = R(e) + R(f)$.

Then necessarily $R(a) = 1$ if and only if a^{-1} exists, $R(a+b) \leqslant R(a) + R(b)$, and the function $d(a, b) \equiv R(a-b)$ determines a metric on $\mathfrak{R}$; $\mathfrak{R}$ is called a *complete* rank ring if $\mathfrak{R}$ is a complete metric space under the metric $R(a-b)$.

If $\mathfrak{R}$ is a complete rank ring which is irreducible† then the values of the rank function are precisely, $0, 1/n, 2/n, \ldots, n/n$, for some positive integer n or precisely all real numbers $\geqslant 0, \leqslant 1$. In the latter case $\mathfrak{R}$ is called *continuous*.

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†An associative ring $\mathfrak{R}$ with unity element is called *irreducible* if whenever two subrings $\mathfrak{R}_1, \mathfrak{R}_2$ are such that: (i) each x in $\mathfrak{R}$ is expressible in the form $x_1 + x_2$ with x_1 in $\mathfrak{R}_1$ and x_2 in $\mathfrak{R}_2$ and (ii) $x_1x_2 = x_2x_1 = 0$ for all x_1 in $\mathfrak{R}_1$, x_2 in $\mathfrak{R}_2$, then one of $\mathfrak{R}_1, \mathfrak{R}_2$ must consist of 0 only.

As von Neumann showed (**3**, Part II, Theorem 18.1) if $\mathfrak{R}$ is a continuous ring then the principal right ideals $(a)_r$, and also the principal left ideals $(a)_l$, form continuous geometries $\bar{R}_{\mathfrak{R}}$, $\bar{L}_{\mathfrak{R}}$ with dimension functions D, D' respectively and:

$$D((a)_r) = D'((a)_l) = R(a).$$

We shall make use of the following facts:

(i) in a continuous geometry, for each element $\mathfrak{r} \neq 0$, there is a decomposition $\mathfrak{r} = \mathfrak{r}_1 \cup \mathfrak{r}_2$ *with* $\mathfrak{r}_1, \mathfrak{r}_2$ perspective and $\mathfrak{r}_1 \cap \mathfrak{r}_2 = 0$ (see **3**, Part I, chapters VI, VII);

(ii) if the continuous geometry is $\bar{R}_{\mathfrak{R}}$ with $\mathfrak{R}$ a regular ring and $\mathfrak{r} = (e)_r$ with e idempotent then for the $\mathfrak{r}_1, \mathfrak{r}_2$ in (i) it is possible to find idempotents e_1, e_2 so that $e = e_1 + e_2$, $\mathfrak{r}_1 = (e_1)_r$, $\mathfrak{r}_2 = (e_2)_r$ and e_1, e_2 are *orthogonal*, that is, $e_1 e_2 = e_2 e_1 = 0$ (see **3**, Part II, chapter III);

(iii) in any regular ring $\mathfrak{R}$ if $(e_1)_r$ and $(e_2)_r$ are perspective and e_1, e_2 are orthogonal there exist elements s_{12} and s_{21} such that $s_{12}s_{21} = e_1$, $s_{21}s_{12} = e_2$, $s_{12} = e_1 s_{12} s_2$, $s_{21} = e_2 s_{21} e_1$ (see **3**, Part II, chapter III);

(iv) in any continuous geometry with dimension function D, if $\mathfrak{r}_1, \ldots, \mathfrak{r}_m$ are independent then $D(\cup_{i=1}^m \mathfrak{r}_i) = \sum_{i=1}^m D(\mathfrak{r})_i$.

We now prove two lemmas from which we shall deduce (★).

LEMMA 1. *Suppose $\mathfrak{R}$ is a continuous ring and e_i, $i = 1, \ldots, N$ are orthogonal idempotents with $R(e_i) = 1/N$ for all i. Suppose that for some c in $\mathfrak{R}$,*

$$(ce_i)_r = (e_{i+1})_r \qquad \text{for} \qquad 1 \leqslant i < N,$$
$$(ce_N)_r = (e_1)_r.$$

Then for every polynomial $p(t) = t^m + z_1 t^{m-1} + \ldots + z_m$ with coefficients z_i in the centre of $\mathfrak{R}$ and degree $m \leqslant N$, it is so that $R((p(c))) \geqslant 1 - m/N$.

Proof. Clearly $D((e_i)_r) = 1/N$ for all i. We shall show below:

(1) if $i \leqslant N - m$ then $(p(c)e_i)_r \subset (e_1 + \ldots + e_{i+m})_r$;

(2) if $i \leqslant N - m$ then $(p(c)e_i)_r \cap (e_1 + \ldots + e_{i+m-1})_r = 0$;

(3) $(p(c)e_i)_r$, $i = 1, \ldots, N - m$ are independent;

(4) $(p(c)e_i)_r$ is perspective to $(e_{i+m})_r$ if $i \leqslant N - m$.

Then (3) and (4) will imply that

$$R(p(c)) = D((p(c))_r) \geqslant D\left(\bigcup_{i=1}^{N-m} (p(c)e_i)_r\right) = \sum_{i=1}^{N-m} D((p(c)e_i)_r)$$
$$= \sum_{i=1}^{N-m} D((e_{i+m})_r) = \sum_{i=1}^{N-m} \frac{1}{N} = 1 - \frac{m}{N},$$

that is, $R(p(c)) \geqslant 1 - m/N$, as required.

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To prove (1): note that if $i + m \leqslant N$ then

$$\begin{aligned} p(c)e_i &= c^m e_i + z_1 c^{m-1} e_i + \ldots + z_m e_i \\ &\in (e_{i+m})_r \cup (e_{i+m-1})_r \cup \ldots \cup (e_i)_r \\ &\leqslant (e_1 + \ldots + e_{i+m})_r. \end{aligned}$$

Thus $(p(c)e_i)_r \subset (e_1 + \ldots + e_{i+m})_r$ as required.

To prove (2): note first that $(c)_r \supset (e_i)_r$ for $1 \leqslant i \leqslant N$ so $(c)_r = \mathfrak{R}$. Thus c^{-1} exists.

Now if $x \in (p(c)e_i)_r \cap (e_1 + \ldots + e_{i+m-1})_r$ then

$$x = (c^m e_i + z_1 c^{m-1} e_i + \ldots + z_m e_i)y = (e_1 + \ldots + e_{i+m-1})x$$

for some y in $\mathfrak{R}$. Since $c^j e_i \in (e_{i+j})_r$ for $i + j \leqslant N$ and $(e_1)_r, \ldots, (e_N)_r$ are independent, so $c^m e_i y = 0$. Since c^{-1} exists, so $e_i y = 0$. Now it follows that $x = 0$. This proves:

$$(p(c)e_i)_r \cap (e_1 + \ldots e_{i+m-1})_r = 0, \text{ that is, } (2).$$

To prove (3): This follows from (1) and (2) since

$$(p(c)e_i)_r \cap \cup_{j<i} (p(c)e_j)_r \subset ((p(c)e_i)_r \cap \cup_{j<i+m} (e_j)_r = 0$$

for $i = 2, \ldots, N - m$.

To prove (4): note (with the help of (2)) that:

$$(p(c)e_i)_r \cap (e_1 + \ldots + e_{i+m-1})_r = 0 = (e_{i+m})_r \cap (e_1 + \ldots + e_{i+m-1})_r$$

and

$$\begin{aligned} (p(c)e_i)_r &\cup (e_1 + \ldots + e_{i+m-1})_r \\ &= (c^m e_i)_r \cup (e_1 + \ldots + e_{i+m-1})_r \\ &= (e_{i+m})_r \cup (e_1 + \ldots + e_{i+m-1})_r. \end{aligned}$$

This shows that $(p(c)e_i)_r$ is perspective to $(e_{i+m})_r$ with axis of perspectivity $(e_1 + \ldots + e_{i+m-1})_r$. This proves (4).

COROLLARY TO LEMMA 1. *The element c is purely transcendental, if for every integer $N = 1, 2, \ldots$ there exists an integer $M \geqslant N$ and orthogonal indempotents $e_1, \ldots, e_M$ with $R(e_i) = 1/M$ for $1 \leqslant i \leqslant M$ and*

$$\begin{aligned} (ce_i)_r &= (e_{i+1})_r \qquad \textit{for} \quad 1 \leqslant i < M, \\ (ce_M)_r &= (e_1)_r. \end{aligned}$$

Proof. Let $p(t)$ be any polynomial $t^m + z_1 t^{m-1} + \ldots + z_m$ with coefficients in the centre of $\mathfrak{R}$. Then Lemma 1 shows that for every $N \geqslant m$, we have $R(p(c)) \geqslant 1 - m/M \geqslant 1 - m/N$. Hence $R(p(c)) = 1$ and so $p(c)^{-1}$ exists. This means that c is purely transcendental.

LEMMA 2. *Suppose N is a positive integer and $e_i^{(N)}$, $i = 1, \ldots, 2^N$ are orthogonal idempotents such that $R(e_i^{(N)}) = 1/2^N$ for all i. Suppose also that $c^{(N)}$ is an element in $\mathfrak{R}$ such that*

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$$(c^{(N)}e_i^{(N)})_r = (e_{i+1}^{(N)})_r \quad \textit{for } 1 \leqslant i < 2^N,$$
$$c^{(N)}e_{2^N}^{(N)} = 0.$$

Then there exist orthogonal idempotents $e_i^{(N+1)}$, $i = 1, \ldots, 2^{N+1}$ *and an element* $c^{(N+1)}$ *such that:*

(5) $$R(e_i^{(N+1)}) = \frac{1}{2^{N+1}} \textit{ for all } i,$$

(6) $$(c^{(N+1)}e_i^{(N+1)})_r = (e_{i+1}^{(N+1)})_r \quad \textit{for } 1 \leqslant i < e^{N+1},$$

(7) $$c^{(N+1)}e_{2^{N+1}}^{(N+1)} = 0,$$

(8) $$e_i^{(N)} = e_i^{(N+1)} + e_{2^N+i}^{(N+1)},$$

(9) $$c^{(N+1)}e_i^{(N)} = c^{(N)}e_i^{(N)} \quad \textit{for } 1 \leqslant i < 2^N,$$

(10) $$R(c^{(N+1)} - c^{(N)}) = \frac{1}{2^{N+1}}.$$

Proof. We can suppose $e_1^{(N)} = e_1 + e_2$ with e_1, e_2 orthogonal idempotents and

$$R(e_1) = R(e_2) = \tfrac{1}{2}R(e_1^{(N)})$$

(see (i) and (ii), preceding Lemma 1). Define

$$e_1^{(N+1)} \equiv e_1 \qquad e_{2^N+1}^{(N+1)} \equiv e_2.$$

Now for $1 < j \leqslant 2^N$,

$$(e_j^{(N)})_r = ((c^{(N)})^j e_1^{(N)})_r = ((c^{(N)})^j e_1^{(N+1)})_r \cup ((c^{(N)})^j e_{2^N+1}^{(N+1)})_r,$$
$$((c^{(N)})^j e_1^{(N+1)})_r \cap ((c^{(N)})^j e_{2^N+1}^{(N+1)})_r = 0.$$

Hence (see (ii) preceding Lemma 1) there exist orthogonal idempotents

$$e_j^{(N+1)}, e_{2^N+j}^{(N+1)}$$

such that their sum is $e_j^{(N)}$ (so (8) holds) and

$$(e_j^{(N+1)})_r = ((c^{(N)})^j e_1^{(N+1)})_r,$$
$$(e_{2^N+j}^{(N+1)})_r = ((c^{(N)})^j e_{2^N+1}^{(N+1)})_r.$$

Note that

$$R(e_j^{(N)}) = R(e_j^{(N+1)}) + R(e_{2^N+j}^{(N+1)})$$
$$\leqslant R(e_1^{(N+1)}) + R(e_{2^N+1}^{(N+1)}) = R(e_1^{(N)}) = R(e_j^{(N)}).$$

It follows that equality holds throughout, so $R(e_i^{(N+1)}) = 1/2^{N+1}$ for all i, so (5) holds.

Now

$$(e_{2^N}^{(N+1)})_r$$

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and

$$(e_{2^N+1}^{(N+1)})_r$$

are perspective (since they have equal ranks); hence by (iii), preceding Lemma 1, there exist elements $s^{(N)}$, $s_1^{(N)}$ such that:

$$s^{(N)} = e_{2^N+1}^{(N+1)} s^{(N)} e_{2^N}^{(N+1)} ;$$

$$s_1^{(N)} = e_{2^N}^{(N+1)} s_1^{(N)} e_{2^{(N)}+1}^{(N+1)} ;$$

$$s^{(N)} s_1^{(N)} = e_{2^N+1}^{(N+1)},\ s_1^{(N)} s^{(N)} = e_{2^N}^{(N+1)} ;$$

$$(s^{(N)} e_{2^N}^{(N+1)})_r = (e_{2^N+1}^{(N+1)})_r,\ s^{(N)} e_{2^N+1}^{(N+1)} = 0.$$

Define $c^{(N+1)} \equiv c^{(N)} + s^{(N)}$. Then if $1 \leqslant i < 2^N$,

$$c^{(N+1)} e_i^{(N+1)} = c^{(N)} e_i^{(N+1)},\ c^{(N+1)} e_{2^N+i}^{(N+1)} = c^{(N)} e_{2^N+i}^{(N+1)} ;$$

if $i = 2^N$,

$$c^{(N+1)} e_{2^N}^{(N+1)} = s^{(N)} e_{2^N}^{(N+1)} ;\ c^{(N+1)} e_{2^N+1}^{(N+1)} = s^{(N)} e_{2^N+1}^{(N+1)} = 0.$$

It follows that (6), (7), (9) also hold. Since

$$R(s^{(N)}) = R(e_{2^N}^{(N+1)}) = \frac{1}{2^{N+1}}$$

it follows that all of (5), (6), (7), (8), (9), (10) hold.

Conclusion from Lemmas 1 and 2. The hypotheses of Lemma 2 are satisfied, for $N = 0$, if we choose $e_1^{(0)} = 1$ and $c^{(0)} = 0$.

Now we can define $c^{(N)}$ and the $e_i^{(N)}$, $i = 1, \ldots, 2^N$ by induction on N as in Lemma 2. Then

$$R(c^{(M)} - c^{(N)}) = \sum_{i=N+1}^{M} \frac{1}{2^i} < \frac{1}{2^N}$$

so $c^{(N)}$ is a rank-convergent sequence. Since $\mathfrak{R}$ is assumed to be complete in the rank metric there will exist an element c in $\mathfrak{R}$ such that $R(c - c^{(N)}) \to 0$ as $N \to \infty$.

Now for any $N \geqslant 1$,

$$c\, e_i^{(N)} = \lim_{M\to\infty} c^{(M)} e_i^{(N)} = e_{i+1}^{(N)} \quad \text{if } 1 \leqslant i < 2^N.$$

Also,

$$(c\, e_{2^N}^{(N)})_r = \lim_{M\to\infty} (c^{(M+1)} e_{2^N}^{(N)})_r =$$
$$= \lim_{M\to\infty} ((e_{2^N+1}^{(N+1)})_r \cup (e_{2^{N+1}+1}^{(N+2)})_r \cup \ldots \cup (e_{2^{M-1}+1}^{(M)})_r).$$

The idempotents $e_{2^i+1}^{(i+1)}$, $i = N, \ldots, M-1$ are orthogonal and $e_{2^i+1}^{(i+1)} = e_1^{(N)} e_{2^i+1}^{(i+1)}$ for $N \leqslant i < M$. Thus (see (iv) preceding Lemma 1),

$$D(ce_{2^N}^{(N)})_r = \lim_{M\to\infty} \sum_{i=N+1}^{M} \frac{1}{2^i} = \frac{1}{2^N}.$$

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Since

$$(ce_{2N}^{(N)})_r \subset (e_1^{(N)})_r$$

and both of these principal right ideals have the same dimension namely, $1/2^N$, they must be identical.

Thus this element c satisfies the hypotheses of the Corollary to Lemma 1 and so c must be purely transcendental.*

*Other types of transcendental elements have been constructed by I. Amemiya and the author.

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