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DISTANCE-GENERICITY FOR REAL ALGEBRAIC HYPERSURFACES

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One of the original applications of catastrophe theory envisaged by Thom was that of discussing the local structure of the focal set for a (generic) smooth submanifold $M \subset \mathbf{R}^{n+1}$. Thom conjectured that for a generic M there would be only finitely many local topological models, a result proved by Looijenga in [**4**]. The objective of this paper is to extend Looijenga's result from the smooth category to the algebraic category (in a sense explained below), at least in the case when M has codimension 1.

Looijenga worked with the compactified family of distance-squared functions on M (defined below), thus including the family of height functions on M whose corresponding catastrophe theory yields the local structure of the focal set at infinity. For the family of height functions the appropriate genericity theorem in the smooth category was extended to the algebraic case in [**1**], so that the present paper can be viewed as a natural continuation of the first author's work in this direction.

We need to state rather more precisely just what Looijenga proved. First, we recall definitions. The family of distance-squared functions on M is the mapping $\Delta: M \times \mathbf{R}^{n+1} \rightarrow \mathbf{R}$ given by

$$\Delta(x, a) = |x - a|^2.$$

A natural way to extend the definition to the case when a is a point at infinity is to take S to be the unit sphere in $\mathbf{R}^{n+1} \times \mathbf{R}$, and define $\Delta^*: M \times S \rightarrow \mathbf{R}$ by the formula

$$\Delta^*(x, a, \lambda) = \lambda|x|^2 - 2(x \cdot a),$$

with $\cdot$ the standard scalar product on $\mathbf{R}^{n+1}$. We shall refer to Δ^* as the compactified family of distance-squared functions on M. Note that when $\lambda \neq 0$ we have

$$\Delta^*(x, a, \lambda) = \lambda\Delta\left(x, \frac{a}{\lambda}\right) - \frac{|a|^2}{\lambda}$$

so that the restriction of Δ^* to the subset given by $\lambda \neq 0$ is a family of

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functions on M equivalent to Δ. Given positive integers r, k we have an induced multi-jet extension

$$ {}_rj_1^k\Delta^*:M^{(r)} \times S \to {}_rJ^k(M, \mathbf{R}) $$

and one calls M (*locally*) *distance-generic* when this mapping is transverse to the canonical stratification of the jet-space defined in [**4**] for ($r = 1$) all r and for all k. The construction of the canonical stratification need not concern us here: all we need to know is that it is finite, semi-algebraic, and right invariant. In fact the results of this paper hold for any stratification of the jet-space having these properties.

Looijenga proved that almost all smooth submanifolds $M \subseteq \mathbf{R}^{n+1}$ are distance-generic, via a fairly standard type of transversality theorem. It is this step which we wish to mimic in the algebraic category. More precisely, we are interested in the case when M is a (non-singular) hypersurface, defined by an equation $F = 0$, with $F:\mathbf{R}^{n+1} \to \mathbf{R}$ a polynomial function of degree d, say. The question to which we address ourselves is whether M can be made distance-generic by arbitrarily small deformations of F, by terms of degree $\leqq d$. The crucial point to notice here is that the space of deformations is given a priori, whereas in the smooth case the deformation space is constructed to make the transversality theorem work. We do not know of any transversality theorem which applies to the situation just described, and therefore have to argue from first principles. What we prove is the following. Write V_d for the real vector space of all polynomial functions $F:\mathbf{R}^{n+1} \to \mathbf{R}$ of degree $\leqq d$: and for such an F set $M_F = F^{-1}(0)$.

THEOREM. *If d is sufficiently large then for almost all non-singular F in V_d (in the sense of Lebesgue measure) the zero-set M_F is distance-generic.*

We do not know whether the hypothesis, that d be sufficiently large, can be removed. The techniques of this paper are unlikely to provide such an extension. In the case $n = 1$ of plane curves, and $n = 2$ of surfaces, local distance genericity holds without any restriction on d [**2**], but for $n \geqq 3$ that is no longer known to be the case. It would be interesting to obtain analogous results for general varieties. The point of keeping to the hypersurface case is that one then has a clear candidate for the deformation space, whereas for general varieties we see no obvious choice.

The plan of the paper is as follows. In Section 1 we consider the effect of a deformation of F at a single point on M_F, and obtain a key technical fact, namely that a certain mapping is a submersion. This allows us in Section 2 to obtain a weak version of the above theorem in which "distance-generic" is replaced by "local distance-generic". Finally, in Section 3 we show how a well-known result in the geometry of hypersurfaces allows one to proceed to multi-germs in much the same way as in Section 2.

1. Deforming germs of hypersurfaces. We consider a real algebraic hypersurface $M_F \subseteq \mathbf{R}^{n+1}$ associated to a polynomial function $F:\mathbf{R}^{n+1} \to \mathbf{R}$ of degree d. We assume F is non-singular, i.e., there are no singular points either in $\mathbf{R}^{n+1}$, or at infinity. The standard co-ordinates in $\mathbf{R}^{n+1}$ will be written as $x_1, \ldots, x_n, y$. Consider a fixed point P on M_F. By applying a rigid motion of $\mathbf{R}^{n+1}$ we can suppose that $P = 0$, and that the tangent hyperplane to M at 0 is $y = 0$. By the Implicit Function Theorem we can write M, locally at 0, as a graph, i.e., there exists an analytic function g, defined on a neighbourhood of the origin in $\mathbf{R}^n$, for which

$$F(x, g(x)) \equiv 0 \quad \text{where } x = (x_1, \ldots, x_n).$$

Now we consider the effect of deforming F by terms of degree $\leqq d$. To this end we consider the mapping

$$\widetilde{F}:\mathbf{R}^{n+1} \times V_d \to \mathbf{R}$$

defined by

$$\widetilde{F}(x, y, f) = F_f(x, y) = F(x, y) + f(x, y).$$

By the assumptions above

$$\frac{\partial \widetilde{F}}{\partial y}(0, 0, 0) = \frac{\partial F}{\partial y}(0, 0) \neq 0$$

so again by the Implicit Function Theorem there exists an analytic function $\widetilde{g}(x, f) = g_f(x)$, defined on a neighbourhood of the origin in $\mathbf{R}^n \times V_d$, for which

$$F_f(x, g_f(x)) \equiv 0.$$

Thus, in a neighbourhood of 0 in $\mathbf{R}^{n+1}$ we have a family of parametrized hypersurfaces $g_f(x)$ nearby to $g(x)$ for f near the zero polynomial in V_d. This gives rise to a family (parametrized by f) of families of distance-squared functions on these hypersurfaces, namely a map

$$\Delta:V_d \times \mathbf{R}^n \times \mathbf{R}^{n+1} \rightarrowtail \mathbf{R}$$

defined by

(1) $\quad \Delta(f, x, a) = (x_1 - a_1)^2 + \ldots + (x_n - a_n)^2 + (g_f(x) - a_{n+1})^2$

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where $a = (a_1, \dots, a_n, a_{n+1})$. Given a positive integer k this mapping induces a single jet-extension $j_1^k\Delta$ into $J^k(\mathbf{R}^n, \mathbf{R})$. Our key technical fact is

(1.1) *Let F in V_d be non-singular. Provided $d \geqq k$, the germ of the jet-extension $j_1^k\Delta$ at any point* $(0, 0, a)$ *is submersive.*

Proof. The vector space V_d is spanned by monomials

$$x_1^{p_1} \dots x_n^{p_n} y^q \quad \text{with } p_1 + \dots + p_n + q \leqq d.$$

We ask which vectors (in the image of the differential of $j_1^k\Delta$) will be produced by individual monomials. Thus we consider the restrictions of F to a 1-dimensional subspace of V_d given by

$$f(x, y) = vx_1^{p_1} \dots x_n^{p_n}y^q \quad (v \in \mathbf{R})$$

for fixed $p_1, \dots, p_n, q$. The corresponding deformed hypersurface $\widetilde{g}(x, f)$ satisfies identically the relation

$$(2) \quad F(x, \widetilde{g}(x, v)) + vx_1^{p_1} \dots x_n^{p_n} \widetilde{g}(x, v)^q = 0.$$

To obtain $j_1^k\Delta$ we take the k-jet of the right hand side of (1), thinking of $a = (a_1, \dots, a_n, a_{n+1})$ as fixed. We are only interested in the part involving v, evaluated at $x = 0$. If we differentiate the formula in (1) with respect to v, we obtain

$$(3) \quad 2\{\widetilde{g}(x, f) - a_{n+1}\}\frac{\partial \widetilde{g}}{\partial v}(x, v).$$

We wish to evaluate the partial derivative in this expression. For this, differentiate (2) with respect to v, and then set $v = 0$: that yields

$$(4) \quad \frac{\partial F}{\partial y}(x, \widetilde{g}(x, 0))\frac{\partial \widetilde{g}}{\partial v}(x, 0) + x_1^{p_1} \dots x_n^{p_n}g(x)^q = 0.$$

We need the leading terms in the expansion of $\dfrac{\partial \widetilde{g}}{\partial v}(x, 0)$. By supposition we can write

$$F(x, y) = cy + O(2),$$

with $c \neq 0$, so

$$\frac{\partial F}{\partial y} = c + O(1),$$

and (4) reduces to

$$(5) \quad \frac{\partial \tilde{g}}{\partial v}(x, 0) = -\frac{x_1^{p_1} \ldots x_n^{p_n} g(x)^q}{c + O(1)}.$$

Next, write

$$g(x) = \frac{1}{2}(\kappa_1 x_1^2 + \ldots + \kappa_n x_n^2) + O(3)$$

with $\kappa_1, \ldots, \kappa_m$ the principal curvatures of M at 0. Then expanding $g(x)^q$, and the denominator, in (5) we obtain

$$\frac{\partial g}{\partial v}(x, 0) = -\frac{1}{2^q c} x_1^{p_1} \ldots x_n^{p_n} (\kappa_1 x_1^2 + \ldots + \kappa_n x_n^2)^q + O(r)$$

where $r = p_1 + .. + p_n + 2q + 1$. And (3), evaluated at $v = 0$, will become (modulo a non-zero scalar)

$$\kappa_1 x_1^2 + \ldots + \kappa_n x_n^2 n - a_{n+1} + O(3)\, x_1^{p_1} \cdot \ldots \cdot x_n^{p_n} (\kappa_1 x_1^2 + \ldots + \kappa_n x_n^2)^q + O(r).$$

We seek the leading term in this expression. We need not consider the case $a_{n+1} = 0$ since then the point a lies on the tangent hyperplane to M at 0, and the distance-squared function has at worst an A_1 singularity at 0. When $a_{n+1} \neq 0$ our expression has the form (modulo a non-zero scalar)

$$(6) \quad x_1^{p_1} \ldots x_n^{p_n} (\kappa_1 x_1^2 + \ldots + \kappa_n x_n^2)^q + O(r).$$

This then will be the result of deforming $F(x, y)$ by a monomial $x_1^{p_1} \ldots x_n^{p_n} y^q$. In the particular case when $q = 0$ the result will be

$$x_1^{p_1} \cdot \ldots \cdot x_n^{p_n} + O(r), \quad \text{with } r = p_1 + \ldots + p_n + 1.$$

(A careful reading of the above will make it clear that this holds even when $k_1, \ldots, \kappa_n$ all vanish.) Clearly, the totality of such vectors, modulo terms of degree $> d$, span the space of all polynomials in $x_1, \ldots, x_n$ of degree $\leqq d$. Thus our jet extension $j_1^k \Delta$ will be submersive at any point $(0, 0, a)$, provided $d \geqq k$.

In fact we have proved rather more than is stated in (1.1), since the same argument shows that the germ of $j_1^k \Delta$ is submersive at any point in its domain. Thus

(1.2) *Let F in V_d be non-singular. Provided $d \geqq k$, the jet extension $j_1^k \Delta$ is a submersion.*

2. Local distance-genericity. The objective of this section is to establish the following special case of the theorem announced in the introduction.

(2.1) *For almost all non-singular F in V_d (in the sense of Lebesgue measure) the hypersurface $M_F \subseteq \mathbf{R}^{n+1}$ is locally distance-generic, provided d is sufficiently large.*

Before proceeding we should explain exactly what we mean by "sufficiently large". In $J_0^k(n, 1)$ the set of jets which are not k-determined forms a semialgebraic set of codimension $\sigma(k)$, say. Moreover it is well-known that

$$\lim_{k \to \infty} \sigma(k) = \infty.$$

Suppose we choose $k = \tau(n)$, say, so that $\sigma(k) \geqq 2n + 2$: then we claim that if

$$j_1^k \Delta_F^*: M_F \times S \to J^k(M_F, \mathbf{R})$$

is transverse to Looijenga's canonical stratification then so is $j_1^l \Delta_F^*$ for all $l \geqq k$. Indeed $j_1^k \Delta_F^*(M_F \times S)$ will consist only of k-determined jets, and since the strata are right-invariant the stratum S_l to which $j_1^l \Delta_F^*(x, a, \lambda)$ belongs in $J^l(M_F, \mathbf{R})$ will be the product of the stratum S_k containing $j_1^k \Delta_F^*(x, a, \lambda)$ with the space of polynomials ϕ for which

$$k + 1 \leqq \deg \phi \leqq l.$$

Thus $j_1^l \Delta_F^*$ is transverse to S^l if $j_1^k \Delta_F^*$ is transverse to S_k, as asserted. The computation of $\tau(n)$ is fairly easy for small values of n, but we know of no general expression for it, nor more importantly for $\sigma(k)$. We certainly have $\sigma(k) \leqq n + k - 1$, which one can clearly see by computing the dimension of the orbit for the jet $x_2^2 + x_3^2 + \ldots + x_n^2$ in $J_0^k(n, 1)$. Moreover any orbit of smaller codimension in $J_0^k(n, 1)$ would have to be that of a k-determined jet, for by Nakayama's Lemma

$$\dim (\mathfrak{M}/\mathfrak{M}J_f + \mathfrak{M}^{k+1}) < k$$

implies that

$$\mathfrak{M}^k \subseteq \mathfrak{M}J_f + \mathfrak{M}^{k+1},$$

hence that f is k-determined. (Here $\mathfrak{M}$ is the maximal ideal in the local algebra of germs at 0 of smooth functions f on $\mathbf{R}^n$, and J_f is the Jacobian ideal.) One might hope that $\sigma(k) = n + k - 1$ (though this is not a conjecture) and consequently that $\tau(n) = n + 3$. Thus the exact meaning of "sufficiently large" in the statement of (2.1) is that $d \geqq \tau(n)$.

Let us return to (2.1). A more precise statement, which clearly implies (2.1), goes as follows. Write Σ for the algebraic subset of V_d comprising singular hypersurfaces.

(2.2) *Let $F \in V_d - \Sigma$. There exists a neighbourhood U of F in $V_d - \Sigma$, and a subanalytic set $B \subseteq U$, of codimension $\geqq 1$, such that for all $G \in U - B$ the hypersurface M_G is locally distance generic, provided that d is sufficiently large.*

Now let us assume (temporarily) that $M_F \subseteq \mathbf{R}^{n+1}$ is compact. Then (2.2) will follow from

(2.3) *Let $F \in V_d - \Sigma$, and let $P \in M_F$. There exists a neighbourhood U of F in $V_d - \Sigma$, a subanalytic set $B \subseteq U$ of codimension $\geqq 1$, and a neighbourhood A of P in $\mathbf{R}^{n+1}$, such that for $G \in U - B$, and $Q \in A \cap M_G$, the germ of $j_1^k\Delta_G^*$ at Q is transverse to the stratification.*

Proof. As in Section 1 we can suppose that $P = 0$, and that the tangent hyperplane to M at 0 is $y = 0$ with $x_1, \ldots, x_n, y$ the standard orthogonal co-ordinate system at 0. Further, we can write M_F, close to 0, as the graph of an analytic function $g(x_1, \ldots, x_n)$. And, deforming F to $F_f = F + f$, with $f \in V_d$, we obtain a family of analytic functions $g_f(x_1, \ldots, x_n)$, analytically parametrized by a neighbourhood of the origin in V_d, such that we can write M_{F_f}, locally, as the graph of $g_f(x_1, \ldots, x_n)$. That yields a family (parametrized by f) of compactified families of distance-squared functions on these graphs, namely a mapping

$$\Delta^*: V_d \times \mathbf{R}^n \times S \rightarrow \mathbf{R},$$

with a corresponding induced jet-extension $j_1^k\Delta^*$ into the jet-space $J^k(\mathbf{R}^n, \mathbf{R})$. Now Δ^* is submersive at any point (f, x, s). When s is a "finite point" that follows immediately from (1.2), bearing in mind our introductory remarks concerning the family of distance-squared functions, and its natural compactification: and when s is an "infinite point" a very similar computation, written out in full in [**1**], establishes the same result. Thus Δ^* will be submersive on a neighbourhood $U \times N \times S$ of $(0, 0, s)$ in $V_d \times \mathbf{R}^n \times S$. Choose U, N to be closed balls, and W to be an open neighbourhood of $U \times N \times S$ in $V_d \times \mathbf{R}^n \times S$ on which $j_1^k\Delta^*$ is submersive, hence transverse to the canonical stratification in $J^k(\mathbf{R}^n, \mathbf{R})$. Let X be one of the (finitely many, semi-algebraic) strata thereof, so $j_1^k(\Delta^*)^{-1}(X) \cap W$ is an analytic submanifold of W. Write π for the projection of $V_d \times \mathbf{R}^r \times S$ onto its first factor, and Σ_π for the singular set of π, restricted to $j_1^k(\Delta^*)^{-1}(X) \cap W$. Evidently Σ_π is subanalytic. Now $\pi: U \times N \times S \rightarrow U$ is proper (as its domain is compact) so

$$B_X = \pi(\Sigma_\pi \cap (U \times N \times S))$$

is subanalytic: by Sard's Theorem it has Lebesgue measure zero, so must have codimension $\geqq 1$. And the union B of the B_X, with X running

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through the strata, will have the same properties. It now follows from the proof of Thom's Basic Transversality Lemma that for $G \in U - B$ the jet-extension mapping

$$j_1^k \Delta_G^*: N \times S \to J^k(\mathbf{R}^n, \mathbf{R})$$

is transverse to the canonical stratification. The result follows on choosing A to be contained in the image of $N \times U$ under the mapping $\mathbf{R} \times V_d \to \mathbf{R}^n \times \mathbf{R}$ given by

$$(x, f) \to (x, g_f(x)).$$

In order to establish (2.2) we have still to remove the assumption that M_F is compact, a point of some importance, since any real algebraic hypersurface of odd degree fails to be compact. We can do this by arguing more carefully. The crux of the matter is that at every point P in M_F we have neighbourhoods U_P of 0 in V_d and N_P of 0 in $\mathbf{R}^n$, and an analytic function defined on $U_P \times N_P$ which serves to parametrize simultaneously all the nearby hypersurfaces M_{F_f} close to P. In fact this holds equally well for a point P at infinity on M_F, since we have assumed that M_F has no singular points either in $\mathbf{R}^{n+1}$, or at infinity. The compactness of the projective closure of M_F in $\mathbf{P}^{n+1}$ allows us to conclude that there exist neighbourhoods U of 0 in V_d, and N of 0 in $\mathbf{R}^n$, such that at every point P in M_F we have an analytic function defined on $U \times N$ which serves to parametrize simultaneously all the nearby hypersurfaces M_{F_f}, close to P. Our work in Section 1 shows that the corresponding mapping Δ^* is defined, and a submersion, on $U \times N \times S$, so that the U and N of (2.3) do not depend on the choice of P. It follows that the compactness assumption is indeed redundant.

3. The transition to multi-germs. Let $M_F \subseteq \mathbf{R}^{n+1}$ be a non-singular hypersurface of degree d. We showed in Section 1 that the space V_d of deformations was sufficiently large to achieve our goal at a single point P on M_F. To extend the proposition of that section to multi-germs we must achieve the same result simultaneously at r distinct points $P_1, \dots, P_r$. The key to this extension is the following geometric lemma.

(3.1) *Let $t_1, \dots, t_r$ be distinct points in $\mathbf{R}^{n+1}$, and $d_1, \dots, d_r$ positive integers. If d is sufficiently large then the subspace of hypersurfaces in V_d which pass through $t_1, \dots, t_r$ with multiplicities $\geqq d_1, \dots, \geqq d_r$ has codimension* $\dim V_{d_1} + \dots + \dim V_{d_r}$.

In other words the codimension is precisely what one expects by "counting constants", provided d is large enough. The proof of (3.1) is by induction on $d_1 + \dots + d_r$, and shows that it suffices to take $d \geqq d_1 +$

$\ldots + d_r$. In the case $n = 1$ of plane curves the proof is written out in complete detail in [**3**], starting on page 111, and requires only minor changes for the general case. We shall require (3.1) in a slightly different guise. For $d \geqq d_i$ we have a linear projection $\pi_i: V_d \to V_{d_i}$ given by deleting terms of degree $> d_i$. Now let $\alpha_1, \ldots, \alpha_r$ be affine isomorphisms of $\mathbf{R}^{n+1}$ with distinct associated translations. Then, for $d \geqq d_1, \ldots, d_r$ we have linear maps

$$V_d \to V_d \times \ldots \times V_d \to V_{d_1} \times \ldots \times V_{d_r}$$

$$f \to (f_0\alpha_1, \ldots, f_0\alpha_r) \to (\pi_1(f_0\alpha_1), \ldots, \pi_r(f_0,\alpha_r)\,).$$

(3.2) *If d is sufficiently large the above composite has maximal rank.*

Proof. One has to show that the kernel has dimension

$$\dim V_{d_1} + \ldots + \dim V_{d_r}.$$

Suppose first that the α_i are just translations, say $\alpha_i(x) = x + t_i$ with $t_i \in \mathbf{R}^{n+1}$. Then the kernel comprises those f in V_d for which

$$\pi_i(f(x + t_i)\,) = 0 \quad \text{for } 1 \leqq i \leqq r:$$

expanding $f(x + t_i)$ by Taylor's Theorem we see that this is precisely the condition that $f = 0$ passes through t_i with multiplicity $\geqq d_i$, so the result follows from (3.1). In the general case the corresponding condition is that $f_0\lambda_i = 0$ passes through $\lambda_i^{-1}(t_i)$ with multiplicity $\geqq d_i$, where λ_i is the linear part of α_i. Since multiplicity is invariant under linear isomorphisms this is the same thing as saying that $f = 0$ passes through t_i with multiplicity $\geqq d_i$, and the proof finishes as before.

We are now in a position to extend the work of the preceding sections to multi-germs. Our objective is to establish the theorem of the introduction, namely

(3.3) *For almost all non-singular F in V_d (in the sense of Lebesgue measure) the hypersurface $M_F \subseteq \mathbf{R}^{n+1}$ is distance-generic, provided d is sufficiently large.*

The condition for M_F to be distance-generic is that for the compactified family $\overset{*}{\Delta}_F: M_F \times S \to \mathbf{R}$ of distance-squared functions on M_F the induced multi-jet extension

$${}_rj_1^k\overset{*}{\Delta}_F: M_F^{(r)} \times \mathbf{R}^{n+1} \to {}_rJ^k(M_F, \mathbf{R})$$

is transverse to Looijenga's canonical stratification of the multi-jet space, for all r and all k. In fact to obtain a generic focal set we need only consider $r = n + 1$. Indeed this clearly implies ${}_rj_1^k\delta_F$ is transverse to the strata for $r \leqq n + 1$. On the other hand, for $r > n + 1$ any relevant stratum in ${}_rJ^k(M_F, \mathbf{R})$ is obtained either from the trivial product of one appearing in $J^k(n, 1)^{n+1}$ by $J^k(n, 1)^{r-n-1}$, or from a product with a factor in $J^k(n, 1)^{n+1}$ of codimension $\geqq n + 1$. Either way, transversality in ${}_{n+1}J^k(n, 1)$ implies transversality in ${}_rJ^k(n, 1)$ for $r \geqq n + 1$. (Note that the situation is not quite so simple for the Maxwell set where one considers the question of whether or not we have equality of values of the distance-squared functions.) Following the philosophy of Section 2 we shall establish the following, rather more precise version of (3.3).

(3.4) *Let $F \in V_d - \Sigma$. There exists a neighbourhood U of F in $V_d - \Sigma$, and a countable collection of subanalytic sets $B_i \subseteq U$, of codimension $\geqq 1$, such that for all $G \in U - \cup_{i=1}^{\infty} B_i$ the hypersurface M_G is distance-generic, provided d is sufficiently large.*

In fact, see the comments above, $d \geqq (n + 1)\tau(n)$ will suffice, with $\tau(n)$ defined as in Section 2, though this is far from a minimal choice. (3.4) will follow from

(3.5) *Let $F \in V_d - \Sigma$, and let $P_1, \dots, P_r$ be distinct points on M_F. There exists a neighbourhood U of F in $V_d - \Sigma$, a countable collection of subanalytic sets $B_i \subseteq U$ of codimension $\geqq 1$, and neighbourhoods $A_1, \dots, A_r$ of $P_1, \dots, P_r$ in $\mathbf{R}^{n+1}$, such that for*

$$G \in U - \bigcup_{i=+}^{\infty} B_i, \quad \textit{and}$$

$$Q_1, \dots, Q_r \textit{ in } A_1 \cap M_G, \dots, A_r \cap M_G,$$

the germ of ${}_rj_1^k\Delta_G^*$ at $Q_1, \dots, Q_r$ is the transverse to the stratification.

The proof follows the same lines as (2.3). We shall only indicate the two changes necessary to make it work.

At each of the points $P_1, \dots, P_r$ we can find orthogonal co-ordinate systems, related to the standard one at the origin by rigid motions $\alpha_1, \dots, \alpha_r$ (say) in which M_F appears (locally) as the graphs of analytic functions $g_1, \dots, g_r$, defined on neighbourhoods of 0 in $\mathbf{R}^n$. Moreover, as we explained in detail in Section 1 for the case $r = 1$, if we deform F to $F_f = F + f$ we have M_{F_f} appearing (locally) as the graphs of analytic families of functions $g_{1,f}, \dots, g_{r,f}$ parametrized by deformation f in V_d close to the zero polynomial. (It will be no restriction to take mutually disjoint neighbourhoods of $P_1, \dots, P_r$ in $\mathbf{R}^{n+1}$, and to suppose that the graphs of $g_{1,f}, \dots, g_{r,f}$ all lie inside the corresponding neighbourhoods.) That yields a family (parametrized by f) of compactified families of distance-squared functions on these graphs, namely a mapping

$$\Delta_r^*: V_d \times (\mathbf{R}^n)^r \times S \to \mathbf{R}$$

with a corresponding jet extension ${}_rj_1^k\Delta_r^*$ into $J_0^k(\mathbf{R}^n, \mathbf{R})^{n+1}$. We need this jet-extension to be submersive. To achieve this we extend the argument of (1.1), using (3.2). Consider a deformation f in V_d, given in terms of the standard orthogonal co-ordinate system at 0. The corresponding deformations in the orthogonal co-ordinate systems at $P_1, \ldots, P_r$ are $f_0\alpha_1, \ldots, f_0\alpha_r$. We need to know what these correspond to in the jet-space. For this we use (3.2) with $r = n + 1$ and $d_1 = \ldots = d_r = \tau(n) = k$ (say): then, given any polynomials $f_1, \ldots, f_r$ in V_k we can (provided d is sufficiently large) find a polynomial f in V_d for which

$$f_0\alpha_i = f_i + O(k + 1) \quad \text{for } 1 \leqq i \leqq r.$$

Now the computation in the proof of (1.1) showed that monomials of degree $\geqq k + 1$ in V_d correspond to terms of degree $\geqq k + 1$ in the space of α-jets, and hence produce nothing in the space of k-jets. On the other hand the same computation showed that a monomial $x_1^{p_1} \cdot \ldots \cdot x_n^{p_n}$ of degree $p_1 + \ldots + p_n = p \leqq k$ in V_d corresponds in the space of k-jets to a non-zero scalar multiple of

$$x_1^{p_1} \cdot \ldots \cdot x_n^{p_n} + O(p + 1).$$

Thus by choosing all but one of $f_1, \ldots, f_r$ to be zero, and the remaining one to be monomial $x_1^{p_1} \cdot \ldots \cdot x_n^{p_n}$ of degree $\leqq k$, we obtain a spanning set of vectors for the product of jet-spaces, as was required.

The rest of the proof finishes off as before. The only slight change is that whereas in the proof of (2.3) we were able to find a finite cover of M_F by neighbourhoods N, in the present situation we will only have a countable cover of $M_F^{(n+1)}$ by neighbourhoods $N_1 \times \ldots \times N_{n+1}$. It is for this reason that the single "bad set" B in the statement of (2.3) is replaced by a countable family B_i in the statement of (3.5).

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