ON STRONGLY PI-REGULAR RINGS OF STABLE RANGE ONE

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Dedicated to Professor Victor P. Camillo
on the occasion of his fiftieth birthday

An associative ring $R$ is said to have stable range one if for any $a, b \in R$ satisfying $aR + bR = R$, there exists $y \in R$ such that $a + by$ is right (equivalently, left) invertible. Call a ring $R$ strongly $\pi$-regular if for every element $a \in R$ there exist a number $n$ (depending on $a$) and an element $x \in R$ such that $a^n = a^{n+1}x$. It is an open question whether all strongly $\pi$-regular rings have stable range one.

The purpose of this note is to prove the following Theorem: If $R$ is a strongly $\pi$-regular ring with the property that all powers of every nilpotent von Neumann regular element are von Neumann regular in $R$, then $R$ has stable range one.

Let $R$ be an associative ring with identity. $R$ is said to have stable range one if for any $a, b \in R$ satisfying $aR + bR = R$, there exists $y \in R$ such that $a + by$ is right (equivalently, left) invertible. Call a ring $R$ strongly $\pi$-regular if for every element $a \in R$ there exist a number $n$ (depending on $a$) and an element $x \in R$ such that $a^n = a^{n+1}x$. This is in fact a two-sided condition [2].

It is an open question whether all strongly $\pi$-regular rings have stable range one. Many special classes of strongly $\pi$-regular rings have been proved to have stable range one (see [1, 4, 6, 7]). Goodearl and Menal [3] proved that strongly $\pi$-regular regular rings are unit-regular, hence have stable range one (Theorem 5.8, p.278). Here a ring $R$ is called unit-regular if for each element $x \in R$ there is a unit $u$ in $R$ such that $xux = x$, that is, $x$ is a unit-regular element. We say also that an element $a$ of $R$ is (von Neumann) regular if $a = axa$ for some element $x$ in $R$.

The main purpose of this note is to prove the following Theorem:

**THEOREM 1.** Let $R$ be a strongly $\pi$-regular ring. If all powers of every nilpotent regular element of $R$ are regular in $R$, then $R$ has stable range one.

To prove Theorem 1, we need the following characterization by Goodearl and Menal [3, Theorem 6.1] of stable range one for strongly $\pi$-regular rings.
PROPOSITION 2. (Goodearl and Menal) For a strongly n-regular ring \( R \), \( R \) has stable range one if and only if every nilpotent regular element of the ring \( eRe \) is unit-regular in \( eRe \), where \( e \in R \) is any idempotent of \( R \).

The proof we are going to give is motivated by a proof of Goodearl and Menal [3, Theorem 5.8].

PROOF OF THEOREM 1: Let \( e^2 = e \in R \) be any idempotent of \( R \), and \( x \in eRe \) be a nilpotent regular element of \( eRe \), with \( xyz = z, y \in eRe \). It suffices to prove, by Proposition 2, that \( x \) is unit-regular in \( eRe \).

Set \( K_i = r.ann_{eRe}(x^i) \), the right annihilator of \( x^i \) in \( eRe \) for all \( i = 0,1,2, \ldots \).

CLAIM 1. There exists an integer \( n \geq 1 \) such that \( xeRe + K_n = eRe \) and \( x^n eRe \cap K_1 = 0 \).

This is clear, since we assume that \( x^n = 0 \) for some \( n \geq 1 \).

CLAIM 2. \( xeRe + K_i \) are direct summands of \( eRe eRe \) for all \( i \geq 1 \).

It is easy to check that an element of \( eRe \) is regular in \( eRe \) if and only if it is regular in \( R \). So \( x^i \) is regular in \( eRe \) for all \( i \geq 2 \) by our assumption on \( R \). We may assume \( x^i y_i x^i = x^i \) for some \( y_i \in eRe \) for \( i \geq 2 \). Then \( K_i = (e - y_i x^i) eRe \). It is easy to check that

\[
xeRe + (e - y_i x^i) eRe = y_i x^i xeRe + (e - y_i x^i) eRe.
\]

We check below that the element \( y_i x^i x \) is actually von Neumann regular in \( eRe \):

\[
y_i x^i x \cdot y_{i+1} x^i \cdot y_i x^i x = y_i x^i x y_{i+1} x^i x = y_i x^i x.
\]

Put \( e_i = y_i x^i x y_{i+1} x^i \) and \( f_i = e - y_i x^i \), then \( e_i f_i = f_i e_i = 0 \). We see that \( e_i \) and \( f_i \) are orthogonal idempotents, hence \( e_i + f_i \) is an idempotent. But \( y_i x^i xeRe = e_i eRe \), so \( xeRe + K_i = e_i eRe + f_i eRe = (e_i + f_i) eRe \) is a direct summand of \( eRe eRe \).

Recall that we assume \( xyz = x \), so \( xeRe + K_1 \) is a direct summand of \( eRe eRe \) for the same reason.

CLAIM 3. \( x^i eRe \cap K_1 \) are direct summands of \( eRe eRe \) for all \( i \geq 1 \).

First, we show \( x^i eRe \cap K_1 = x^i K_{i+1} \). Since \( x^i K_{i+1} \subseteq x^i eRe \) and \( x^i K_{i+1} \subseteq K_1 \), \( x^i K_{i+1} \subseteq z^i eRe \cap K_1 \); on the other hand, pick any \( z^i r \in z^i eRe \cap K_1 \), then \( z^i z^i r = z^i + 1 r = 0 \) so that \( r \in K_{i+1} \); thus \( z^i r \in z^i K_{i+1} \) and so \( z^i eRe \cap K_1 \subseteq z^i K_{i+1} \).

Second, recall that we assume \( x^i y_i x^i = x^i \), so that \( K_{i+1} = (e - y_{i+1} x^i) eRe \), and we see that \( x^i eRe \cap K_1 = x^i K_{i+1} = x^i (e - y_{i+1} x^i + 1) eRe \). We check below that
$x^i(e - y_{i+1}x^{i+1})$ is von Neumann regular in $eRe$:

$$x^i(e - y_{i+1}x^{i+1}) \cdot y_i \cdot x^i(e - y_{i+1}x^{i+1}) = (x^i - x^i y_{i+1}x^{i+1}) y_i x^i(e - y_{i+1}x^{i+1})$$

$$= (e - x^i y_{i+1}x^{i+1}) x^i y_i x^i(e - y_{i+1}x^{i+1})$$

$$= (e - x^i y_{i+1}x^{i+1}) x^i(e - y_{i+1}x^{i+1})$$

$$= (x^i - x^i y_{i+1}x^{i+1})(e - y_{i+1}x^{i+1})$$

$$= x^i(e - y_{i+1}x^{i+1})(e - y_{i+1}x^{i+1})$$

$$= x^i(e - y_{i+1}x^{i+1})$$

Therefore $x^i eRe \cap K_1 = x^i K_{i+1}$ is a direct summand of $eRe eRe$.

Inasmuch as $xyz = x$, $xeRe \cap K_1 = xK_2$ is a direct summand of $eRe eRe$.

**Claim 4.** $(xeRe + K_m)/xeRe \cong K_1/(x^m eRe \cap K_1)$ for all $m$.

The right ideals of $eRe$ involved here are all direct summands of $eRe eRe$ by Claim 2 and Claim 3. We have the ascending and descending chains of direct summands

$$xeRe \supseteq xeRe + K_1 \supseteq xeRe + K_2 \supseteq \ldots \supseteq xeRe + K_m$$

$$K_1 \supseteq xeRe \cap K_1 \supseteq xeRe \cap K_2 \supseteq \ldots \supseteq xeRe \cap K_m$$

which give us the decompositions

$$(xeRe + K_m)/xeRe \cong \bigoplus_{i=0}^{m-1} (xeRe + K_{i+1})/(xeRe + K_i)$$

$$K_1/(x^m eRe \cap K_1) \cong \bigoplus_{i=0}^{m-1} (x^i eRe \cap K_1)/(x^{i+1} eRe \cap K_1)$$

so if we can show that

$$(xeRe + K_{i+1})/(xeRe + K_i) \cong (x^i eRe \cap K_1)/(x^{i+1} eRe \cap K_1)$$

for all $i$, we are done.

First we note that

$$(xeRe + K_{i+1})/(xeRe + K_i) = (xeRe + K_i + K_{i+1})/(xeRe + K_i)$$

$$\cong K_{i+1}/[(xeRe + K_i) \cap K_{i+1}] = K_{i+1}/[(xeRe \cap K_{i+1}) + K_i].$$

As $z^i K_{i+1} \subseteq z^i eRe \cap K_1$ and $z^i[(xeRe \cap K_{i+1}) + K_i] \subseteq z^{i+1} eRe \cap K_1$, left multiplication by $z^i$ gives a module homomorphism

$$f : K_{i+1}/[(xeRe \cap K_{i+1}) + K_i] \to (z^i eRe \cap K_1)/(z^{i+1} eRe \cap K_1).$$
$f$ is epic: Pick any $r \in z_i eRe \cap K_1$, $r = z_i a$ for some $a \in eRe$. But, since $z_{i+1} a = x r = 0$, $a \in K_{i+1}$. So $f(a) = r$.

$f$ is monic: If $z \in K_{i+1}$ and $z_i z \in z_{i+1} eRe \cap K_1$, then we have $z_i z = z_{i+1} b$ for some $b \in eRe$ and $z_{i+1} b = x (z_i z) = 0$, whence $x b \in K_{i+1} \cap xeRe$. Since $x_i(z - xb) = 0$, $z - xb \in K_i$, thus $z \in (xeRe \cap K_{i+1}) + K_i$, that is, $f$ is monic.

We have proved that $f$ is an isomorphism.

**CLAIM 5.** $x$ is unit-regular in $eRe$.

It follows from Claim 1 and Claim 4 that

$$(xeRe + K_n)/xeRe = eRe/xeRe \cong K_1/(x^n eRe \cap K_1) = K_1/0 = K_1.$$  

It is assumed that $xyz = x$, hence

$$eRe = yzeRe \oplus K_1 = xeRe \oplus (e - xy)eRe.$$  

So $K_1 \cong (e - xy)eRe$. Denote this isomorphism by $\alpha$. Also, the restriction of the left multiplication by $x$ gives an isomorphism $\beta$ from $yzeRe$ to $xeRe$. Define $u \in end(eRe_{eRe}) = eRe$ to be the direct sum of $\alpha$ and $\beta^{-1}$. Then it is easy to check that $u$ is a unit in $eRe$ and $xux = x$.  

Yu [6] proved, among other things, that strongly $\pi$-regular rings whose idempotents are all central have stable range one. This result now can be easily deduced as a corollary of Theorem 1.

**COROLLARY 3.** (Yu, [6]) Strongly $\pi$-regular rings whose idempotents are all central have stable range one.

**PROOF:** It is known that a ring $R$ has stable range one if and only if $R/J(R)$ has stable range one, where $J(R)$ denotes the Jacobson radical of $R$ [5, Theorem 2.2)]. Let $R$ be a strongly $\pi$-regular ring whose idempotents are all central. We need only to show that $R/J(R)$ has stable range one.

It is trivial to see that for a strongly $\pi$-regular ring $S$ with $J(S) = 0$, all idempotents of $S$ are central if and only if $S$ contains no nonzero nilpotent element. Applying this equivalence to the factor ring $R/J(R)$, we see that $R/J(R)$ contains no nonzero nilpotent element, hence all powers of every nilpotent regular element are regular. So the conclusion follows from Theorem 1.  

We conclude this note by giving an example, which shows that the converse of Theorem is false.

**EXAMPLE.** Let $R$ be the $2 \times 2$ matrix ring over $F[z]/(z^2)$, where $F$ is any field and $F[z]$ is the polynomial ring over $F$.  


Clearly, $R$ is a finite dimensional algebra, hence strongly $\pi$-regular. Of course, $R$ has stable range one. But not all the powers of every nilpotent regular element of $R$ are regular in $R$. Taking $a = \begin{pmatrix} 0 & 1 \\ 0 & x \end{pmatrix}$ and $u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, it is easy to see that $a$ is nilpotent and $aua = a$. But $a^2 = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in J(R)$ is not regular. So the condition that all powers of every nilpotent regular element are regular is sufficient but not necessary for strongly $\pi$-regular rings to have stable range one.

REFERENCES


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