9

Incorporating hadrons

9.1 The mixing matrix

The weak interaction for the leptons was introduced into the theory by arranging the left-handed leptons (chirality -1) in three generations of doublets and the right-handed charged leptons into three singlets:

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L; \; e_R^-, \quad \mu_R^-, \quad \tau_R^-. \tag{9.1}$$

The similarities of the interactions of leptons to those of quarks suggest that one should similarly introduce for the quarks left-handed doublets and right-handed singlets. The situation for the quarks is different, since all of them are massive. For this reason each quark field has a right-handed component. The fields are classified as three doublets,

$$q_L^{1'} = \begin{pmatrix} u' \\ d' \end{pmatrix}_L, \quad q_L^{2'} = \begin{pmatrix} c' \\ s' \end{pmatrix}_L, \quad \text{and} \quad q_L^{3'} = \begin{pmatrix} t' \\ b' \end{pmatrix}_L; \tag{9.2}$$

and six right-handed singlets, u'_R, d'_R, c'_R, s'_R, t'_R, and b'_R. The superscripts denote three generations and the primes indicate that they are gauge quarks. The part of the Lagrangian which contains the kinetic terms and the couplings of the quarks to $W^\pm$, Z^0, and photons is written as follows:

$$\mathcal{L} = \bar{q}'_{Li}\gamma^\mu \left\{ i\partial_\mu + \frac{g}{2}\vec{\tau}\vec{W}_\mu + \frac{g'}{2}YB_\mu \right\} q'_{Li} + \bar{q}'_{Ri}\gamma^\mu \left\{ i\partial_\mu + \frac{g'}{2}YB_\mu \right\} q'_{Ri}. \tag{9.3}$$

The operator Y denotes the weak hypercharge, which has been defined already,

$$Y = 2(Q - I_3). \tag{9.4}$$

At this stage it is not clear whether the fields u', d', ... stand for the physical states because Eq. (9.3) contains only kinetic and interaction terms. Physical fields are eigenstates of the mass matrix which will be introduced below. This is the reason

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why the quark fields in Eq. (9.3) have a prime and we referred to them as gauge eigenstates or gauge quarks.

Masses for the quarks are generated through quark–Higgs Yukawa couplings. A Yukawa interaction invariant under SU(2) ⊗ U(1) gauge transformations is easily constructed:

$$\mathcal{L}_{\text{mass}} = h^{ij}_{(\text{d})}(\bar{u}'_i\,,\ \bar{d}'_i)_{\text{L}}\begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix} d_{j\text{R}} + h^{ij}_{(\text{u})}(\bar{u}'_i\,,\ \bar{d}'_i)\begin{pmatrix}-\bar{\phi}^0\\ \phi^-\end{pmatrix} u_{j\text{R}} + \text{h.c.}, \tag{9.5}$$

where $i,\ j = 1,\ 2,\ 3$ and the Higgs fields are the same fields as those introduced in Chapters 5 and 7. The matrices $h^{ij}_{(\text{u})}$ and $h^{ij}_{(\text{d})}$ denote couplings of i and j quarks of the up and down types, respectively. The symmetry is broken by giving a vacuum expectation value to ϕ^0:

$$\phi = \begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix} \underset{\text{breaking}}{\longrightarrow} \phi = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v+\eta\end{pmatrix} \tag{9.6}$$

and

$$\phi_{\text{c}} = -\text{i}\tau_2\phi^* = \begin{pmatrix}-\bar{\phi}^0\\ \phi^-\end{pmatrix} \underset{\text{breaking}}{\longrightarrow} -\frac{1}{\sqrt{2}}\begin{pmatrix}v+\eta\\ 0\end{pmatrix}, \tag{9.7}$$

with η the field fluctuation around the minimum. Spontaneous breaking of the symmetry generates the mass terms

$$\mathcal{L}_{\text{mass}} = \frac{v}{\sqrt{2}}\left(\bar{u}'_{\text{L}i}h^{(\text{u})}_{ij}u'_{\text{R}j} + \bar{d}'_{\text{R}j} + \bar{d}'_{\text{L}i}h^{(\text{d})}_{ij}d'_{\text{R}j}\right) + \text{h.c.} \tag{9.8}$$

The expressions

$$M^{(\text{u})}_{ij} = \frac{v}{\sqrt{2}}h^{(\text{u})}_{ij} \qquad \text{and} \qquad M^{(\text{d})}_{ij} = \frac{v}{\sqrt{2}}h^{(\text{d})}_{ij} \tag{9.9}$$

occurring above are called the mass matrices. From the way they were introduced, there is no reason for them to be either symmetric or Hermitian. In fact the Lagrangian in (9.5) is manifestly gauge-invariant and the mass matrices are to a certain extent arbitrary. The mass matrices are very important because they determine the masses and the flavor mixing of the quarks.

The quark fields that have been investigated up to now are, as has already been mentioned, non-physical gauge eigenstates. To find the physical or mass eigenstates, we must transform the quark-mass matrices into diagonal form.

Any square matrix can be diagonalized by a bi-unitary transformation. Therefore it is always possible to find four matrices $\text{U}_{\text{L,R}}$ and $\text{D}_{\text{L,R}}$ that diagonalize the mass

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matrices,

$$\mathcal{M}^{(\mathrm{u})} = U_{\mathrm{L}}^{+} M^{(\mathrm{u})} U_{\mathrm{R}} = \begin{pmatrix} m_{\mathrm{u}} & 0 & 0 \\ 0 & m_{\mathrm{c}} & 0 \\ 0 & 0 & m_{\mathrm{t}} \end{pmatrix}, \tag{9.10}$$

$$\mathcal{M}^{(\mathrm{u})} = D_{\mathrm{L}}^{+} M^{(\mathrm{d})} D_{\mathrm{R}} = \begin{pmatrix} m_{\mathrm{d}} & 0 & 0 \\ 0 & m_{\mathrm{s}} & 0 \\ 0 & 0 & m_{\mathrm{b}} \end{pmatrix}. \tag{9.11}$$

The mass eigenstates, to be denoted as unprimed fields, are related to the gauge eigenstates by the transformations

$$\begin{aligned} u_{\mathrm{L}i} &= (U_{\mathrm{L}}^{+})_{ij} u'_{\mathrm{L}j}, \quad u_{\mathrm{R}i} = (U_{\mathrm{R}}^{+})_{ij} u'_{\mathrm{R}j}, \\ d_{\mathrm{L}i} &= (D_{\mathrm{L}}^{+})_{ij} d'_{\mathrm{L}j}, \quad d_{\mathrm{R}i} = (D_{\mathrm{R}}^{+})_{ij} d'_{\mathrm{R}j}. \end{aligned} \tag{9.12}$$

In terms of the mass eigenstates the mass term is now diagonal. Therefore we can substitute the physical fields everywhere in the Lagrangian and deduce the physical couplings. The neutral couplings expressed in terms of physical quarks retain the same form as they had with gauge quarks. The charge current after the substitution becomes

$$j_{\mu}^{+} = \bar{u}'_{\mathrm{L}i} \gamma_{\mu} d'_{\mathrm{L}i} = \bar{u}_{\mathrm{L}i} \gamma_{\mu} V_{ij} d_{\mathrm{L}j} \tag{9.13}$$

and the charged-current interaction is

$$\mathcal{L}_{\mathrm{cc}} = \frac{g}{\sqrt{2}} \left(j_{\mu}^{+} W_{\mu}^{-} + j_{\mu}^{-} W_{\mu}^{+} \right), \tag{9.14}$$

where $V = U_{\mathrm{L}}^{+} D_{\mathrm{L}}$ and summation over repeated indices is understood. This matrix is one of the most important quantities in the standard model, because it contains information on all possible flavor-transitions and CP violation. It is called the flavor-mixing matrix or the Cabibbo (1963)–Kobayashi–Maskawa (1973) (CKM) matrix.

By construction the flavor matrix is unitary, a property that will be used extensively in the next section. The mixing matrix is derived directly from the mass matrices, which shows that all information about it is included in the mass matrices. A determination of the mass matrices from experimental data is impossible because they contain 36 real parameters (9 complex numbers for each charge sector). By contrast, there are only ten quantities that can be determined by experiment: six quark masses and four independent mixing parameters. The fact that only four parameters of the mixing matrix are relevant can be understood as follows: a unitary $N \times N$ matrix may be expressed by N^2 real parameters. Among them $N(N-1)/2$ can be chosen to be the rotation angles of an orthogonal matrix and the remaining

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$N(N+1)/2$ taken as phase angles. Not all phases, however, are physical. Each quark field has an arbitrary phase that can be used to eliminate a phase of the CKM matrix, except for an overall phase. This means that one can arrange the phases of the quark fields in such a way that they eliminate phases in V_{ij} of Eq. (9.14). An exception to this rule is an overall phase that is lost when we square matrix elements in order to produce probabilities. Thus $N \times N$ flavor mixing can be parametrized by $N(N-1)/2$ rotation angles and

$$N(N+1)/2 - (2N-1) = (N-1)(N-2)/2$$

phase angles. For three quark generations ($N = 3$) there remain three rotation angles and one phase, which is responsible for CP violation. So in all there are four parameters that describe the mixing matrix.

The unitarity of the mixing matrix that is required in gauge theories is a consequence of the unitarity of the matrices U_L and D_L:

$$V_{ik}(V_{kj})^{+} = V_{ik}V_{jk}^{*} = \delta_{ij}. \tag{9.15}$$

This relation expresses the orthogonality of rows and columns within the matrix.

As was shown above, the mixing matrix can be parametrized with four quantities, but this does not determine the functional form of the matrix. The first explicit parametrization was given by Kobayashi and Maskawa. They used Euler-type angles for three-dimensional rotations in the flavor space and one phase:

$$V_{\mathrm{KM}} = \begin{pmatrix} c_1 & -s_1 c_3 & -s_1 s_3 \\ s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 \mathrm{e}^{\mathrm{i}\delta} & c_1 c_2 s_3 + s_2 c_3 \mathrm{e}^{\mathrm{i}\delta} \\ s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 \mathrm{e}^{\mathrm{i}\delta} & c_1 s_2 s_3 - c_2 c_3 \mathrm{e}^{\mathrm{i}\delta} \end{pmatrix}, \tag{9.16}$$

where the abbreviations $s_i = \sin\theta_i$ and $c_i = \cos\theta_i$ are used. The parameters are chosen so that, for $\theta_2 = \theta_3 = \delta = 0$, the three-dimensional mixing matrix is reduced to the corresponding one for just two doublets. The angles θ_i may without loss of generality be chosen to lie in the first quadrant, $0 \leq \theta_i < \pi/2$. The phase angle δ may take any value within the interval $[-\pi, \pi]$.

This parametrization is just one possibility. Another one that is very customary was given by Maiani. It is quite suitable for investigations of B-meson decays:

$$V_{\mathrm{M}} = \begin{pmatrix} c_\beta c_\theta & c_\beta s_\theta & s_\beta \\ -s_\beta s_\gamma c_\theta \mathrm{e}^{\mathrm{i}\delta'} - s_\theta c_\gamma & c_\gamma c_\theta - s_\beta s_\gamma s_\theta \mathrm{e}^{\mathrm{i}\delta'} & s_\gamma c_\beta \mathrm{e}^{\mathrm{i}\delta'} \\ -s_\beta c_\gamma c_\theta + s_\gamma s_\theta \mathrm{e}^{-\mathrm{i}\delta'} & -s_\beta s_\theta c_\gamma - s_\gamma c_\theta \mathrm{e}^{-\mathrm{i}\delta'} & c_\beta c_\gamma \end{pmatrix}. \tag{9.17}$$

The quantity s_γ is mainly the coupling for $\mathrm{b} \to \mathrm{c}$ and s_β is mainly for $\mathrm{b} \to \mathrm{u}$. Ranges for angles and the phase can be chosen as in the Kobayashi–Maskawa parametrization.

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A further parametrization is the one by Wolfenstein. In this case the elements are expanded in terms of a small parameter $\lambda = \sin\theta_c$ exploiting the experimental information about the smallness of the mixing angles. The structure of the matrix is determined by the unitary conditions of the mixing matrix:

$$V_W = \begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta + i\eta\frac{1}{2}\lambda^2) \\ -\lambda & 1-\frac{1}{2}\lambda^2 - i\eta A^2\lambda^4 & A\lambda^2(1+i\eta\lambda^2) \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix}. \tag{9.18}$$

In contrast to the parametrizations discussed so far, the one by Wolfenstein is only an approximation, being an expansion in a small parameter. The unitarity condition is satisfied to a given order in λ. The real parts of the elements are correct to the order λ^3 and the imaginary parts to order λ^5. The parameters A, η, and ρ are of order unity or even smaller:

$$\lambda = 0.221 \pm 0.002, \quad A = 1.0 \pm 0.1, \quad \sqrt{\rho^2+\eta^2} = 0.46 \pm 0.23. \tag{9.19}$$

Magnitudes for elements of the mixing matrix are determined directly from experiments. They in turn are translated into values for the rotation angles. We will review the experiments and the corresponding values. It is much more difficult to obtain values for δ, since it is related to CP-violating quantities. We shall return to its determination in Chapters 15 and 16.

9.2 Flavor-changing neutral couplings (FCNCs)

The structure of the mixing matrix has another important consequence. The neutral couplings of the theory preserve flavor to a large degree of accuracy. The suppression of flavor-changing neutral couplings is required by many experimental results. For instance, the decay $K_L \to \mu^+\mu^-$ is highly suppressed. The branching ratio is

$$\mathrm{Br}(K_L \to \mu^+\mu^-) = (7.2 \pm 0.2) \times 10^{-9}.$$

The $D^0\bar{D}^0$ mixing has not been observed at the 10^{-3} level. These properties and others are incorporated into the theory by the construction described in Section 9.1. In fact, FCNCs are absent at the tree level. The proof of this follows from the structure of neutral currents. We can begin again with the Lagrangian in Eq. (8.10) and substitute the ψs with quark fields. The couplings of the quarks to the Z and γ are

$$\mathcal{L}_{nc} = -e\sum_{i=1}^{3}\bar{q}^{\,i}Q\gamma^\mu q_i A_\mu + \frac{g}{c}\sum_{l=1}^{3}\{\bar{q}^{\,i}\tau_3\gamma^\mu q_i - s^2\bar{q}^i Q\gamma^\mu q_i\}Z_\mu. \tag{9.20}$$

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Table 9.1. *Couplings of quarks and leptons to* Z^0

States	g_V	g_A
Up quarks	$\frac{1}{2} - \frac{4}{3}\sin^2\theta_W$	$\frac{1}{2}$
Down quarks	$-\frac{1}{2} + \frac{2}{3}\sin^2\theta_W$	$-\frac{1}{2}$
Neutrinos	$\frac{1}{2}$	$\frac{1}{2}$
Charged leptons	$-\frac{1}{2} + 2\sin^2\theta_W$	$-\frac{1}{2}$

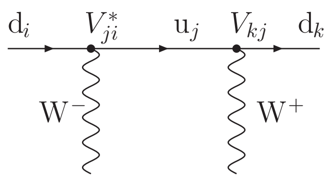

Figure 9.1. Vertices appearing in box diagrams.

At first sight the quarks occurring in (9.20) should have a prime, since they are still gauge quarks, but Eq. (9.20) is diagonal in the quark fields and the unitary matrices $U_{L,R}$ and $V_{L,R}$ will disappear when the quark fields are replaced by physical states. Thus the omission of the prime is justified.

Next we introduce a convenient notation and write the neutral-current couplings in the form

$$\mathcal{L}_{nc} = \frac{g}{2\sqrt{2}\,c}\bar{q}^i\gamma^\mu(g_V - g_A\gamma_5)q^i Z_\mu, \tag{9.21}$$

with q^i representing general states of up and down quarks or leptons. The couplings are given in Table 9.1, where we also include the neutral couplings to neutrinos and leptons. In this way neutral couplings are diagonal at the tree level.

The suppression that has been introduced so far is not sufficient. Flavor-changing effects will now appear through higher-order corrections that involve charged currents. Higher-order effects are $O(G\alpha)$ and this suppression is not sufficient. However, the method introduced so far suppresses FCNC to the level $O(G\alpha m_q^2/M_W^2)$, where m_q is the mass of the quark in the intermediate state. We can see this by considering the upper line of the box diagram shown in Fig. 9.1. The V_{KM} matrix elements which occur in this line include a mass-independent term,

$$V^*_{ij}V_{kj} = V_{kj}V^+_{ji} = \delta_{ki}, \tag{9.22}$$

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and a mass-dependent term

$$V_{ij}^* V_{kj} \frac{m_{qj}}{M_W} = V_{kj} V_{ji}^+ \frac{m_{qj}}{M_W}. \tag{9.23}$$

The leading term vanishes by virtue of the unitarity of the mixing matrix and the next term is proportional to the mass of the intermediate quark. This is the famous Glashow–Iliopoulos–Maiani cancellation scheme (Glashow *et al.*, 1970). Processes involving quarks in intermediate states, which are light relative to M_W, give a very small contribution. The mechanism has important consequences for box and penguin diagrams. These diagrams occur for K^0–$\bar{K}^0$ mixing and the ε_K parameter.

The above requirements for flavor conservation in neutral couplings determines to a large extent the representation assignment of the fermion fields. There is a general theorem, which states that, for a gauge theory based on the group SU(2) $\times$ U(1), the bounds of FCNC are satisfied if we classify (Paschos, 1977; Glashow and Weinberg, 1977) the quarks into representations of the group in such a way that quarks of the same charge and the same helicity have the same T (total weak isospin) and T_3 (third component of isospin). For quarks of only two charges $(2/3, \ -1/3)$ it implies that there must be equal numbers of up and down quarks.

We illustrate the implications of the result with some examples.

Example 1 Models with three quarks are not allowed, since they will produce strangeness-changing neutral currents. To solve this problem Glashow, Iliopoulos, and Maiani (Glashow *et al.*, 1970) introduced a charmed quark. The matrix M is

$$M = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{9.24}$$

and the charged current

$$J_\mu^+ = \left(\bar{u}\,\bar{c}\,\bar{d}\,\bar{s}\right) \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ c \\ d \\ s \end{pmatrix}. \tag{9.25}$$

Example 2 Models with five quarks, u, c, d, s, and b, produce flavor-changing couplings. To eliminate these couplings the top quark was introduced. The charged-current interactions are now described by Eqs. (9.13) and (9.14).

9.3 The elements of the mixing matrix

There are many processes that determine values for the elements of the mixing matrix. They involve products of the weak couplings times hadronic matrix

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elements. Estimates of the latter require methods of strong interactions as they apply to low or high energies. For the sake of brevity, the description of hadronic methods given here is short. The aim is to give a general impression of the methods and arrive quickly at the relevant numerical results. The interested student will find more details in the references or in the following chapters, especially Chapters 11–14.

9.3.1 Determination of V_{ud}

For the weak interaction we introduced in Chapter 2 the Fermi coupling constant G_{F}, which is related to the SU(2) coupling by Eq. (8.14). It was also mentioned there that its numerical value is determined by the muon lifetime. To obtain a precise value for G_{F} it is necessary to include radiative corrections from the exchange of photons and gauge bosons, as well as the emission of photons. Such diagrams in general introduce infinities, which must be treated with special care. The electroweak theory is renormalizable and the infinities can be absorbed into a few coupling constants. In this book we do not cover the method of renormalization, but refer to an article and a book (Sirlin, 1978; Bardin and Passarino, 1999). The precise value for the Fermi coupling constant is

$$G_{\mu} = (1.166\,32 \pm 0.000\,04) \cdot 10^{-5}\,\mathrm{GeV}^{-2}, \tag{9.26}$$

with the subscript indicating that it is obtained from the muon lifetime.

The method which was used to construct the hadronic Lagrangian requires that the charged currents for quarks have the same coupling constant multiplied by the quark mixing matrix, as is seen in Eq. (9.14). This property is called universality. Thus the V_{ud} coupling is given by the ratio of the coupling constant measured in β-decay, to be denoted by G_V, to the muon decay constant:

$$V_{\mathrm{ud}} = \frac{G_V}{G_{\mu}}. \tag{9.27}$$

The most accurate experiments for β-decay, so far, were done in nuclei and involve $0^+ \to 0^+$ transitions, also known as superallowed transitions. Their measurements and analyses have a long history. Precise determination of G_V must include radiative and in addition nuclear corrections. It is beyond the scope of this chapter to describe the corrections in detail. The result of the analyses is a very precise value,

$$V_{\mathrm{ud}} = 0.9740 \pm 0.0003 \pm 0.0015, \tag{9.28}$$

where the first error is statistical and the second represents the theoretical uncertainty.

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There are two other elementary transitions that are also relevant. The first is pion β-decay,

$$\pi^+ \to \pi^0 + e^+ + \nu.$$

The branching ratio for this decay has been measured to be

$$\mathrm{Br}\big(\pi^+ \to \pi^0 + e^+ + \nu\big) = (1.025 \pm 0.034) \times 10^{-8}, \tag{9.29}$$

which has a 3% error and is not as accurate as ratios from nuclear β-decays. The determination of V_{ud} from this decay has no nuclear corrections but carries a larger statistical error,

$$V_{\mathrm{ud}} = 0.965 \pm 0.016. \tag{9.30}$$

The second elementary transition is the decay of neutrons: $n \to p + e^- + \bar{\nu}$. This decay depends both on the vector current and on the axial current, in contrast to the previous two cases, to which only the vector current contributes. Precise measurements of the neutron lifetime,

$$\tau_{\mathrm{n}} = 888.5 \pm 0.8\,\mathrm{s}, \tag{9.31}$$

give the value

$$V_{\mathrm{ud}} = 0.9801 \pm 0.0030. \tag{9.32}$$

We note that the three determinations are consistent with each other. The most accurate one from the superallowed nuclear transitions will be used later on.

9.3.2 Determination of V_{us}

There are several ways to determine V_{us}, among which the K_{l_3} decays are the cleanest. Hyperon decays give values that are almost as accurate. For the K-meson decays we use the reactions

$$K_L^0 \to \pi^- e^+ \nu_e \quad \text{and} \quad K^+ \to \pi^0 e^+ \nu_e.$$

The transition is from a pseudoscalar to another pseudoscalar particle and only the vector current contributes. Its matrix element can be written as

$$\langle \pi(p')|J_\mu|K(p)\rangle = C\big[\big(p_\mu + p'_\mu\big) f_+(q^2) + \big(p_\mu - p'_\mu\big) f_-(q^2)\big], \tag{9.33}$$

where $q^2 = (p - p')^2$, C is an isospin Clebsch–Gordan coefficient, and $f_\pm(q^2)$ are the form factors. In the SU(3)-symmetry limit at $q^2 = 0$ the form factor is known: $f_+(0) = 1$. Corrections to this value were computed to account for

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symmetry-breaking effects. Then the width

$$\Gamma = \frac{G_\mu^2 M_K^5}{192\pi^3} C^2 |f_+(0)|^2 |V_{us}|^2 \tag{9.34}$$

determines V_{us}. The result including all corrections is

$$V_{us} = 0.220 \pm 0.002. \tag{9.35}$$

The analysis of hyperon decays studies the decays of many hyperons, for which a χ^2 fit is performed. They involve form factors of both vector and axial currents, which are assumed to satisfy the SU(3) symmetry. The general fit is an impressive success of the SU(3) symmetry since there is a single value V_{us} consistent with all the hyperon data and the above value. It carries a slightly larger error arising from the theoretical uncertainties.

9.3.3 Determination of $|V_{cd}|$ and $|V_{cs}|$

One way to obtain the couplings $|V_{cd}|$ and $|V_{cs}|$ is to study the production of charmed particles in deep inelastic neutrino–nucleon scattering. The particles produced decay semileptonically and appear as events with opposite-sign dimuons. The elementary interactions are

$$\begin{aligned}\nu + d \to \mu^- + {}& c \\ & \searrow \mu^+ + \nu + s,\end{aligned} \tag{9.36}$$

$$\begin{aligned}\bar{\nu} + \bar{d} \to \mu^+ + {}& \bar{c} \\ & \searrow \mu^- + \bar{\nu} + \bar{s}.\end{aligned} \tag{9.37}$$

The semileptonic decays involve a mixture of charmed particles whose branching ratio is taken as $B_e = 7.1 \pm 1.3\%$.

Thus it remains to compute the production rates for charm quarks, which are discussed in Chapter 11. Here we mention that the original reactions are computed in the parton model as follows

$$\sigma(\nu N \to cX) = \frac{G^2 ME}{\pi}\left[r_d(U + D)|V_{cd}|^2 + 2r_s S|V_{cs}|^2\right], \tag{9.38}$$

$$\sigma(\bar{\nu} N \to \bar{c}X) = \frac{G^2 ME}{\pi}\left[r_{\bar{d}}(\bar{U} + \bar{D})|V_{cd}|^2 + 2r_{\bar{s}}\bar{S}|V_{cs}|^2\right]. \tag{9.39}$$

The r-coefficients measure the suppressions in the production of charmed quarks due to phase-space restrictions. Estimates for the experiments at energies $E_\nu =$ 220 GeV and $E_{\bar{\nu}} = 150$ GeV gave the values

$$\begin{aligned} r_d\,(220\text{ GeV}) &= 0.91, & r_s\,(220\text{ GeV}) &= 0.72, \\ r_{\bar{d}}\,(150\text{ GeV}) &= 0.70, & r_{\bar{s}}\,(150\text{ GeV}) &= 0.66. \end{aligned} \tag{9.40}$$

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The capital letters $U, D, \ldots$ denote integrals of the quark distribution functions in the proton. They are extracted from data on high-energy neutrino–nucleon scattering.

In Eqs. (9.38) and (9.39) we have two equations with two unknowns and thus we can solve for V_{cd} and V_{cs}. The results are

$$|V_{\mathrm{cd}}| = 0.22 \pm 0.03, \tag{9.41}$$

$$|V_{\mathrm{cs}}| \geq 0.75. \tag{9.42}$$

Another source of information on $|V_{\mathrm{cs}}|$ is the semileptonic D-meson decays. They are proportional to $f_+^{\mathrm{D}\to\mathrm{K}}(q^2)|V_{\mathrm{cs}}|$ and require estimates of the form factor $f_+^{\mathrm{D}\to\mathrm{K}}(q^2)$.

The values for $|V_{\mathrm{cs}}|$ obtained by these methods have large errors due to theoretical uncertainties. These values have been superseded by measurements of W decays to identified charmed hadrons and the subsequent decays. W bosons decay to the pairs $(\mathrm{u}\bar{\mathrm{q}})$ and $(\mathrm{c}\bar{\mathrm{q}})$ with $\bar{\mathrm{q}} = \bar{\mathrm{d}}, \bar{\mathrm{s}}, \bar{\mathrm{b}}$ antiquarks. The sum of the squares of the couplings for the six decays should add up to the value of 2. Since five of the six couplings are well measured or they are very small, LEP measurements can be converted into a precise value of

$$|V_{\mathrm{cs}}| = 0.996 \pm 0.016.$$

Without the use of unitarity the central value from all measurements is consistent, with 0.97 ± 0.10. With these values the upper-left-hand corner of the CKM matrix is known to a high degree of accuracy.

9.3.4 B-Meson decays and the determination of V_{cb} *and* V_{ub}

The relatively long-lived B mesons made possible the determination of two more elements in the mixing matrix, V_{cb} and V_{ub}. The decays of the B mesons proceed in the spectator model with the decay of b quarks into c and u quarks. The total width is the incoherent sum of the contributions from the above two decays, corrected, of course, for the exchange of gluons as described by QCD. The method accounts for the semileptonic and non-leptonic decays. Both decays were used in determining V_{cb}, while the semileptonic spectrum is used for constraining the element V_{ub}. In these estimates theoretical uncertainties enter the calculation and we shall discuss them in some detail.

The total width is given by

$$\Gamma_{\mathrm{tot}} = \Gamma_0(r|V_{\mathrm{ub}}|^2 + s|V_{\mathrm{cb}}|^2), \tag{9.43}$$

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where r and s are products of phase space, color factors, and QCD corrections, and

$$\Gamma_0 = \frac{G^2 m_\mathrm{b}^5}{192\pi^3}. \tag{9.44}$$

There are theoretical uncertainties for $r,\ s$, and Γ_0. For instance, the factor Γ_0 is sensitive to the b-quark mass:

$$\frac{1}{\Gamma^0} = \begin{cases} 0.93 \times 10^{-14}\,\mathrm{s} & \text{for} \quad m_\mathrm{b} = 5.00\,\mathrm{GeV}, \\ 1.22 \times 10^{-14}\,\mathrm{s} & \text{for} \quad m_\mathrm{b} = 4.75\,\mathrm{GeV}. \end{cases} \tag{9.45}$$

The spectator and parton models were used for analyzing, along these lines, the lepton spectra of semileptonic decays. A consistent analysis determines two more matrix elements, V_cb and V_ub, with ≈20% error.

An alternative method considers exclusive B decays in the heavy-quark effective theory (HQET). This is a systematic expansion in inverse powers of the heavy-quark mass. When the mass of the heavy quark is taken to infinity, the decays $\mathrm{B} \to \mathrm{D}^*\ell\bar{\nu}$ and $\mathrm{B} \to \mathrm{D}\ell\nu$ become equal. Eperimentally the two branching ratios are different, so corrections of $\mathcal{O}(1/m_\mathrm{q})$ must be included. Consequently, specific final states are selected to determine

$$|V_\mathrm{cb}| = 0.041 \pm 0.002 \tag{9.46}$$

and

$$|V_\mathrm{ub}| = 0.004 \pm 0.001. \tag{9.47}$$

Some details for the calculations are included in Section 14.5. Finally, the discovery of the top quark was achieved by observing semileptonic decays that provide an approximate estimate of $|V_\mathrm{tb}|$.

With the B-meson decays we close the discussion concerning the elements of the CKM matrix, of which six elements are directly determined by experiments. Values for the remaining three elements, involving couplings of the top quark, are deduced from the unitarity of the matrix.

9.3.5 Summary and unitarity

In this chapter we were able to derive accurate values for the matrix elements from tree-level constraints. We emphasize that we can determine only their magnitudes, not their relative phases. In summary,

$$\begin{aligned} &|V_\mathrm{ud}| = 0.9740 \pm 0.0020, \quad && |V_\mathrm{us}| = 0.220 \pm 0.002, \\ &|V_\mathrm{cd}| = 0.22 \pm 0.03, && |V_\mathrm{cs}| = 0.97 \pm 0.10, \\ &|V_\mathrm{cb}| = 0.041 \pm 0.002, && |V_\mathrm{ub}| = 0.004 \pm 0.001. \end{aligned} \tag{9.48}$$

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The unitarity of the mixing matrix restricts the matrix elements even further. The tree constraints together with unitarity give the following ranges reported by the Particle Data Group (Gilman *et al.*, 2002):

$$|V_{ij}| = \begin{pmatrix} 0.9741\text{–}0.9756 & 0.219\text{–}0.226 & 0.0025\text{–}0.0048 \\ 0.219\text{–}0.226 & 0.9732\text{–}0.9748 & 0.0038\text{–}0.004 \\ 0.004\text{–}0.014 & 0.037\text{–}0.044 & 0.9990\text{–}0.9993 \end{pmatrix}. \tag{9.49}$$

The small values for many of the elements justify the small-angle approximation and the Wolfenstein parametrization described in Section 9.1. Similarly, one obtains values for the other parametrizations as was done in Eq. (9.19). The phase δ is still undetermined. Its determination requires measurements of the CP parameters, which we postpone until Chapters 15 and 16.

We are now in a position to test the unitarity of the mixing matrix. There are two types of constraints.

(i) The sum of the squares of absolute values of the elements for each row or each column must sum up to unity. This can be tested for the first row,

$$|V_{\rm ud}|^2 + |V_{\rm us}|^2 + |V_{\rm ub}|^2 = 0.9970 \pm 0.0036, \tag{9.50}$$

which is consistent with unity. The radiative corrections for the $V_{\rm ud}$ element are very crucial because without them the right-hand side in (9.50) would be greater than unity, in fact $\sum_i |V_{{\rm u}i}|^2 = 1.020 \pm 0.004$.

(ii) A convenient and pictorial way to summarize the content of the CKM matrix is in terms of unitarity triangles. Consider the entries of each row or column of the matrix as the components of a vector. Then the unitarity condition applied to any two columns is the dot product of one column with the complex conjugate of another column. For the first and third columns the condition yields

$$V_{\rm ud}V_{\rm ub}^* + V_{\rm cd}V_{\rm cb}^* + V_{\rm td}V_{\rm tb}^* = 0. \tag{9.51}$$

The unitarity triangle is a geometrical representation of this equation in the complex plane. Each term in the equation is proportional to $A\lambda^3$ and, to leading order,

$$V_{\rm cd}V_{\rm cb}^* \approx -A\lambda^3, \qquad V_{\rm ud}V_{\rm ub}^* \approx A\lambda^3(\rho + {\rm i}\eta), \quad \text{and} \quad V_{\rm td}V_{\rm tb}^* \approx A\lambda^3(1-\rho-{\rm i}\eta). \tag{9.52}$$

We can choose to orient the triangle so that $V_{\rm cd}\,V_{\rm cb}^*$ lies on the x-axis and scale out the common factor $A\lambda^3$ which is of order 1%. Now the coordinates for the vertices are shown in Fig. 9.2. The angles α, β, and γ of the triangle are also referred to as ϕ_2, ϕ_1, and ϕ_3, respectively. It is evident from the construction of the triangle that β and γ are the phases of the elements $V_{\rm td}$ and $V_{\rm ub}$, respectively:

$$V_{\rm td} = |V_{\rm td}|{\rm e}^{-{\rm i}\beta}, \quad V_{\rm ub} = |V_{\rm ub}|{\rm e}^{-{\rm i}\gamma}. \tag{9.53}$$

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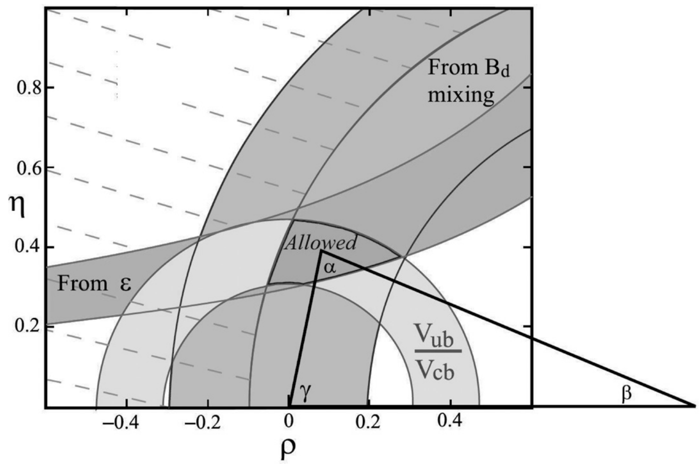

Figure 9.2. The unitarity triangle.

Once $A\lambda^3$ is factored out, the triangle depends on ρ and η. Let us select the x-axis to be ρ and the y-axis η. The shape of the triangle is now determined by three measurements. The first constraint comes from the magnitude of $V_{\rm ub}$ which determines a ring centered at the origin. The CP parameter $\varepsilon_{\rm K}$ determines a second region. Finally, the mass difference of the $\rm B_d$ mesons defines another ring; this time its center is at $\rho = 1$ (see Eqs. (15.53), (16.26) and (16.27)). The three regions are shown in Fig. 9.2, where their intersection defines the apex of the triangle. All additional measurements that depend on parameters of the triangle must reproduce the unitarity triangle (see Section 16.5).

We have mentioned already that CP violation is attributed to the phase in the CKM matrix. Quantitative predictions for CP asymmetries always contain

$$s_1 s_2 s_3 \sin\delta \qquad \text{or} \qquad s_\beta s_\gamma s_\theta \sin\delta' \tag{9.54}$$

as a multiplicative factor. For three generations of quarks there is a rephasing-invariant measure of CP violation. In terms of the elements, it is given by

$$J_{i\alpha} \equiv \mathrm{Im}\left\{V_{j\beta}V_{k\gamma}(V_{j\gamma}V_{k\beta})^*\right\}, \tag{9.55}$$

where i, j, k and α, β, γ are cyclic permutations of 1, 2, 3, i.e. once we give numerical values to i and α, the other indices are determined (Jarlskog, 1985). There are nine such invariants, which are all equal to each other. Their explicit form in the Maiani parametrization is

$$J_{i\alpha} \approx \beta\gamma s_\theta \sin\delta'. \tag{9.56}$$

For the central values of the angles

$$J_{i\alpha} \approx 2.5 \times 10^{-4} s_\theta \sin\delta'. \tag{9.57}$$

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The smallness of this quantity implies that CP parameters will in general be small. There are also exceptions to this rule, which happen when the CP-violating quantity, which is proportional to $J_{i\alpha}$, is divided by another small quantity. It is evident from this discussion that the CP asymmetries manifest themselves in two ways:

(i) processes in which the rates are large have small asymmetries; and
(ii) large asymmetries occur for observables when the branching ratios are small.

Both situations appear in K- and B-meson decays, for which CP asymmetries have been observed.

Beyond the estimates of CKM elements discussed in this chapter, there are additional limits from observations related to loop diagrams. The theoretical analyses are now more complicated and involve additional theoretical assumptions. The advantage, however, is that they investigate the quantum nature of the theory and lead to a consistent picture. In fact, there are additional checks for the angles (CP phases) of the unitarity triangle. We shall cover several of these exciting topics in later chapters of the book.

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