ON FINITE COLLINEATION GROUPS OF F_5

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Our aim in this paper is to fill one gap left in (**1**) and to prove that if H is a finite collineation group of F_5, the free plane generated by a finite open configuration of rank 9, then $|H| \leqq 12$. Alltop has shown that $|H| \leqq 24$ and that there exist finite collineation groups of F_5 which have order 12, so that the argument in this paper shows that $|H| \leqq 12$ is the best estimate which can be given. In (**1**), Alltop has completely settled the question for F_n, $n \neq 5$. The notation of this paper will generally be that of (**1**).

Most of the arguments used here will consist of case analyses of degenerate planes of ranks 7 and 8 and will be sketched rather than given in detail. The one theorem of some interest in this paper, other than the main result, is the following theorem which yields some information about the collineation group of F_4 (π^2 in the notation of (**4**)).

THEOREM 1. *Let F_4 be generated by the four points (or the four lines), A, B, C, D, and let H be the collineation group of F_4 generated by extending to collineations of F_4 the 24 elements of the symmetric group on A, B, C, and D. Then there are no fixed elements in F_4 under H.*

Proof. Observe first that the elements of H are stage-preserving as collineations of F_4, and that H is transitive on the four elements of stage 0, the six elements of stage 1, the three elements of stage 2, the three elements of stage 3, and the six elements of stage 4. If there is a fixed element under H, let us assume for definiteness that a line, L, is a fixed element of minimal stage, $n \geqq 4$. Then L has two predecessors, say Q_1 and Q_2, which must constitue an orbit (under H) since neither can be fixed, while L is. Thus, we can conclude, since H is stage-preserving, that Q_1 and Q_2 are of stage $n - 1$. Similarly, Q_1 has two predecessors, L_1 and L_2, while Q_2 has two predecessors, L_3 and L_4. Now, since Q_1 and Q_2 are interchanged by H, the orbit of L_1 must include at least one of L_3 and L_4, say L_3. Assume that the orbit of L_1 does not include L_4. Then, L_1 and L_3 form one orbit, and L_2 and L_4 another, and the point $Q = L_1 \cap L_3$ is fixed under H and has stage less than or equal to $n - 1$, a contradiction. Thus, we have shown that L_1, L_2, L_3, and L_4 constitute an orbit under H; therefore, each L_i has stage $n - 2$, and we have the situation of Figure 1. Next, let H' be the permutation group on $\{L_1, L_2, L_3, L_4\}$ induced by H. Then H' must be a homomorphic image of H. But H' can have no elements of odd order (since

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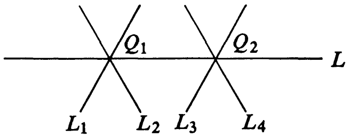

FIGURE 1

Q_1 and Q_2 are fixed or interchanged), so $|H'| = 4$ or 8. However, neither of these cases is possible since S_4 has no normal subgroups of order 6 or 3.

It is apparent that the proof of this theorem also suffices to prove the following slightly more general theorem.

THEOREM 1'. *Let F_{2n} be the free plane generated by $m = \frac{1}{2}(2n + 4)$ points (lines), and let H be the collineation group induced on F_{2n} by S_m, the symmetric group on the m points (lines). Then there are no fixed points in F_{2n} under H.*

We proceed now to the proof of the main theorem of this paper.

THEOREM 2. *Let H be a finite collineation group of F_5, the free plane generated by an open configuration of rank* 9. *Then* $|H| \leqq 12$.

Proof. We know by (**1**, Theorems 5.1 and 5.2) that there exists a finite free generator, D, of F_5 such that $DH = D$, and D contains *no element of degree* 2 (i.e., incident with exactly two elements of D). Thus, H is a subgroup of the collineation group of D. Since D is a finite free generator of F_5, the results of (**2**) imply that $r(D)$, the rank of D, is 9. We claim now that D contains at least one element of degree 1. For if not, since $r(D)$ is odd, not all elements can be of degree 0, so there would have to be an element, z, of odd degree greater than or equal to 3. If there are no elements of degree 1 or 2, the "connected component" of D generated by z would be a confined configuration, which is not possible, since D is open. Let x be an element of degree 1 in D, and let $D' = D - \{x\}$. Then we know that $r(D') = 8$. Now, there are two possibilities: (a) D' is a degenerate incidence structure, and (b) D' is a non-degenerate incidence structure. We first consider case (a). We know that the free completion of D', $F(D')$, is a degenerate plane of rank 8, and these are listed in (**2**; **3**). The two possible types are shown in Figures 2 and 3.

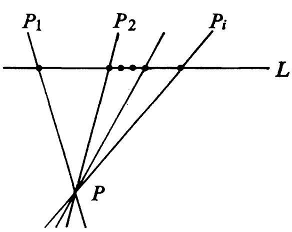

FIGURE 2

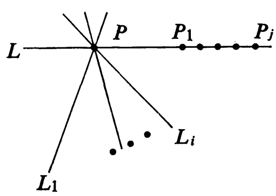

FIGURE 3

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Since $r(F(D')) = 8$, if $F(D')$ is of the first kind (Figure 2), then $i = 4$, and if $F(D')$ is of the second kind, then $i + j = 5$. Now, D' is a free contraction of $F(D')$, and $D = D' + \{x\}$ is a non-degenerate incidence structure of rank 9 with no element of degree 2, where x is an element of degree 1. By using the principle of duality, we may assume that x is a point. But this situation can arise only in a limited number of ways, and these can be enumerated and the collineation group of D determined for each case. If $F(D')$ is of the first kind, then $x \notin L$ (since D must be non-degenerate). Thus, one of the other lines, say $P \cdot P_1$, is in D' and $x \in P \cdot P_1$. However, if $L \in D$, we could not have $P_1 \in D$, for P_1 would be of degree 2. Since $P \cdot P_1$ cannot be of degree 2 in D, $P_1 \in D \Leftrightarrow P \in D$. If $P \in D$, then $L \notin D$, and two points, say, $P_3, P_4 \notin D$, $L \notin D$, and D is as in Figure 4.

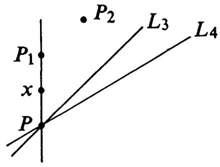

FIGURE 4

Clearly, the collineation group of $D, G(D)$, is of order 4. If $P_1 \notin D$, then $P \notin D$ and D can only be as in Figure 5.

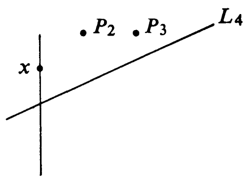

FIGURE 5

In this case, $|G(D)| = 2$. We next examine the situation where $F(D')$ is of the second kind (Figure 3). We know that $i + j = 5$, and it is easy to see that if $i \leqq 1$, or $j \leqq 1$, it is impossible to add an element of degree 1 and obtain a non-degenerate configuration. Thus, $i = 2$ or 3, and the point x cannot be adjoined to L. It is easy to see that this situation cannot occur, however, since if x is added to L_1, then L_1 will be of degree 2 and cannot be in D; therefore, x cannot have been removed from L_1. Thus, if $F(D')$ is of the first kind, $|G(D)| \leqq 4$, and $F(D')$ cannot be of the second kind, and we have disposed of case (a). We now assume that D' is non-degenerate, and again have two possibilities: (b_1) D' has an element y of degree 1, and (b_2) D' has no element of degree 1. In case (b_1), let $D'' = D' - \{y\}$, and we know that $r(D'') = 7$ and D'' must be degenerate. In this case, we must examine all possible structures obtained by adjoining two elements of degree 1 to a free contraction of a

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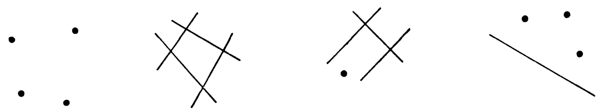
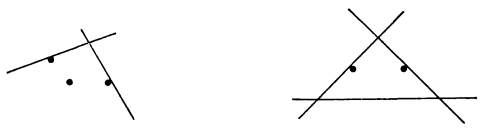
structure of rank 7, and determine the collineation groups of those resultant configurations which are non-degenerate and which have no element of degree 2. Again, we refer to Figures 2 and 3, where now in Figure 2, $i = 3$, and in Figure 3, $i + j = 4$, since $r(D'') = 7$. A case analysis is required, as above, but again it is true that every structure D obtained is degenerate or has $|G(D)| \leqq 12$. The final case is (b_2), where D' has no element of degree 1, and D' is non-degenerate, $r(D') = 8$. If x were a point, then the (unique) line L, of D' incident with x, must be of degree 0 or 2 in D'. If the degree of L is 0, then the stage of every element in D' is 0, and D' must be as in Figure 6. Otherwise, it is possible to take a free contraction of D' for at least one step. In fact, we can take a sequence of free contractions of D' until we have arrived at an open non-degenerate configuration of rank 8, E, with no elements of degree 2. Now, if there are no elements of degree 1 in E, then all elements are of degree 0 (otherwise, we could obtain a confined configuration) and E must be one of the types pictured in Figure 6.



(1) (2) (3) (4)

FIGURE 6

On the other hand, if E contains an element y of degree 1, then $E' = E - \{y\}$ has rank 7, and a simple case analysis shows that E must be one of the following two types:



(1) (2)

FIGURE 7

Now, by (**1**, Theorem 5.2), we know that $|G(E)| \geqq |G(D')|$. Also, since x is the unique element in D of degree 1, then x must be fixed by $G(D)$, and therefore $G(D) \cong H(D')$, where $H(D')$ is the subgroup of $G(D')$ fixing L, the unique element of D' incident with $x \in D$. Finally, if E is as in Figure 7, or Figure 6, numbers (3) and (4), then $|G(E)| \leqq 6$. Otherwise, $G(E) = S_4$, and $|G(E)| = 24$. Now, in this last case, Theorem 1 shows that $H(E)$ (the restriction of $H(D')$ to E) must be a *proper* subgroup of $G(E)$; therefore,

$$|H(E)| = |G(D)| \leqq 12,$$

as asserted in the theorem.

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