TWO EXPONENTIAL DIOPHANTINE EQUATIONS

by ERIK DOFS

(Received 17 April, 1996)

Introduction. In [3], two open problems were whether either of the diophantine equations

$$3n^2+3n+1=f^\omega, \qquad n^2+3n+9=f^\omega \tag{1}$$

with $n\in\mathbf{Z}$ and f a prime number, is solvable if $\omega>3$ and $3\nmid\omega$, but in this paper we allow f to be any (rational) integer and also $3\mid\omega$. Equations of this form and more general ones can effectively be solved [5] with an advanced method based on analytical results, but the search limits are usually of enormous size. Here both equations (1) are norm equations in $K(\sqrt{-3}):N(a+b\rho)=f^\omega$ with $\rho=(-1+\sqrt{-3})/2$, which makes it possible to treat them arithmetically.

The first problem. The equation $3n^2+3n+1=f^\omega$ is easily (in fact already) solved, for writing $x=3n+1$ reduces it to $x^2+x+1=3f^\omega$, and Nagell [4] proved a long time ago that this equation has no integer solutions with $\omega>2$, except the trivial cases with $|f|=1$. If $\omega=2$, the equation is Pellian and has infinitely many solutions.

The second problem. Cohn [1] showed that $x^2+2x+4=y^m$ $(m>2)$ has no solutions but the second equation (1) has probably not been solved earlier so we give a proof of the following theorem.

THEOREM. *The equation*

$$x^2+3x+9=y^m \tag{2}$$

has integer solutions if and only if $m=2$ *or* 3, *and the solutions are given by:*

$$x=-8,-3,0,5 \quad \textit{if} \quad m=2 \quad \textit{and} \quad x=-6,3 \quad \textit{if} \quad m=3.$$

Proof. It is sufficient to study the cases where m is a prime number and we first solve equation (2) for $m=2,3$: If $m=2$, it can be written as $(2x+3)^2+27=(2y)^2$ and clearly there is only a finite number of x-values that satisfy this. If $m=3$, we write it as $(x+6)^3-(x-3)^3=(3y)^3$ and then necessarily $x=-6,3$.

The remaining cases $m\equiv\pm1\pmod 6$ are considered now. We can assume $x\equiv\pm1\pmod 3$ as $x\equiv0\pmod 3$ in equation (2) leads to the impossible congruence $x_1^2+x_1+1\equiv0\pmod 9$ if $m>3$. Then $\gcd(x-3\rho,x-3\bar\rho)=1$ and we can write $x-3\rho=(a+b\rho)^m$ if $a+b\rho$ is chosen to be primary/semi-primary depending on $x\pmod 3$. By defining $u_m=\dfrac{(a+b\rho)^m-(a+b\bar\rho)^m}{(a+b\rho)-(a+b\bar\rho)}$, we find (from $x-3\rho=(a+b\rho)^m$)

$$u_m=-3/b, \tag{3}$$

i.e. $b=\pm3$, as $u_m\in\mathbf{Z}$ and $b\equiv0\pmod 3$. Further, by developing u_m we see that the sign of b is determined by $m\pmod 3): ma^{m-1}\equiv m\equiv-3/b\pmod 3)$.

https://doi.org/10.1017/S0017089500032122 Published online by Cambridge University Press

First assume that m is a prime number congruent to $1 \pmod 6$. Now $b=-3$ and equation (3) becomes

$$\binom{m}{1}a^{m-1}+\binom{m}{2}3a^{m-2}-\binom{m}{4}3^3a^{m-4}-\binom{m}{5}3^4a^{m-5}+\ldots-\binom{m}{m-2}3^{m-3}a^2+3^{m-1}=1 \quad (4)$$

where the coefficient of a^{m-k} is $A_k=\pm\binom{m}{k}3^{k-1}$ if $3\nmid k$ and $A_k=0$ if $3\mid k$. Assuming $m\equiv 1 \pmod{3^g}$ for some maximum g, we now prove that $3^{g+2}\mid A_k$ when $k\geq 4$. For $k=4$ this is clearly true. Now $3^g \parallel m-1$ and if $3^h \parallel k!$ then $h<k/2$, thus $3^{g+k-1-[k/2]}\mid A_k$, i.e. $3^{g+2}\mid A_k$ if $5\leq k<m$. Finally, if $k=m$ clearly $m\geq g+3$, and consequently $3^{g+2}\mid 3^{m-1}$. We have thus proved that $m\equiv 1 \pmod{3^g}$ implies

$$ma^{m-1}+\binom{m}{2}3a^{m-2}\equiv 1 \pmod{3^{g+2}}. \quad (5)$$

Writing (5) as $(m-1)a^{m-1}+(a^{m-1}-1)+\binom{m}{2}3a^{m-2}\equiv 0 \pmod{3^{g+2}}$ one sees that $3^g \parallel (m-1)a^{m-1}$ while 3^{g+1} divides both the other terms, which is impossible.

Next assume that m is a prime number congruent to $-1 \pmod 6$. Then however, equation (3) is unsolvable and one way of seeing this is to use the periodicity of $u_m \pmod 7$ [**2**]. The period is a factor of 6 as $(-3\mid 7)=1$ and for any a (and $b=3$), $u_m\equiv u_5\equiv 1, \pm 3 \pmod 7$, contradicting (3). Thus the equation $x^2+3x+9=y^m$ $(m>1)$ has integer solutions if and only if $m=2,3$.

Acknowledgement. The author is sincerely grateful to the referee for suggesting several simplications of the proof.

REFERENCES

1. J. H. E. Cohn, The Diophantine equation $x^2+3=y^n$, *Glasgow Math. J.* **35** (1993), 203–206.
2. J. H. E. Cohn, The Diophantine equation $x^2+C=y^n$, *Acta Arith.* **65** (1993), 367–381.
3. E. Dofs, On some classes of homogeneous ternary cubic diophantine equations, *Ark. Mat.* **13** (1975), 29–72.
4. T. Nagell, Des équations indéterminées $x^2+x+1=y^n$ et $x^2+x+1=3y^n$, *Norsk Mat. For. Skr. Series I* **2** (1921).
5. T. N. Shorey, A. J. van der Poorten, R. Tijdeman and A. Schinzel, Applications of the Gel'fond-Baker method to diophantine equations, *Transcendence theory: advances and applications* (Academic Press, 1977), 59–77.

Royal Institute of Technology
S-10044 Stockholm
Sweden