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High Reynolds number flow between torsionally oscillating disks

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In this paper the problem solved is that of unsteady flow of a viscous incompressible fluid between two parallel infinite disks, which are performing torsional oscillations about a common axis. The solution is restricted to high Reynolds numbers, and thus extends an earlier solution by Rosenblat for low Reynolds numbers.

The solution is obtained by the method of matched asymptotic expansions. In the main body of the fluid the flow is inviscid, but may be rotational, and in the boundary layers adjacent to the disks the non-linear convection terms are small. These two regions do not overlap, and it is found that in order to match the solutions a third region is required in which viscous diffusion is balanced by steady convection. The angular velocity is found to be non-zero only in the boundary layers adjacent to the disks.

1. Introduction

The problem considered here is that of incompressible viscous flow between two parallel infinite plane disks, which perform torsional oscillations about a common axis. The amplitude of oscillation, ε , is taken to be small, and the two disks oscillate with the same frequency, ω , and amplitude but π out of phase. An important parameter is the Reynolds number $R = \omega d^2/\nu$, where $2d$ is the distance between disks. The solution here is restricted to high Reynolds numbers.

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Rosenblat [1] has attacked this problem by expanding the solution in powers of ε . He has obtained expressions for the velocities for both large and small values of Reynolds number. However, as he points out, his work is subject to the limitation $\varepsilon R^{\frac{1}{2}} \ll 1$, and is strictly speaking, the limiting solution as $\varepsilon R^{\frac{1}{2}} \to 0$. Here we consider the case of $\varepsilon R^{\frac{1}{2}} \gg 1$, and the solution obtained here is the limiting solution as $\varepsilon R^{\frac{1}{2}} \to \infty$. As the Reynolds number becomes large, the flow near each disk approaches that of a single disk oscillating in an unbounded fluid. This problem was solved by Rosenblat [2], and later in an improved manner by Benney [3]. More recently Riley [4] pointed out an error in Rosenblat's paper [2], and considered large amplitude torsional oscillations of a single disk.

In Section 2 the equations are put into non-dimensional form appropriate to the various regions of the flow. In Sections 3, 4 and 5 the equations are solved in different regions, and the solution is completed by asymptotic matching procedures in Section 5. In Section 6 possible extensions of the work are briefly considered.

The dependent variables are written as functions of x, t, ε and $\lambda = \varepsilon R^{\frac{1}{2}}$, and are expanded as asymptotic series in λ

$$f(x,\ t,\ \varepsilon,\ \lambda) \sim \sum_{n=0}^{\infty} \lambda^{-n} f_n(x,\ t,\ \varepsilon)\ ,$$

and then f_n is expanded as a power series in ε . Except in Section 3 only the first term in the asymptotic series is considered. If no further terms are desired it would be possible to use an asymptotic series in $R^{\frac{1}{2}}$, namely

$$f \sim \sum_{n=0}^{\infty} R^{-\frac{1}{2}n} \tilde{f}_n(x,\ t,\ \varepsilon)\ .$$

However at one stage in the matching procedure (eq. 5.16) a coefficient λ^{-1} occurs, and if higher order terms are required the asymptotic series in λ is the natural one to use.

It is found that there are three distinct regions of flow, in each of which different physical effects dominate. These regions are

(i) a region occupying the main body of the fluid, denoted by I in fig. 1;

(ii) boundary layer regions, of thickness $O\left(R^{-\frac{1}{2}}\right)$, adjacent to the disks, denoted by II in fig. 1;

(iii) an intermediate region, of thickness $O\left(\varepsilon^{-1}R^{-\frac{1}{2}}\right)$, between regions I and II , denoted by III in fig. 1.

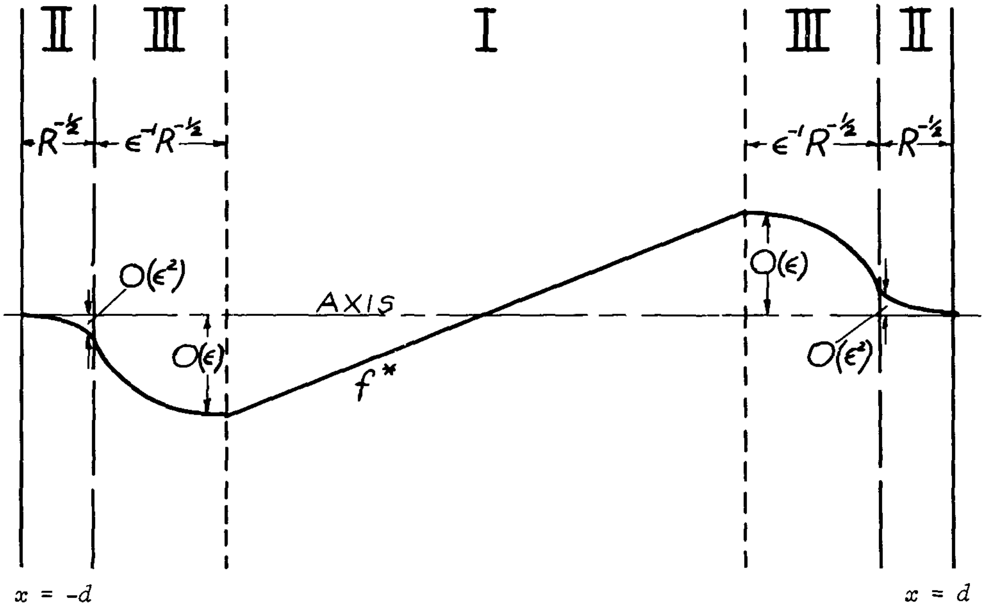

FIGURE 1.

The steady axial flow component is shown as f^* in fig. 1.

(a) Region I

In this region the angular velocity is zero, and each term in the axial momentum equation (unsteady term, convection term, diffusion term) is zero. Thus in this region the solution behaves like steady convection in an inviscid fluid.

(b) Region II

In the boundary layers on the disks, the convection terms are small, and the balance is essentially between the unsteady terms and the diffusion terms. The small steady axial flow arises from interaction between periodic terms in the angular velocity (physically, from centrifugal force effects).

(c) Region III

In the intermediate region the angular velocity is again zero. The dominant part of the axial flow is steady, and there is a balance between convection and viscous diffusion.

2. Equations of motion

We denote dimensional quantities by a superscript asterisk. As the motion is axisymmetric we shall write the equations of motion using Stokes's stream function Ψ and the angular velocity h^*. The disks being of infinite radius, we may write $\Psi = r^{*2}f^*$, where r^* is the radial coordinate, and f^* and h^* will now be functions of the axial coordinate x^* and the time t^* only. The axial velocity is $2f^*$, and the radial velocity is $-r^*f^*_{x^*}$, the subscript denoting a partial derivative. The equations of motion may now be written as

$$f^*_{x^*x^*t^*} = -2h^*h^*_{x^*} - 2f^*f^*_{x^*x^*x^*} + \nu f^*_{x^*x^*x^*x^*} , \tag{2.1}$$

and

$$h^*_{t^*} = 2h^*f^*_{x^*} - 2f^*h^*_{x^*} + \nu h^*_{x^*x^*} , \tag{2.2}$$

where ν is the kinematic viscosity.

If the disks lie in the planes $x^* = \pm d$, and are performing torsional oscillations of equal angular frequency ω and amplitude 2ε,

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but π out of phase, the no slip conditions at the disks may be written

$$(2.3)\qquad f^* = f^*_{x^*} = 0\ ,\quad h^* = \pm\varepsilon\omega\left(e^{i\omega t^*}+e^{-i\omega t^*}\right)\quad \text{at}\quad x^* = \pm d\ .$$

We have to solve equations (2.1) and (2.2) with boundary conditions (2.3).

Two dimensionless parameters are involved in this problem, the amplitude of oscillation ε and the Reynolds number of the flow $R = \omega d^2/\nu$. We shall restrict ourselves to the case $\varepsilon \ll 1$, $\varepsilon R^{\frac{1}{2}} \gg 1$.

Equations (2.1) and (2.2), with boundary conditions (2.3), can be transformed into dimensionless equations in a number of distinct ways. We shall, for convenience, collect the various forms of equations here.

The following remarks provide some justification for the dimensionless transformations used. From (2.3) we see that h^* is $O(\varepsilon\omega)$ in at least part of the flow field, and we shall assume that h^* is $O(\varepsilon\omega)$ everywhere. The axial velocity f^* is induced by centrifugal forces in the viscous boundary layers adjacent to the disks. These boundary layers we expect to be of thickness $O\left(R^{-\frac{1}{2}}\right)$, and hence expect f^* to be $O\left(\varepsilon R^{-\frac{1}{2}}\right)$. Thus we adopt the following dimensionless forms.

(i) Interior region

$$(2.4)\qquad \begin{cases} x^* = d\bar{x}\ , \\ t^* = \omega^{-1}t\ , \\ h^* = \varepsilon\omega H(\bar{x},\ t)\ , \\ f^* = \varepsilon R^{-\frac{1}{2}}\omega d F(\bar{x},\ t)\ , \\ \quad = \varepsilon(\nu\omega)^{\frac{1}{2}}F(\bar{x},\ t)\ . \end{cases}$$

(2.1), (2.2) and (2.3) now become

$$(2.5)\qquad F_{\bar{x}\bar{x}t} = -2\varepsilon R^{\frac{1}{2}}HH_{\bar{x}} - 2\varepsilon R^{-\frac{1}{2}}FF_{\bar{x}\bar{x}\bar{x}} + R^{-1}F_{\bar{x}\bar{x}\bar{x}\bar{x}}\ ,$$

$$(2.6)\qquad H_t = 2\varepsilon R^{-\frac{1}{2}}HF_{\bar{x}} - 2\varepsilon R^{-\frac{1}{2}}FH_{\bar{x}} + R^{-1}H_{\bar{x}\bar{x}}\ ,$$

and

$$(2.7)\qquad F = F_{\bar{x}} = 0\ ,\quad H = \pm\left(e^{it}+e^{-it}\right)\quad \text{at}\quad \bar{x} = \pm 1\ .$$

These boundary conditions are not in fact used, as the solution is matched to the solution in the intermediate region.

(ii) Boundary layer regions

For simplicity we shall only consider the boundary layer near $x^* = -d$, as that near $x^* = d$ has the same form.

$$\text{(2.8)} \qquad \begin{cases} x^* = d\left(-1+R^{-\frac{1}{2}}x\right) , \\ t^* = \omega^{-1}t , \\ h^* = \varepsilon\omega h(x, t) , \\ f^* = \varepsilon(\nu\omega)^{\frac{1}{2}}f(x, t) . \end{cases}$$

(2.1) and (2.2) now become

$$\text{(2.9)} \qquad f_{xxt} = -2\varepsilon hh_x - 2\varepsilon ff_{xxx} + f_{xxxx} ,$$

$$\text{(2.10)} \qquad h_t = 2\varepsilon hf_x - 2\varepsilon fh_x + h_{xx} .$$

The boundary condition at $x^* = -d$ becomes

$$\text{(2.11)} \qquad f = f_x = 0 , \quad h = -\left(e^{it}+e^{-it}\right) \quad \text{at} \quad x = 0 .$$

The boundary condition at $x^* = d$ is not used as the solution at the outer edge of the boundary layer $(x \to \infty)$ is matched to the solution in the intermediate region.

(iii) Intermediate regions

The transformation for the intermediate regions is most simply written by use of the results for the boundary layer region.

$$\text{(2.12)} \qquad \begin{cases} X = \varepsilon x , \\ f(x, t) = F(X, t) , \\ h(x, t) = H(X, t) . \end{cases}$$

(2.1) and (2.2) [or, alternatively, (2.9) and (2.10)] now become

$$\text{(2.13)} \qquad F_{XXt} = -2HH_X - 2\varepsilon^2 FF_{XXX} + \varepsilon^2 F_{XXXX} ,$$

and

$$\text{(2.14)} \qquad H_t = 2\varepsilon^2 HF_X - 2\varepsilon^2 FH_X + \varepsilon^2 H_{XX} .$$

There are no boundary conditions used directly for these equations, the

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solution being obtained by matching into the boundary layer solution as $X \to 0$ and into the solution in the interior region as $X \to \infty$.

Before proceeding with the solution of the equations in the various regions, we make some remarks which simplify the later work.

In solving the equations for the angular velocity, we find that the various expansions used give rise to terms which are independent of time. On physical grounds we can immediately say that such terms must be zero, as sinusoidal torsional oscillations of the disks cannot give rise to a steady angular velocity of the fluid. This applies not only to the particular boundary conditions (2.3) used here, but to other cases where the amplitudes and frequencies of oscillation of the disks may be different.

The particular boundary conditions (2.3) being considered here enable us to further simplify the solution of the equations. From (2.3) we see that the angular velocity h^* may be expected to be an odd function of x^*. If this is assumed, then from (2.2) we see that f^* will also be an odd function of x^*. These two results greatly simplify the work in Section 3.

3. Solution in interior region

We are interested in obtaining the solution of the problem for the case $\varepsilon R^{\frac{1}{2}} \gg 1$ (strictly speaking, we wish to obtain an asymptotic solution valid as $\varepsilon R^{\frac{1}{2}} \to \infty$). Introducing the abbreviation $\lambda = \varepsilon R^{\frac{1}{2}}$, we may rewrite (2.5) and (2.6) as

$$F_{\bar{x}\bar{x}t} = -2\lambda H H_{\bar{x}} - 2\varepsilon^2\lambda^{-1} F F_{\bar{x}\bar{x}\bar{x}} + \varepsilon^2\lambda^{-2} F_{\bar{x}\bar{x}\bar{x}\bar{x}} , \tag{3.1}$$

and

$$H_t = 2\varepsilon^2\lambda^{-1} H F_{\bar{x}} - 2\varepsilon^2\lambda^{-1} F H_{\bar{x}} + \varepsilon^2\lambda^{-2} H_{\bar{x}\bar{x}} . \tag{3.2}$$

We assume that F and H can be expanded in a series of inverse powers of λ,

$$F(\bar{x}, t, \varepsilon, \lambda) = \sum_{n=0}^{\infty} \lambda^{-n} F_n(\bar{x}, t, \varepsilon) , \tag{3.3}$$

and

$$H(\bar{x}, t, \varepsilon, \lambda) = \sum_{n=0}^{\infty} \lambda^{-n} H_n(\bar{x}, t, \varepsilon) . \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.1) and (3.2), and equating powers of λ, we obtain the following results.

(i) Coefficient of λ^1

$$H_o H_{o\bar{x}} = 0 . \tag{3.5}$$

Hence $H_o = H_o(t)$. ˙owever, from Section 2, H_o is an odd function of $\bar{x}$. Thus we must have $H_o \equiv 0$.

(ii) Coefficient of λ^o

$$F_{o\bar{x}\bar{x}t} = 0 \tag{3.6}$$

$$H_{o_t} = 0 . \tag{3.7}$$

(3.7) is automatically satisfied. The general solution of (3.6) is

$$F_o(\bar{x}, t, \varepsilon) = F_{oo}(\bar{x}, \varepsilon) + \bar{x}F_{o1}(t, \varepsilon) + F_{o2}(t, \varepsilon) , \tag{3.8}$$

where F_{oo}, F_{o1} and F_{o2} are functions to be determined. As F_o is an odd function of $\bar{x}$, F_{oo} must be an odd function of $\bar{x}$, and $F_{o2} \equiv 0$. Thus

$$F_o(\bar{x}, t, \varepsilon) = F_{oo}(\bar{x}, \varepsilon) + \bar{x}F_{o1}(t, \varepsilon) . \tag{3.9}$$

From the boundary conditions F_{o1} must be a periodic function of t, and it can be chosen so that

$$\int_o^{2\pi} F_{o1}(t, \varepsilon) dt = 0 , \tag{3.10}$$

any constant in F_{o1} being absorbed into F_{oo}.

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(iii) Coefficient of λ^{-1}

(3.11) $$F_{1_{\bar{x}\bar{x}t}} = -2H_1 H_{1_{\bar{x}}} - 2\varepsilon^2 F_o F_{o_{\bar{x}\bar{x}\bar{x}}} ,$$

(3.12) $$H_{1_t} = 0 .$$

From (3.12) $H_1 = H_1(\bar{x}, \varepsilon)$, and hence, from Section 2, $H_1 = 0$. Substituting this result and (3.9) into (3.11) we get

(3.13) $$F_{1_{\bar{x}\bar{x}t}} = -2\varepsilon^2 F_{oo} F_{oo_{\bar{x}\bar{x}\bar{x}}} - 2\varepsilon^2 \bar{x} F_{oo_{\bar{x}\bar{x}\bar{x}}} F_{o1}(t) .$$

Integration of (3.13) once with respect to t gives

(3.14) $$F_{1_{\bar{x}\bar{x}}} = -2\varepsilon^2 t F_{oo} F_{oo_{\bar{x}\bar{x}\bar{x}}} - 2\varepsilon^2 \bar{x} F_{oo_{\bar{x}\bar{x}\bar{x}}} \int_o^t F_{o1}(u)du + F_{1o_{\bar{x}\bar{x}}}(\bar{x}, \varepsilon) ,$$

$F_{1o_{\bar{x}\bar{x}}}(\bar{x}, \varepsilon)$ being the unknown function of integration. Now as F_{o1} has period 2π , (3.10) shows that $\int_o^t F_{o1}(u)du$ is also periodic, with period 2π . Now $F_{1_{\bar{x}\bar{x}}}$ cannot increase indefinitely with t , so that the first term in (3.14) must be zero. Hence

(3.15) $$F_{oo} F_{oo_{\bar{x}\bar{x}\bar{x}}} = 0 ,$$

the solution of which, restricting ourselves to odd functions of $\bar{x}$, is

(3.16) $$F_{oo} = a_{oo}(\varepsilon)\bar{x} ,$$

where a_{oo} is some function of ε only, which is found during the matching procedure.

Equation (3.13) now becomes

$$F_{1_{\bar{x}\bar{x}t}} = 0 ,$$

with solution

(3.17) $$F_1(\bar{x}, t, \varepsilon) = F_{1o}(\bar{x}, \varepsilon) + \bar{x} F_{11}(t, \varepsilon) ,$$

as we are restricting ourselves to odd functions of $\bar{x}$.

(iv) Coefficients of higher powers of λ^{-1}

From (3.2) we see that taking successive values of n we have

$$H_{n_t} = 0 ,$$

and hence $H_n = H_n(x, \varepsilon) = 0$ from Section 2. Thus in the interior region

(3.18) $$H \equiv 0 .$$

From (3.1) it is easy to show that for all n we will have

$$F_{n_{\bar{x}\bar{x}t}} = 0 ,$$

$$F_{n_{\bar{x}\bar{x}\bar{x}}} = 0 ,$$

so that F_n will take the same form (3.8) as F_0 .

If $a_{00}(\varepsilon)$ and $F_{01}(t, \varepsilon)$ are now expanded in powers of ε ,

$$a_{00}(\varepsilon) = \sum_{n=0}^{\infty} a_{00n}\varepsilon^n ,$$

$$F_{01}(t, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n F_{01n}(t) ,$$

we see from (3.9) and (3.16) that F_0 may be written

(3.19) $$F_0(\bar{x}, t, \varepsilon) = \bar{x} \sum_{n=0}^{\infty} \varepsilon^n \left[a_{00n}+F_{01n}(t)\right] .$$

4. Solution in boundary layers

We rewrite (2.9), (2.10) and (2.11) as

(4.1) $$f_{xxt} - f_{xxxx} = -2\varepsilon\left(hh_x+ff_{xxx}\right) ,$$

(4.2) $$h_t - h_{xx} = 2\varepsilon\left(hf_x-fh_x\right) ,$$

(4.3) $$f = f_x = 0 , \quad h = -\left(e^{it}+e^{-it}\right) \quad \text{at} \quad x = 0 .$$

The Reynolds number does not appear in these equations, as they are essentially those governing the flow due to torsional oscillations of an

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infinite disk in an unbounded fluid. This problem has been attacked by Rosenblat [2], Benney [3] and Riley [4], and is rather simpler as (4.1) may be integrated once with respect to x, and the function of integration is known to be zero. For this problem of two oscillating disks the function of integration is not zero, and it is preferable to deal with (4.1) as it stands.

Although the Reynolds number does not appear in (4.1), (4.2) and (4.3), we expand f and h in inverse powers of λ in order to match the solution with that in the interior.

(4.4) $$f = \sum_{n=0}^{\infty} \lambda^{-n} f_n(x, t, \varepsilon) ,$$

(4.5) $$h = \sum_{n=0}^{\infty} \lambda^{-n} h_n(x, t, \varepsilon) .$$

Substitution of (4.4) and (4.5) into (4.1), (4.2) and (4.3), gives, if only f_0 and h_0 are considered,

(4.6) $$f_{0_{xxt}} - f_{0_{xxxx}} = -2\varepsilon\left\{h_0 h_{0_x} - f_0 f_{0_{xxx}}\right\} ,$$

(4.7) $$h_{0_t} - h_{0_{xx}} = 2\varepsilon\left\{h_0 f_{0_x} - f_0 h_{0_x}\right\} ,$$

(4.8) $$f_0 = f_{0_x} = 0 , \quad h_0 = -2\cos t \quad \text{at} \quad x = 0 .$$

The form of these equations, with the non-linear terms multiplied by the small parameter ε, suggests that we expand f_0 and h_0 as a power series in ε, namely

(4.9) $$f_0(x, t, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n f_{0n}(x, t) ,$$

(4.10) $$h_0(x, t, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n h_{0n}(x, t) .$$

Substituting these into (4.6), (4.7) and (4.8), and equating powers of ε we obtain the following sets of equations.

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(i) Coefficients of ε^0

$$f_{00_{xxt}} - f_{00_{xxxx}} = 0 ,$$

$$h_{00_t} - h_{00_{xx}} = 0 ,$$

$$f_{00} = f_{00_x} = 0 , \quad h_{00} = -2\cos t \quad \text{at} \quad x = 0 .$$

The solution of these equations is

$$f_{00} = a_{002}x^2 + a_{003}x^3 ,$$

$$h_{00} = b_{001}x - 2e^{-x/\sqrt{2}} \cos(t-x/\sqrt{2}) ,$$

where a_{002} , a_{003} and b_{001} are functions of ε to be determined. In Section 5 it is found that the functions F_0 and H_0 , into which f_0 and h_0 are matched, are finite. Thus f_{00} and h_{00} , which form the dominant parts of f_0 and h_0 as $\varepsilon \to 0$, should be finite as $x \to \infty$. Hence $a_{002} = a_{003} = b_{001} = 0$. Thus

(4.11) $$f_{00} = 0 ,$$

(4.12) $$h_{00} = -2e^{-x/\sqrt{2}} \cos(t-x/\sqrt{2}) .$$

(ii) Coefficient of ε^1

$$f_{01_{xxt}} - f_{01_{xxxx}} = -2h_{00}h_{00_x} ,$$

$$= 2\sqrt{2}\, e^{-x\sqrt{2}}\{-\sin(t-x/\sqrt{2})\cos(t-x/\sqrt{2}) + \cos^2(t-x/\sqrt{2})\} ,$$

by using (4.12). Thus

(4.13) $$f_{01_{xxt}} - f_{01_{xxxx}} = \sqrt{2}\, e^{-x\sqrt{2}}\{-\sin(2t-x\sqrt{2}) + \cos(2t-x\sqrt{2}) + 1\} ,$$

(4.14) $$h_{01_t} - h_{01_{xx}} = 0 ,$$

(4.15) $$f_{01} = f_{01_x} = h_{01} = 0 \quad \text{at} \quad x = 0 .$$

The solution of (4.14) which satisfies (4.15) is

$$h_{01} = b_{011}x \ ,$$

and from Section 2 we must have $b_{011} = 0$. Thus

$$h_{01} = 0 \ . \tag{4.16}$$

From (4.13) it is evident that f_{01} will be composed of two parts, one independent of time and one periodic with twice the angular frequency of oscillation of the disks. We write $f_{01} = f_{01s} + f_{01u}$, subscripts s and u denoting steady and unsteady parts, respectively, and consider these separately, as it is clear that they must each satisfy boundary conditions (4.15).

(a) Steady part

This is obtained by solving

$$f_{01s_{xxxx}} = -\sqrt{2}\, e^{-x\sqrt{2}} \ ,$$

$$f_{01s} = f_{01s_x} = 0 \quad \text{at} \quad x = 0 \ .$$

The solution is

$$f_{01s} = -\frac{1}{2\sqrt{2}}\, e^{-x\sqrt{2}} + \frac{1}{2\sqrt{2}} - \tfrac{1}{2}x + a_{012s}x^2 + a_{013s}x^3 \ . \tag{4.17}$$

At this stage the necessity for introducing the intermediate region, to be considered in Section 5, appears. Comparing (4.11) and (4.17) with (4.9), we see that the series in (4.9) will only converge if εx is bounded, and thus the straightforward approach of matching this solution with that in the interior region (Section 3), by letting $x \to \infty$, is not valid. Thus an additional region is required, which we shall call the intermediate region, in which the dimensionless space variable is $X = \varepsilon x$. This region is considered in Section 5.

Section 5 shows that if the boundary layer solution is to be matched into the solution in the intermediate region, then the terms of the expansions (4.9) and (4.10) must be such that $f_{0n} = O(x^n)$ and $h_{0n} = O(x^n)$ as $x \to \infty$. Hence, if we anticipate these results,

$a_{o12s} = a_{o13s} = 0$, and

(4.18) $$f_{o1s} = -\frac{1}{2\sqrt{2}}e^{-x/2} + \frac{1}{2\sqrt{2}} - \tfrac{1}{2}x\ .$$

(b) Unsteady part

This is obtained by solving

(4.19) $$f_{o1u_{xxt}} - f_{o1u_{xxxx}} = \sqrt{2}\, e^{-x\sqrt{2}} \{\cos(2t-x\sqrt{2}) - \sin(2t-x\sqrt{2})\}\ ,$$

(4.20) $$f_{o1u} = f_{o1u_x} = 0 \quad \text{at} \quad x = 0\ .$$

The solution is obtained by writing

(4.21) $$f_{o1u} = f_{o1uc}\cos 2t + f_{o1us}\sin 2t + f_{o1uo}(t) + xf_{o1u1}(t)\ ,$$

the terms $f_{o1uo}(t)$ and $xf_{o1u1}(t)$, where f_{o1uo} and f_{o1u1} are arbitrary functions of time, being solutions of the homogeneous equation. Substituting (4.21) into (4.19), equating coefficients of $\cos 2t$ and $\sin 2t$ and solving we obtain

$$f_{o1uc} = \frac{1}{4\sqrt{2}}\, e^{-x\sqrt{2}}\, [\cos(x\sqrt{2}) + \sin(x\sqrt{2})] + e^{-x}[A\cos x + B\sin x]\ ,$$

$$f_{o1us} = \frac{1}{4\sqrt{2}}\, e^{-x\sqrt{2}}\, [-\cos(x\sqrt{2}) + \sin(x\sqrt{2})] + e^{-x}[-B\cos x + A\sin x]\ ,$$

where terms increasing exponentially with x have been excluded, and A and B are arbitrary constants. Substitution of these into (4.21) gives

(4.22) $$\begin{cases} f_{o1u} = \frac{1}{4\sqrt{2}}\, e^{-x\sqrt{2}}\, [\cos(x\sqrt{2}-2t) + \sin(x\sqrt{2}-2t) \\ \qquad + e^{-x}[A\cos(x-2t) + B\sin(x-2t)] + f_{o1uo}(t) + xf_{o1u1}(t)\ . \end{cases}$$

Applying boundary conditions (4.20) to (4.22) we obtain

(4.23) $$\frac{1}{4\sqrt{2}}\,(\cos 2t - \sin 2t) + A\cos 2t - B\sin 2t + f_{o1uo}(t) = 0$$

and

(4.24) $$\begin{cases} -\frac{1}{4}(\cos 2t - \sin 2t) + \frac{1}{4}(\sin 2t + \cos 2t) - A\cos 2t \\ \quad + B\sin 2t + A\sin 2t + B\cos 2t + f_{o1u1}(t) = 0\ , \end{cases}$$

as equations determining f_{o1uo} and f_{o1u1}. The solution is still incomplete at this stage as A and B are still unknown. They are found during the process of matching into the solution in the intermediate region.

(iii) Coefficient of ε^2

From the foregoing it appears that the higher order terms in the expansions for f_o and h_o become complicated and unwieldy. We content ourselves with quoting the asymptotic form of f_{o2} and h_{o2} for large x.

(a) As Riley [4] has pointed out the correct form for f_{o2} is

(4.25) $$f_{o2} = a_{o22}x^2 + a_{o23}x^3 ,$$

where a_{o22} and a_{o23} are found during the matching procedure.

(b) Rosenblat [2] has given the expression for h_{o2}, using slightly different non-dimensional variables. Using the variables introduced in Section 2 we have

(4.26) $$h_{o2} = O\left[x^2 e^{-x/\sqrt{2}} \cos(t-x/\sqrt{2})\right] .$$

(4.25) and (4.26) provide further evidence for the existence of the intermediate region in which the appropriate length variable is $X = \varepsilon x$.

It is easy to see from (4.7), (4.12), (4.16) and (4.26) that all terms in the expansion (4.10) for h_o will tend to zero exponentially at the outer edge of the boundary layer region. We shall see in the next section that in the intermediate region the angular velocity is identically zero, and thus the angular velocity is matched automatically.

5. Solution in the intermediate region, and matching between regions

The equations governing the flow in the intermediate region are

(2.13) $$F_{XXt} = -2HH_X - 2\varepsilon^2 FF_{XXX} + \varepsilon^2 F_{XXXX} ,$$

(2.14) $$H_t = 2\varepsilon^2 HF_X - 2\varepsilon^2 FH_X + \varepsilon^2 H_{XX} .$$

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Expanding F and H in inverse powers of λ ,

$$F(X,\ t,\ \varepsilon,\ \lambda) = \sum_{n=0}^{\infty} \lambda^{-n} F_n(X,\ t,\ \varepsilon) \ ,$$

$$H(X,\ t,\ \varepsilon,\ \lambda) = \sum_{n=0}^{\infty} \lambda^{-n} H_n(X,\ t,\ \varepsilon) \ ,$$

and considering only the dominant terms we obtain

$$F_{o_{XXt}} = -\ 2H_o H_{o_X} + \varepsilon^2 \left(-2F_o F_{o_{XXX}} + F_{o_{XXXX}}\right) \ , \tag{5.1}$$

$$H_{o_t} = \varepsilon^2 \left(2H_o F_{o_X} - 2F_o H_{o_X} + H_{o_{XX}}\right) \ . \tag{5.2}$$

F_o and H_o are now expanded as power series in ε ,

$$F_o = \sum_{n=0}^{\infty} \varepsilon^n F_{on}(X,\ t) \ , \tag{5.3}$$

$$H_o = \sum_{n=0}^{\infty} \varepsilon^n H_{on}(X,\ t) \ . \tag{5.4}$$

We now prove, by induction, that $H_{on} = 0$. Substitution of (5.4) into (5.2), and extraction of the coefficient of ε^o gives $H_{oo_t} = 0$. Hence $H_{oo} = H_{oo}(X,\ \varepsilon) = 0$, from Section 2. If we now assume that $H_{or} = 0$ for $r = 0,\ 1,\ \ldots,\ n-1$, we get from (5.2), $H_{on_t} = 0$. Hence $H_{on} = H_{on}(X,\ \varepsilon) = 0$, from Section 2. Thus we have $H_o \equiv 0$, and (5.1) reduces to

$$F_{o_{XXt}} = \varepsilon^2 \left(-2F_o F_{o_{XXX}} + F_{o_{XXXX}}\right) \ . \tag{5.5}$$

We now substitute (5.3) into (5.5), and equate coefficients of various powers of ε .

(i) Coefficient of ε^o

$$F_{oo_{XXt}} = 0 \ .$$

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The solution of this equation is

$$F_{00} = F_{000}(X) + XF_{001}(t) + F_{002}(t) \ . \tag{5.6}$$

It is convenient at this stage to carry out part of the matching procedure. From the non-dimensional forms (2.4), (2.8) and (2.12) used, continuity of the axial velocity implies

$$\lim_{x\to\infty} [f_0(x, t, \varepsilon)] = \lim_{X\to 0} [F_0(X, t, \varepsilon)] \ , \tag{5.7}$$

and

$$\lim_{X\to\infty} [F_0(X, t, \varepsilon)] = \lim_{\bar{x}\to -1} [F_0(\bar{x}, t, \varepsilon)] \ . \tag{5.8}$$

If we consider now only the dominant terms in ε, the right hand side of (5.7) is finite, so that $\lim_{x\to\infty} f_{00}$ must be finite, as mentioned in Section 4. This implies that $f_{00} = 0$, which in turn implies that $\lim_{X\to 0} F_{00}(X, t) = 0$. Hence

$$F_{000}(0) = 0 \ , \tag{5.9}$$

$$F_{002}(t) = 0 \ . \tag{5.10}$$

Now from (3.19) the right hand side of (5.8) is finite, which, from (5.6), implies that

$$\lim_{X\to\infty} F_{000}(X) = - a_{000} \ , \tag{5.11}$$

$$F_{001}(t) = 0 \ . \tag{5.12}$$

(5.12) now implies, from (5.8) and (3.19)

$$F_{010}(t) = 0 \ . \tag{5.13}$$

We now have

$$F_{00} = F_{000}(X) \ . \tag{5.14}$$

(ii) **Coefficient of** ε^1

$$F_{01_{XXt}} = 0 \ ,$$

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the solution of which is

$$F_{01} = F_{010}(X) + XF_{011}(t) + F_{012}(t) \; . \tag{5.15}$$

At this stage we need to use the full matching procedure. The non-dimensional form of the axial coordinate used implies that near $\bar{x} = -1$ we have

$$\begin{cases} \bar{x} = -1 + R^{-\frac{1}{2}}x \; , \\ \quad = -1 + R^{-\frac{1}{2}}\varepsilon^{-1}X \; , \\ \quad = -1 + \lambda^{-1}X \; . \end{cases} \tag{5.16}$$

Throughout this work we have considered only the first term in expansions in inverse powers of λ , so that (5.16) gives, in the overlap between the interior region and the intermediate region, $\bar{x} = -1$. Thus (5.8) becomes, after use of (5.3) and (3.19),

$$\lim_{X\to\infty} [F_{00}+\varepsilon F_{01}+\varepsilon^2F_{02}+ \ldots] = - \sum_{n=0}^{\infty} \varepsilon^n[a_{00n}+F_{01n}(t)] \; . \tag{5.17}$$

We now match powers of ε in (5.17). The coefficient of ε^0 gives (5.11), (5.12) and (5.13). The coefficient of ε^1 gives

$$\lim_{X\to\infty} F_{01} = - a_{001} - F_{011}(t) \; .$$

Hence

$$\lim_{X\to\infty} F_{010}(X) = - a_{001} \; , \tag{5.18}$$

$$F_{011}(t) = 0 \; , \tag{5.19}$$

$$F_{012}(t) = - F_{011}(t) \; . \tag{5.20}$$

Coefficients of higher powers of ε are matched similarly.

We now match the solution to that in the boundary layer region, obtained in Section 4. The matching procedure (5.7) may be written

$$f_{00} + \varepsilon f_{01} + \varepsilon^2 f_{02} + \ldots \sim F_{00} + \varepsilon F_{01} + \varepsilon^2 F_{02} + \ldots \; . \tag{5.21}$$

Here ε occurs implicitly on the right hand side, and this dependence must be made explicit before powers of ε are matched. If derivatives of

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F_{on} with respect to X are denoted by primes, (5.21) may be written

(5.22) $$\begin{cases} f_{oo} + \varepsilon f_{o1} + \varepsilon^2 f_{o2} + O(\varepsilon^3) \sim \left(F_{oo}\right)_{X=0} + \varepsilon x\left(F_{oo}'\right)_{X=0} \\ + \dfrac{\varepsilon^2 x^2}{2!}\left(F_{oo}''\right)_{X=0} + \varepsilon\left(F_{o1}\right)_{X=0} + \varepsilon^2 x\left(F_{o1}'\right)_{X=0} + \varepsilon^2\left(F_{o2}\right)_{X=0} + O(\varepsilon^3) . \end{cases}$$

We now equate coefficients of the various powers of ε .

(a) ε^0. This has been done, and gives (5.9) and (5.10).

(b) ε^1. Substitution of (4.17), (4.22), (5.6) and (5.15) into (5.22), and use of (5.12) gives, when exponentially small terms on the left hand side are ignored,

$$\frac{1}{2\sqrt{2}} - \tfrac{1}{2}x + a_{o12s}x^2 + a_{o13s}x^3 + f_{o1uo}(t) + xf_{o1u1}(t) \sim x\left(F_{ooo}'\right)_{X=0} + \left(F_{o1o}\right)_{X=0} + F_{o12}(t).$$

Powers of x must match asymptotically, so that we have

(5.23) $$a_{o13s} = 0 ,$$

(5.24) $$a_{o12s} = 0 ,$$

(5.25) $$-\tfrac{1}{2} + f_{o1u1}(t) = \left(F_{ooo}'\right)_{X=0} ,$$

(5.26) $$\frac{1}{2\sqrt{2}} + f_{o1uo}(t) = \left(F_{o1o}\right)_{X=0} + F_{o12}(t) .$$

Equations (5.23) and (5.24) have been used earlier to obtain (4.18). The steady and the time dependent parts of (5.25) may be separated to give

(5.27) $$\left(F_{ooo}'\right)_{X=0} = -\tfrac{1}{2} ,$$

(5.28) $$f_{o1u1}(t) = 0 .$$

Substituting (5.28) into (4.24) gives

$$(\tfrac{1}{2}+B+A)\sin 2t + (B-A)\cos 2t = 0 ,$$

and hence $A = B = -\frac{1}{4}$. (4.23) now gives

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$$(5.29)\qquad \begin{cases} f_{01uo} = \frac{1}{4}\left(1-\frac{1}{\sqrt{2}}\right)\cos 2t - \frac{1}{4}\left(1-\frac{1}{\sqrt{2}}\right)\sin 2t \ , \\ \qquad = \frac{1}{4}(\sqrt{2}-1)\cos\left(2t+\frac{1}{4}\pi\right) \ . \end{cases}$$

Separating the steady and the time dependent parts of (5.26) gives

$$(5.30)\qquad \left(F_{010}\right)_{X=0} = \frac{1}{2\sqrt{2}} \ ,$$

$$(5.31)\qquad F_{012}(t) = f_{01uo}(t) \ .$$

Comparing (5.31) with (5.29) and (5.20) we see that the dominant part of the fluctuating flow is now known everywhere.

(c) ε^2. Substitution of (4.25), (5.6) and (5.15) into (5.22), and use of (5.19) gives

$$a_{022}x^2 + a_{023}x^3 \sim \tfrac{1}{2}x^2\left(F_{000}''\right)_{X=0} + x\left(F_{010}'\right)_{X=0} + \left(F_{02}\right)_{X=0} \ .$$

When powers of x are matched, we obtain

$$(5.32)\qquad a_{023} = 0 \ ,$$

$$(5.33)\qquad a_{022} = \tfrac{1}{2}\left(F_{000}''\right)_{X=0} \ ,$$

$$(5.34)\qquad \left(F_{010}'\right)_{X=0} = 0 \ ,$$

$$(5.35)\qquad \left(F_{02}\right)_{X=0} = 0 \ .$$

(iii) **Coefficient of** ε^2

$$F_{02_{XXt}} = -\ 2F_{00}F_{00_{XXX}} + F_{00_{XXXX}} \ .$$

In view of (5.6), (5.10) and (5.12) this may be written

$$F_{02_{XXt}} = -\ 2F_{000}F_{000_{XXX}} + F_{000_{XXXX}} \ ,$$

and integration of this once with respect to t gives

$$F_{02_{XX}} = t\left(-2F_{000}F_{000_{XXX}} + F_{000_{XXXX}}\right) + F_{020}''(X) \ ,$$

$F_{020}''(X)$ being the function of integration. Now $F_{02_{XX}}$ cannot increase indefinitely with t, so that we must have

$$F_{000_{XXXX}} - 2F_{000}F_{000_{XXX}} = 0 ,$$

as the equation determining F_{000}. This equation can be integrated once with respect to X to give

(5.32) $$F_{000_{XXX}} - 2F_{000}F_{000_{XX}} + F_{00_X}^{\;2} = \mu ,$$

where μ is some constant. The boundary conditions on F_{000} are given by (5.9), (5.11) and (5.27), i.e.

(5.33) $$\begin{cases} F_{000} = 0 , \quad F_{000}' = -\tfrac{1}{2} & \text{at} \quad X = 0 , \\ F_{000} = -a_{000} & \text{as} \quad X \to \infty . \end{cases}$$

Rasmussen [5] has examined solutions of (5.32), and shown that boundary conditions of the type (5.33) can only be applied if $\mu = 0$. In this case (5.32), with boundary conditions (5.33), is, apart from changes in scale, the same as an equation obtained by Benney (equation 3.35 in [3]). He gave a numerical solution according to which

$$F_{000}''(0) = 0.415 , \quad F_{000}(\infty) = -0.530 ,$$

so that

(5.34) $$a_{000} = 0.530 ,$$

(5.35) $$a_{022} = 0.208 .$$

(iv) Coefficient of ε^3

In order to complete the solution to $O(\varepsilon^2)$ we need an equation for F_{010}, which is obtained by considering the coefficient of ε^3,

(5.36) $$F_{03_{XXt}} = -2F_{00}F_{01_{XXX}} - 2F_{01}F_{00_{XXX}} + F_{01_{XXXX}}$$

Substituting (5.6) and (5.15) into (5.36), using (5.10), (5.12), (5.19), (5.31), and integrating once with respect to t we get

$$F_{03_{XX}} = -\tfrac{1}{4}(\sqrt{2}-1)\sin\left(2t+\tfrac{1}{4}\pi\right)F_{000_{XXX}} + t\left\{-2F_{000}F_{010_{XXX}} - 2F_{010}F_{000_{XXX}} + F_{010_{XXXX}}\right\} .$$

As argued earlier the coefficient of t must be zero, so that the equation to be satisfied by F_{010} is

$$(5.37) \qquad F_{010_{XXXX}} - 2F_{000}F_{010_{XXX}} - 2F_{000_{XXX}}F_{010} = 0 ,$$

with boundary conditions (5.18), (5.30) and (5.34).

It is worth while gathering together the expressions for the axial and angular velocities in the various regions. They are:

(a) Boundary layer region

$$(5.38) \quad \begin{cases} f_0 = \varepsilon\left\{\dfrac{1}{2\sqrt{2}} - \tfrac{1}{2}x - \dfrac{1}{2\sqrt{2}}e^{-x\sqrt{2}} + \tfrac{1}{4}e^{-x\sqrt{2}}\cos\left(x\sqrt{2} - 2t - \tfrac{1}{4}\pi\right)\right. \\ \qquad \left. - \dfrac{1}{2\sqrt{2}}e^{-x}\cos\left(x-2t-\tfrac{1}{4}\pi\right) + \tfrac{1}{4}(\sqrt{2}-1)\cos\left(2t+\tfrac{1}{4}\pi\right)\right\} + O(\varepsilon^2) , \end{cases}$$

$$(5.39) \quad h_0 = -2e^{-x/\sqrt{2}}\cos(t - x/\sqrt{2}) + O(\varepsilon^2) .$$

(b) Intermediate region

$$(5.40) \quad F_0 = F_{000}(X) + \varepsilon\left\{F_{010}(X) + \tfrac{1}{4}(\sqrt{2}-1)\cos\left(2t+\tfrac{1}{4}\pi\right)\right\} + O(\varepsilon^2) ,$$

$$(5.41) \quad H_0 \equiv 0 .$$

F_{000} and F_{010} are solutions of equations (5.32) and (5.37) respectively, with appropriate boundary conditions.

(c) Interior region

$$(5.42) \quad F_0 = \bar{x}\left\{F_{000}(\infty) + \varepsilon\left[F_{010}(\infty) + \tfrac{1}{4}(\sqrt{2}-1)\cos\left(2t+\tfrac{1}{4}\pi\right)\right]\right\} + O(\varepsilon^2) ,$$

$$(5.43) \quad H_0 \equiv 0 .$$

6. Possible extensions of the work herein

Extensions of this work in two directions would be interesting. These are (i) solution of the problem for $\varepsilon R^{\frac{1}{2}} = O(1)$, to cover the region between the results of Rosenblat [1] and those of the present work, and (ii) extension of the solution herein to flow between torsionally oscillating disks whose amplitudes and frequencies are different. Some thoughts on these extensions are presented here.

(i) Solution for $\varepsilon R^{\frac{1}{2}} = O(1)$

In this case the thicknesses of the two intermediate regions would be such that they would overlap, and there would be no distinct interior region. In addition the expansion of the solution in inverse powers of $\lambda = \varepsilon R^{\frac{1}{2}}$ may not be valid. Provided the solution was restricted to $R^{\frac{1}{2}} >> 1$, the boundary layer regions considered in Section 4 would still exist, but the whole region between them would be governed by the equations of Section 5. The basic steady flow in this region would be governed by the equation

$$F_{000_{XXXX}} - 2F_{000}F_{000_{XXX}} = 0 ,$$

as in Section 5, but the boundary conditions on this equation would be

$$F_{000} = 0 , \quad F_{000}' = -\tfrac{1}{2} \quad \text{at} \quad X = 0 ,$$

$$F_{000} = 0 , \quad F_{000}' = -\tfrac{1}{2} \quad \text{at} \quad X = 2\varepsilon R^{\frac{1}{2}} ,$$

and a separate numerical solution of the equation would be needed for each value of $\varepsilon R^{\frac{1}{2}}$.

(ii) Torsional oscillations of disks with different amplitudes and frequencies

The work presented here has been simplified by the particular boundary conditions used, which imply that f^* and h^* are odd functions of x^* . Closer examination of the work of Sections 3, 4 and 5 shows that the solution can be readily extended to allow for an arbitrary phase difference between the disks, the only difference in the solution being in the time dependent part of F_0 . However if the amplitudes and/or frequencies of oscillation of the disks are different f^* and h^* will not be odd functions of x^* , and the solution becomes much more complicated.

In the interior region (Section 3) the equation governing the basic

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steady flow is (3.15), whose solution in this case is

(6.1) $$F_{00} = A + B\bar{x} + C\bar{x}^2 \ .$$

If the amplitudes and frequencies of oscillation of the disks are the same then F_{00} is an odd function of $\bar{x}$, so that $A = C = 0$, and the remaining constant B is determined by matching the axial velocity with that in the intermediate region. For the general case, however, we obtain two conditions by matching the axial velocity with the two intermediate regions at $\bar{x} = \pm 1$, and there is still a degree of indeterminacy. This indeterminacy cannot be overcome by matching the radial velocity at $\bar{x} = \pm 1$, as the radial velocity in the intermediate regions is determined by the second order term in the expansions in inverse powers of λ, and in any case this would give two extra conditions, so that F_{00} would be overdetermined.

The following physical argument is put forward as a possible way out of this impasse. Consider the disk at $\bar{x} = -1$ to be oscillating with angular frequency ω and amplitude 2ε, and the disk at $\bar{x} = 1$ to be stationary. Then the solutions in the boundary layer and intermediate region near $\bar{x} = -1$ would be as given in Sections 4 and 5, with the dominant solution in the interior given by (6.1). Matching the axial velocity at $\bar{x} = -1$ would give one condition on A, B and C. As the disk at $\bar{x} = 1$ is stationary, there would, to first order, be no boundary layer there, and the no slip boundary conditions should be applied to (6.1). This would give

$$F_{00}(1) = F_{00}'(1) = 0 \ ,$$

giving two extra conditions so that A, B and C would be determinate.

If (6.1) is now examined, it is found that $F_{00}'(-1)$ is the same as that obtained in Section 5. In physical terms we may say that the radial velocity at the outer edge of the intermediate region is the same in the two cases. If we make the physical assumption that the radial velocity at the outer edge of the intermediate region near one disk is independent of the amplitude of oscillation of the other disk, we could now obtain the solution for arbitrary amplitudes of oscillation of the disks. A weaker assumption, which would lead to the same result, would be that the above

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radial velocity is a monotonic function of the amplitude of oscillation of the second disk. An argument in favour of this assumption is its symmetric nature, as it is easily shown from (6.1) that if this assumption is true at one disk, it automatically holds for the second disk. Granted the truth of this assumption, the work here can readily be extended to cover different frequencies of oscillation, as well as different amplitudes.

The above argument is mainly physical, and mathematical justification for it should be sought.

Note added on 31 October, 1969. A recent paper by A.F. Jones and S. Rosenblat, "The flow induced by torsional oscillations of infinite planes", *J. Fluid Mech.* 37 (1969), 337-347, treats the same problem, with the same conclusions. They also consider the case $\varepsilon R^{\frac{1}{2}} = O(1)$, for $R \gg 1$.

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