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ON TWO NEW CLASSES OF LOCALLY CONVEX SPACES

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The purpose of this paper is to introduce two new classes of locally convex spaces which contain the classes of semi-Montel and Montel spaces. Further we give some examples and study some properties of these classes. As to permanence properties, these classes have similar properties to semi-Montel and Montel spaces except strict inductive limits and these classes are not always preserved under their completions. We shall call these two classes as β-semi-Montel and β-Montel spaces. A β-semi-Montel space is obtained by replacing the word "bounded" by "strongly bounded" in the definition of a semi-Montel space. If a β-semi-Montel space is infra-barrelled, we call the space β-Montel.

In a locally convex space $E(\tau)$, if each bounded subset is relatively compact, $E(\tau)$ is semi-Montel. If $E(\tau)$ is infra-barrelled and semi-Montel, it is Montel. In this paper we weaken the conditions of being (semi-)Montel and introduce two new classes of locally convex spaces.

One contains the class of all Montel spaces and another contains the class of all semi-Montel spaces. We shall call these two classes β-Montel and β-semi-Montel spaces. Now we explain what we investigate in each section. In Section 1, we give some notations and definitions of

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β-(semi-)Montel space. In Section 2 we give some examples of β-(semi-) Montel spaces which are not (semi-)Montel. In Section 3 we consider some properties of β-(semi-)Montel spaces. Finally we investigate the separability of a metrizable β-Montel space in Section 4.

1. Notations and definitions

Mostly we shall use the notations of [3] and [6]. Let $E(\tau)$ be a Hausdorff topological vector space. Throughout this paper we assume that $E(\tau)$ is a locally convex space over the real or complex field K. For the sake of simplicity, it is denoted by lcs $E(\tau)$. E' denotes the topological dual space of $E(\tau)$. The dual of $E(\tau)$ always means the topological dual space. When two vector spaces E and F over K form a dual pair, $\sigma(E, F)$, $\tau(E, F)$, $\beta(E, F)$ and $\beta^*(E, F)$ are the topology of uniform convergence on the set of all finite subsets, all absolutely convex $\sigma(F, E)$-compact subsets, all $\sigma(F, E)$-bounded subsets and all $\beta(F, E)$-bounded subsets of F on E respectively. Let $E(\tau)$ be a locally convex space and E' be its dual. $\tau_c(E', E)$ means the topology of uniform convergence on the set of all τ-precompact subsets of E on E'. $E(\tau)$ is said to be a countably barrelled space if each $\sigma(E', E)$-bounded subset of E' which is the countable union of equicontinuous subsets of E' is itself equicontinuous [4]. $E(\tau)$ is said to be a W-space if each $\sigma(E', E)$-bounded subset of E' is $\beta(E', E)$-bounded [5]. We say that an lcs $E(\tau)$ possesses a fundamental sequence of bounded subsets if there exists a sequence $B_1 \subset B_2 \subset \ldots \subset B_n \subset \ldots$ of bounded subsets in $E(\tau)$ such that every bounded subset B is contained in some B_k.

DEFINITION. Let $E(\tau)$ be a locally convex space and E' be its dual.

(1) $E(\tau)$ is said to be a *β-semi-Montel space* if each $\beta(E, E')$-bounded subset is relatively τ-compact.

(2) $E(\tau)$ is said to be a *β-Montel space* if it is a β-semi-Montel space and infra-barrelled.

REMARK 1. Clearly every Montel space is β-Montel. Every semi-Montel space is β-semi-Montel and every β-Montel space is β-semi-Montel.

PROPOSITION 1. *Let $E(\tau)$ be a β-semi-Montel space and E' be its* *dual. Then the following conditions are equivalent:*

(1) $E(\tau)$ *is a semi-Montel space;*

(2) $E(\tau)$ *is a W-space;*

(3) $E(\tau)$ *is sequentially complete.*

Proof. Clearly every semi-Montel space is sequentially complete. If $E(\tau)$ is sequentially complete, τ-boundedness is identical with $\beta(E, E')$-boundedness. Hence $E(\tau)$ is a W-space. Finally if $E(\tau)$ is a W-space, it is semi-Montel from the assumption and definition.

2. Examples of β-(semi-)Montel spaces

EXAMPLE 1. Let $E(\tau)$ be an infra-barrelled locally convex space and not barrelled and E' be its dual. Then $E'(\sigma(E', E))$ is β-semi-Montel but not semi-Montel. If B is any $\beta(E', E)$-bounded subset in $E'(\sigma(E', E))$, it is relatively $\sigma(E', E)$-compact since it is a τ-equicontinuous subset. If $E'(\sigma(E', E))$ is semi-Montel, $E(\tau)$ is a barrelled space. Therefore $E'(\sigma(E', E))$ is not a semi-Montel space.

EXAMPLE 2. Let T be a completely regular Hausdorff space. $C_s(T)$ denotes all continuous real valued functions on T with the topology of simple convergence. Then the dual of $C_s(T)$ consists of all bounded Radon measures on T with finite support. We denote this dual by $M_f(T)$.

$C_s(T)$ is always infra-barrelled from Corollary 4 of [2]. If T is $[0, 1]$ with the usual topology, $C_s([0, 1])$ is not barrelled from Corollary 13 of [2]. Hence in the dual pair $(C_s([0, 1]), M_f([0, 1]))$, $M_f([0, 1])(\sigma(M_f([0, 1]), C_s([0, 1])))$ is β-semi-Montel but not semi-Montel.

EXAMPLE 3. We give another example from sequences spaces.

From Example F of [7],

$$\psi = \{x \in K^N : x \text{ has finitely many non-zero coordinates}\},$$

however K is a real or complex field. For each element x of ψ, a norm is given by $\|x\|_\infty = \sup_{i \in N} |x_i|$. $(\psi, \|\ \|_\infty)$ is a normed space and not

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barrelled since $A = \{x : x \in \psi, |x_n| \le 1/n, n = 1, 2, \ldots\}$ is a barrel in $(\psi, \| \|_\infty)$ but not a 0-neighbourhood in $(\psi, \| \|_\infty)$. As $(\psi, \| \|_\infty)$ is a dense subspace of $(c_0, \| \|_\infty)$, the dual of $(\psi, \| \|_\infty)$ is l^1. In the dual pair (ψ, l^1), we make some remarks about $l^1(\sigma(l^1, \psi))$.

(1) $l^1(\sigma(l^1, \psi))$ is β-semi-Montel but not semi-Montel.

(2) $l^1(\sigma(l^1, \psi))$ is metrizable since it has a countable base of 0-neighbourhoods.

From (1) and (2), $l^1(\sigma(l^1, \psi))$ is a β-Montel space but not a Montel space.

Next we give a proposition generalizing Example 3. Before this, we use the following notations.

Let X be a set such that $|X| \ge \aleph_0$ and K be a real or complex field. For an arbitrary positive number p with $1 \le p < \infty$, we put

$$l^p(X) = \left\{(z_x)_{x\in X} : (z_x)_{x\in X} \in K^X, \left(\sum_{x\in X} |z_x|^p\right)^{1/p} < \infty\right\},$$

$$l^\infty(X) = \left\{(z_x)_{x\in X} : (z_x)_{x\in X} \in K^X, \sup_{x\in X} |z_x| < \infty\right\}$$

and

$$\psi(X) = \left\{(z_x)_{x\in X} : (z_x)_{x\in X} \in K^X, \ (z_x)_{x\in X} \text{ has finitely many non-zero coordinates}\right\}.$$

For an arbitrary positive number p with $1 \le p < \infty$, we usually give a norm on $l^p(X)$ and $\psi(X)$ such that

$$\|(z_x)_{x\in X}\|_p = \left(\sum_{x\in X} |z_x|^p\right)^{1/p} \quad \text{for} \quad (z_x)_{x\in X} \in l^p(X),$$

$$\|(z_x)_{x\in X}\|_\infty = \sup_{x\in X} |z_x| \quad \text{for} \quad (z_x)_{x\in X} \in \psi(X).$$

Then $\left(l^p(X), \| \|_p\right)$ is a Banach space with the dual $l^q(X)$, where

$1/p + 1/q = 1$ (if $p = 1$, $q = \infty$). Then we have the following.

PROPOSITION 2. *Let p be a positive number with $1 \le p \le \infty$. Then $l^p(X)\left(\sigma\left(l^p(X), \psi(X)\right)\right)$ is a β-Montel space but not a Montel space.*

Proof. As any $\beta\left(\psi(X), l^p(X)\right)$-bounded subset is finite dimensional, $l^p(X)\left(\sigma\left(l^p(X), \psi(X)\right)\right)$ is infra-barrelled for $1 \le p \le \infty$. Next we shall show that $l^p(X)\left(\sigma\left(l^p(X), \psi(X)\right)\right)$ is β-semi-Montel but not semi-Montel.

For each p with $1 < p \le \infty$, there is a positive q such that $1/p + 1/q = 1$ (if $p = \infty$, $q = 1$). $\left(\psi(X), \|\ \|_q\right)$ is a dense subspace of $\left(l^q(X), \|\ \|_q\right)$ with the dual $l^p(X)$. In case of $p = 1$, $\left(\psi(X), \|\ \|_\infty\right)$ is a normed space with the dual $l^1(X)$. Hence $l^p(X)\left(\sigma\left(l^p(X), \psi(X)\right)\right)$ is β-semi-Montel. On the other hand let Y be a countable subset of X such that $Y = \{x_i : x_i \in X, i \in N\}$ and we consider the sequence $\{z^n\}$ such that $z^n_x = i$ for $x = x_i$, $i = 1, 2, \ldots, n$ and $z^n_x = 0$ for $x \ne x_i$, $i = 1, 2, \ldots, n$ for each $n \in N$. Then in $K^X\left(\sigma\left(K^X, \psi(X)\right)\right)$, $\{z^n\}$ converges to $z = \left(z_x\right)_{x \in X}$ where $z_x = i$ for $x = x_i$, $i = 1, 2, \ldots$, and $z_x = 0$ otherwise. However z does not belong to $l^p(X)$ for $1 \le p \le \infty$.

Consequently $l^p(X)\left(\sigma\left(l^p(X), \psi(X)\right)\right)$ is not semi-Montel. This completes the proof.

3. Some properties of β-(semi-)Montel spaces

First of all we give a few permanence properties of β-(semi-)Montel spaces.

PROPOSITION 3. *The product space $E(\tau) = \prod_{\alpha \in I} E_\alpha\left(\tau_\alpha\right)$ of β-(semi-) Montel spaces $E_\alpha\left(\tau_\alpha\right)$ $(\alpha \in I)$ is β-(semi-)Montel.*

Proof. If $E_\alpha\left(\tau_\alpha\right)$ $(\alpha \in I)$ is infra-barrelled, $\prod_{\alpha \in I} E_\alpha\left(\tau_\alpha\right)$ is infra-barrelled. Let E'_α $(\alpha \in I)$ be the dual of $E_\alpha\left(\tau_\alpha\right)$ and

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$E'_\alpha = \bigoplus_{\alpha\in I} E'_\alpha$ be the dual of $E(\tau)$. As $E\big(\beta(E, E')\big) = \prod_{\alpha\in I} E_\alpha\big(\beta\big(E_\alpha, E'_\alpha\big)\big)$ in the dual pair $\left(\prod_{\alpha\in I} E_\alpha, \bigoplus_{\alpha\in I} E'_\alpha\right)$, for any $\beta(E, E')$-bounded subset B , there exists a B_α $(\alpha \in I)$ which is $\beta\big(E_\alpha, E'_\alpha\big)$-bounded and B is contained in $\prod_{\alpha\in I} B_\alpha$. As each B_α is relatively τ_α-compact, $\prod_{\alpha\in I} B_\alpha$ is relatively compact in $E(\tau)$, so is B .

PROPOSITION 4. *The locally convex direct sum* $E(\tau) = \bigoplus_{\alpha\in I} E_\alpha\big(\tau_\alpha\big)$ *of* β-*(semi-)Montel spaces* $E_\alpha\big(\tau_\alpha\big)$ $(\alpha \in I)$ *is* β-*(semi-)Montel.*

Proof. Let E'_α $(\alpha \in I)$ be the dual of $E_\alpha\big(\tau_\alpha\big)$ and $E' = \prod_{\alpha\in I} E'_\alpha$ be the dual of $E(\tau)$. As $E\big(\beta(E, E')\big) = \bigoplus_{\alpha\in I} E_\alpha\big(\beta\big(E_\alpha, E'_\alpha\big)\big)$ in the dual pair $\left(\bigoplus_{\alpha\in I} E_\alpha, \prod_{\alpha\in I} E'_\alpha\right)$, every $\beta(E, E')$-bounded subset B is contained in $B_{\alpha_1} + \ldots + B_{\alpha_n}$ (each B_{α_i} is $\beta\big(E_{\alpha_i}, E'_{\alpha_i}\big)$-bounded in $E_{\alpha_i}\big(\tau_{\alpha_i}\big)$, $i = 1, 2, \ldots, n$). As $B_{\alpha_1} + \ldots + B_{\alpha_n}$ is relatively compact in $E(\tau)$, so is B . Clearly $E(\tau)$ is infra-barrelled if $E_\alpha\big(\tau_\alpha\big)$ $(\alpha \in I)$ is infra-barrelled.

PROPOSITION 5. *A closed subspace* $H(\bar\tau)$ *of a* β-*semi-Montel space* $E(\tau)$ *is* β-*semi-Montel.*

Proof. Let H' be the dual of $H(\bar\tau)$ and E' be the dual of $E(\tau)$.

If B is an arbitrary $\beta(H, H')$-bounded subset, it is $\beta(E, E')$-bounded in $E(\tau)$ since $\beta(H, H')$ is finer than the topology $\beta(E, E')$ on H .

Hence B is $\bar\tau$-relatively compact.

COROLLARY. *A topological projective limit* $E(\tau) = \varprojlim A_{\alpha\beta}\big(E_\beta\big(\tau_\beta\big)\big)$ *is* β-*semi-Montel if* $E_\alpha\big(\tau_\alpha\big)$ $(\alpha \in I)$ *is* β-*semi-Montel.*

Next we give the other properties related to β-(semi-)Montel spaces.

PROPOSITION 6. *Let* $E(\tau)$ *be a countably barrelled, separable and metrizable locally convex space and* E' *be its dual. If* N *is a*

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countable dense subset in $E(\tau)$ *and* F *is a subspace of* E *which is spanned by* N , *then* $E'(\sigma(E', F))$ *is a* β-*Montel space.*

Proof. $E'(\sigma(E', F))$ is metrizable since it has a countable base of 0-neighbourhoods. $E(\tau)$ is barrelled from the assumption. If B is any $\sigma(E', E)$-bounded subset, it is relatively $\sigma(E', E)$-compact. To show $\sigma(E', E) \leq \beta(E', F)$, it suffices to show that for each element x of E , there is a $\sigma(F, E')$-bounded subset C such that x is an element of τ-closure of C . For an arbitrary element x of E , there is a sequence $\{x_n\}$ such that each x_n is an element of F and $\{x_n\}$ converges to x from its separability and metrizability. If C is the sequence $\{x_n\}$, it is $\sigma(F, E')$-bounded and x is an element of τ-closure of C . Now if B is an arbitrary $\beta(E', F)$-bounded subset, it is $\sigma(E', E)$-bounded.

Thus B is relatively $\sigma(E', F)$-compact. So $E'(\sigma(E', F))$ is β-Montel.

Let $E(\tau)$ be a Montel space and E' be its dual. Then the strong dual $E'(\beta(E', E))$ is also a Montel space. In case of a β-Montel space, a similar proposition holds.

PROPOSITION 7. *Let* $E(\tau)$ *be a* β-*Montel space and* E' *be its dual. Then* $E'(\beta^*(E', E))$ *is* β-*Montel.*

Proof. If B is any absolutely convex and $\beta(E, E')$-bounded subset, it is relatively $\sigma(E, E')$-compact from the assumption. Then $E'(\tau(E', E))$ is infra-barrelled since $\beta^*(E', E) = \tau(E', E)$. Next we show that any $\beta(E', E)$-bounded subset in $E'(\tau(E', E))$ is relatively $\tau(E', E)$-compact.

If B is any $\beta(E, E')$-bounded subset in $E(\tau)$, it is τ-precompact.

From this $\tau(E', E) = \beta^*(E', E) \leq \tau_c(E', E)$. If C is an arbitrary $\beta(E', E)$-bounded subset in $E'(\tau(E', E))$, it is a τ-equicontinuous subset.

Consequently C is relatively $\sigma(E', E)$-compact and relatively $\tau_c(E', E)$-compact from the property of the topology $\tau_c(E', E)$. Hence it is relatively $\tau(E', E)$-compact. Thus $E'(\beta^*(E', E))$ is β-Montel.

Using Proposition 7, we can give a β-Montel space whose completion is not β-semi-Montel.

EXAMPLE 4. From Example 3, $l^1(\sigma(l^1, \psi))$ is β-Montel. $\psi(\beta^*(\psi, l^1))$ is β-Montel from Proposition 7. On the other hand, $(\psi, \| \|_\infty)$ is a normed space with the dual l^1. Thus $(\psi, \| \|_\infty)$ is β-Montel and the completion of $(\psi, \| \|_\infty)$ is $(c_0, \| \|_\infty)$. As $(c_0, \| \|_\infty)$ is an infinite dimensional Banach space, it is not β-semi-Montel.

From Example 4 we obtain the following.

THEOREM 1. *Under the conditions of Proposition 6, the subspace* $F(\bar{\tau})$ *of* $E(\tau)$ *is a* β*-Montel space.*

Proof. From Proposition 6, $E'(\sigma(E', F))$ is a β-Montel space and $F(\beta^*(F, E'))$ is a β-Montel space from Proposition 7. $F(\bar{\tau})$ is a dense subspace of $E(\tau)$, so the dual of $F(\bar{\tau})$ is E'.

As $F(\bar{\tau})$ is metrizable, $F(\bar{\tau}) = F(\beta^*(F, E'))$.

REMARK 2. Let $E(\tau)$ be the strict inductive limit of β-semi-Montel spaces $E_n(\tau_n)$, $n = 1, 2, \ldots$, and E' be the dual of $E(\tau)$ and B be an arbitrary $\beta(E, E')$-bounded subset in $E(\tau)$. Then it is not known whether we can find a space $E_n(\tau_n)$ where B is $\beta(E_n, E_n')$-bounded. (E_n' is the dual of $E_n(\tau_n)$.) About the other construction appeared in [7], as in the case of a (semi-)Montel space, a β-(semi-)Montel space is not always preserved under them.

4. Separability of a metrizable β-Montel space

In general, every Fréchet Montel space is separable. In this section we consider whether every metrizable β-Montel space is separable. In fact the space is not always separable. Here we give an example of the above fact.

EXAMPLE 5. Let X be a set such that $|X| > \aleph_0$. From Proposition 2, $l^1(X)(\sigma(l^1(X), \psi(X)))$ is β-Montel; and from Proposition 7, $\psi(X)(\beta^*(\psi(X), l^1(X)))$ is also β-Montel. However $(\psi(X), \| \|_\infty)$ is a normed space whose dual is $l^1(X)$. Hence $(\psi(X), \| \|_\infty)$ is a metrizable β-Montel space and clearly not separable.

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However we obtain a theorem that a metrizable β-Montel space is separable under the following condition.

THEOREM 2. *Let $E(\tau)$ be a metrizable β-Montel space and E' be its dual. If $E'(\sigma(E', E))$ has a fundamental sequence of bounded subsets, $E(\tau)$ is separable.*

Proof. $E(\beta(E, E'))$ is metrizable from the assumption. Let $\| \|_\tau$ and $\| \|_\beta$ be the F-norms on $E(\tau)$ and $E(\beta(E, E'))$. Then if $E(\tau)$ is not separable, there exist a positive number δ and an uncountable subset A of E such that $\|z-z'\|_\tau > \delta$, for $z, z' \in A$, $z \neq z'$.

For $n \in N$ we put $K_n = \{x : \|x\|_\beta < 1/n, x \in E\}$ and for an arbitrary $x \in E$, we denote $\inf\{t : t > 0, x/t \in K_n\}$ by $[x]_n$. Then there is a positive number M_1 such that $A_1 = A \cap \{x : [x]_1 < M_1, x \in E\}$ is an uncountable set.

Similarly for each $n \in N$, $n \geq 2$, there is a positive number M_n such that $A_n = A_{n-1} \cap \{x : [x]_n < M_n, x \in E\}$ is an uncountable set. We obtain a sequence of uncountable subsets of E . For each A_j we take an element $z_j \in A_j$. $\{z_j\}$ is $\beta(E, E')$-bounded since for each K_n , $\{z_j\} \subset \max\{m_1, m_2, \ldots, m_{n-1}, M_n\} \cdot K_n$ $(z_i \in m_i \cdot K_n, i = 1, 2, \ldots, n-1)$. Hence $\{z_j\}$ is relatively τ-compact. On the other hand, $\|z_i - z_j\|_\tau > \delta$ for $i \neq j$. This leads to a contradiction.

REMARK 3. As an example of Theorem 2, $l^p(\sigma(l^p, \psi))$ $(1 \leq p \leq \infty)$ is given. Conversely let $E(\tau)$ be a separable, metrizable and β-Montel space and E' be its dual. Then $E'(\sigma(E', E))$ does not always have a fundamental sequence of bounded subsets. For example, we put $E(\tau) = (\psi, \| \|_\infty)$ then the weak dual $l^1(\sigma(l^1, \psi))$ does not have a fundamental sequence of bounded subsets.

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