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ON COMMUTATIVITY OF RINGS WITH SOME POLYNOMIAL CONSTRAINTS

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Let R be an associative ring with unity 1, $N(R)$ the set of nilpotents, $J(R)$ the Jacobson radical of R and $n > 1$ be a fixed integer. We prove that R is commutative if and only if it satisfies $(xy)^n = y^n x^n$ for all $x, y \in R \setminus N(R)$ and commutators in R are $n(n+1)$-torsion free. Moreover, we extend the same result in the case when $x, y \in R \setminus J(R)$.

1. Introduction

Throughout R will denote an associative ring with unity 1. Let us denote the centre of ring R by $Z(R)$, the commutator ideal by $C(R)$, the Jacobson radical by $J(R)$, the set of nilpotents by $N(R)$ and for any x, y in R, $[x, y] = xy - yx$. Let $n > 1$ be a fixed integer. We consider the following properties:

(I) $(xy)^n = x^n y^n$, for all x, y in R.
(II) $(xy)^n = x^n y^n$, for all x, y in $R \setminus N(R)$.
(III) $(xy)^n = x^n y^n$, for all x, y in $R \setminus J(R)$.
(IV) $(xy)^n = y^n x^n$, for all x, y in R.
(V) $(xy)^n = y^n x^n$, for all x, y in $R \setminus N(R)$.
(VI) $(xy)^n = y^n x^n$, for all x, y in $R \setminus J(R)$.
(VII) For all x, y in R $n(n+1)[x, y] = 0$ implies $[x, y] = 0$.

A well-known theorem of Herstein [6] asserting that a ring R, which satisfies (I), must have nil commutator ideal, has recently been generalised by Bell [4] as follows: "If R is an n-torsion free ring and satisfies (I) for two consecutive integers $n, n+1$, then R is commutative". Abu-Khuzam [1] proved that if R is an $n(n-1)$-torsion free ring satisfying (I), then R is commutative. Further, Bell and Yaqub [5] established commutativity of the ring R for the case when condition (I) is replaced by either of conditions (II) or (III) in the hypotheses of the last result. In [11] the authors proved that if R satisfies (IV) for two consecutive integers $n, n+1$ together with the condition $(x+y)^n = x^n + y^n$, then R is commutative. In this direction we prove the following:

Theorem 1. *A ring R is commutative if and only if it satisfies (V) and (VII).*

Theorem 2. *A ring R is commutative if and only if it satisfies (VI) and (VII).*

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2. Main results

The following results are pertinent for developing the proofs of the above theorems. Proofs of Lemma 1 and Lemma 2 can be found in [3] and [8] respectively. Moreover, they hold even for rings without unity. Lemma 3 has also been proved in [9], but here we are supplying an independent and elementary proof.

LEMMA 1. *Let R be a ring satisfying an identity $q(X) = 0$, where $q(X)$ is a polynomial in noncommuting indeterminates, its coefficients being integers with highest common factor one. If there exists no prime p for which the ring of 2×2 matrices over $GF(p)$ satisfies $q(X) = 0$, then R has a nil commutator ideal and the nilpotent elements of R form an ideal*

LEMMA 2. *If $x, y \in R$ and $[x,[x,y]] = 0$, then $[x^m, y] = mx^{m-1}[x,y]$ for all positive integers m.*

LEMMA 3. *Let R be a ring and $f : R \to R$ be a function such that $f(x+1) = f(x)$ holds for all x in R. If for some positive integer m, $x^m f(x) = 0$ for all x in R, then necessarily $f(x) = 0$.*

PROOF: We have $f(x) = \{(1+x) - x\}^{2m+1} f(x)$. On expanding the right hand side expression by the binomial theorem and using the fact that $x^m f(x) = 0$ and $(1+x)^m f(x) = 0$, we immediately get $f(x) = 0$. ∎

PROOF OF THEOREM 1: Let u, v be units in R. On replacing x by $u^{-1}v$ and y by u in the given condition, we get

$$u^{-1}v^n u = \left(u^{-1}vu\right)^n = u^n\left(u^{-1}v\right)^n = u^n v^n u^{-n}$$

which implies that

$$[u^{n+1}, v^n] = 0 \quad \text{for all units} \quad u, v \text{ in } R. \tag{1}$$

This readily yields that $[u^{n+1}, (v^n)^{n+1}] = 0$, $[u^n, (v^{n+1})^n] = 0$ and hence

$$[u, v^{n(n+1)}] = 0 \quad \text{for all units } u, v \text{ in } R. \tag{2}$$

Let $a \in N(R)$. Then there exists a minimal positive integer p such that

$$[u, a^k] = 0 \quad \text{for all integers } k \geqslant p. \tag{3}$$

Suppose $p > 1$, then $\left(1 + a^{p-1}\right)$ is a unit in R and hence by (2) and (3) we have, $0 = [u, \left(1 + a^{p-1}\right)^{n(n+1)}] = n(n+1)[u, a^{p-1}]$, which implies that $[u, a^{p-1}] = 0$. This contradicts the minimality of p. Hence $p = 1$ and (3) implies that

$$[u, a] = 0 \text{ for all units } u \text{ in } R \text{ and all nilpotents } a \text{ in } R. \tag{4}$$

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Now suppose $x \in R$ and again u is a unit in R. If $ux \in N(R)$, then by (4), $[u, ux] = 0$ and hence $[u, x] = 0$. Next, suppose that $ux \notin N(R)$, then $ux^n u^{-1} = \left((ux)u^{-1}\right)^n = u^{-n}x^n u^n$ and hence $[u^{n+1}, x^n] = 0$. Using the same arguments as used to get (4) from (2), we have

(5) $$[a, x^n] = 0 \text{ for all } x \text{ in } R \text{ and all nilpotents } a \text{ in } R.$$

Let $S = \langle x^n : x \in R \rangle$ be the subring generated by all nth powers of elements of R. Thus (5) implies that $N(S) \subseteq Z(S)$. Clearly S satisfies the hypotheses placed on R. In fact, all nilpotent elements of S are central and hence S satisfies $(xy)^n = y^n x^n$ for all x, y in S, which is a polynomial identity with coprime integral coefficients. But if we consider $x = e_{12}$, $y = e_{21}$, we find that no 2×2 matrices over $GF(p)$, p a prime satisfies this identity. Hence by Lemma 1, the commutator ideal $C(S)$ of S is nil. Thus in view of the above arguments we get

(6) $$C(S) \subseteq N(S) \subseteq Z(S).$$

For all x, y in S, $(xy)^n x = x(yx)^n$, which implies that $[x^{n+1}, y^n] = 0$.

Now using (6) and Lemma 2, we have $(n+1)x^n[x, y^n] = 0$ and hence $x^n[x, (n+1)y^n] = 0$. Replacing x by $(x+1)$ we see that $(x+1)^n[x, (n+1)y^n] = 0$, and hence, by Lemma 3, $(n+1)[x, y^n] = 0$. Using hypothesis (VII), we get $[x, y^n] = 0$. Again using (6), Lemma 2, Lemma 3 and hypothesis (VIII), we conclude that $[x, y] = 0$ for all x, y in S. Therefore

(7) $$[x^n, y^n] = 0 \text{ for all } x, y \text{ in } R.$$

Now we observe that $x^{n+1}y^{n+1} = x(x^n y^n)y = x(yx)^n y$, that is

(8) $$x^{n+1}y^{n+1} = (xy)^{n+1} \text{ for all } x, y \text{ in } R \setminus N(R).$$

If u, v are units in R, then we find that

(9) $$[u^{n+1}, v^n] = 0 \text{ for all units } u, v \text{ in } R.$$

Similar arguments to those used in getting (5) from (1) yield the following from (6) (use (8) in place of (V)).

(10) $$[a, x^{n+1}] = 0 \text{ for all } x \text{ in } R \text{ and all nilpotents } a \text{ in } R.$$

Thus (5) and (10) yield, $[a, x] = 0$ for all x in R that is $N(R) \subseteq Z(R)$, But by [7], (7) yields that the commutator ideal $C(R)$ of R is nil. Hence we have

(11) $$C(R) \subseteq N(R) \subseteq Z(R).$$

To complete the proof of our theorem, let $x, y \in R$. By (7), $[x^n, y^n] = 0$. Using (11) and Lemma 2, we find that $nx^{n-1}[x, y^n] = 0$. Replace x by $(x+1)$ to get $(x+1)^n[x, ny^n] = 0$. Hence by Lemma 3, $n[x, y^n] = 0$ and using hypothesis (VII) we get $[x, y^n] = 0$. Applying the same argument to $[x, y^n] = 0$, we see that $y^{n-1}[x, ny] = 0 = (y+1)^{n-1}[x, ny]$ and hence by Lemma 3 and (VII), $[x, y] = 0$. This completes the proof. ∎

As a consequence of the above theroem, we can derive the following:

COROLLARY. *Let R be a ring satisfying (IV) and (VII). Then R is commutative.*

PROOF OF THEOREM 2: The arguments used in the proof of Theorem 1 are still valid in the present situation. Hence, we get

(1) $$[u, v^{n(n+1)}] = 0 \text{ for all units } u, v \text{ in } R.$$

(2) $$[u, a] = 0 \text{ for all } a \in N(R) \text{ and all units } u \text{ in } R.$$

Now if $x, y \in J(R)$, then (1) yields

(3) $$[(1+x), (1+y)^{n(n+1)}] = 0 \text{ for all } x, y \text{ in } J(R).$$

Using the structure theory of rings, it can easily be verified that if R is a semisimple ring satisfying $(xy)^n = y^n x^n$, then R is commutative. Thus by (VI), $R/J(R)$ is commutative and hence

(4) $$C(R) \subseteq J(R).$$

Now if $c_1 = [x_1, y_1]$, $c_2 = [x_2, y_2]$, $c_3 = [x_3, y_3]$ be any commutators then by (3) and (4), we get

(5) $$[(1+c_3), (1+c_1+c_2+c_1c_2)^{n(n+1)}] = 0.$$

This is a polynomial identity satisfied by all elements of R. Now consideration of $c_1 = c_3 = [e_{11}, e_{11}+e_{12}]$, $c_2 = [e_{11}+e_{12}, e_{21}]$ shows that no 2×2 matrices over $GF(p)$, p a prime, satisfies (5) and hence by Lemma 1, the commutator ideal of R is nil and the set of nilpotents form an ideal. Combining this with (2), we find that

(6) $$[(1+x), [(1+x), (1+y)]] = 0 \text{ for all } x, y \in J(R).$$

Using (3), (6) and Lemma 2, we get $n(n+1)(1+y)^{n^2+n-1}[1+x, 1+y] = 0$ that is $(1+y)^{n^2+n-1}[1+x, n(n+1)(1+y)] = 0$. Hence by (VII), $[x, y] = 0$ for all x, y in

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$J(R)$. Thus $J(R)$ is commutative and $(J(R))^2$ is central. An easy induction shows that

(7) $$(xy)^k = y^k x^k \text{ for all } k \geqslant 2; x, y \in J(R).$$

Combining (7) together with (VI), we find that R satisfies (IV). Hence the commutativity of R follows by the Corollary to Theorem 1. ∎

The noncommutative ring of 3×3 strictly upper triangular matrices over the ring $\mathbb{Z}$ of integers shows that the existence of unity 1 in both the theorems is essential. Further, we provide the following example to show that condition (VII), where commutators are $n(n+1)$-torsion free cannot be replaced by "n-torsion free" or "$(n+1)$-torsion free" even if the given condition $(xy)^n = y^n x^n$ holds for all x, y in R.

Example. Let

$$R = \left\{ aI + B \mid B = \begin{pmatrix} o & b & c \\ o & o & d \\ o & o & o \end{pmatrix}, I = \begin{pmatrix} 1 & o & o \\ o & 1 & o \\ o & o & 1 \end{pmatrix}, a, b, c, d \in GF(3) \right\}.$$

It can easily be verified that $(xy)^2 = y^2 x^2$ and $(xy)^3 = y^3 x^3$. Thus with $n = 2$, $n[x, y] = 0$ implies $[x, y] = 0$ and $(xy)^n = y^n x^n$. Also with $n = 3$, $(n+1)[x, y] = 0$ implies that $[x, y] = 0$ and $(xy)^n = y^n x^n$ for all x, y in R. However, R is not commutative.

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