b-GENERALIZED DERIVATIONS OF SEMIPRIME RINGS HAVING NILPOTENT VALUES

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Abstract

Let $R$ be a semiprime ring with extended centroid $C$ and with maximal right ring of quotients $Q_{mr}(R)$. Let $d: R \rightarrow Q_{mr}(R)$ be an additive map and $b \in Q_{mr}(R)$. An additive map $\delta: R \rightarrow Q_{mr}(R)$ is called a (left) $b$-generalized derivation with associated map $d$ if $\delta(xy) = \delta(x)y + bxd(y)$ for all $x, y \in R$. This gives a unified viewpoint of derivations, generalized derivations and generalized $\sigma$-derivations with an $X$-inner automorphism $\sigma$. We give a complete characterization of $b$-generalized derivations of $R$ having nilpotent values of bounded index. This extends several known results in the literature.

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1. Results

Throughout the paper, unless specially stated, $R$ is always a semiprime ring with Martindale symmetric ring of quotients $Q_s(R)$. We let $Q_{mr}(R)$ (respectively $Q_{ml}(R)$) denote the maximal right (respectively left) ring of quotients of $R$. It is known that $R \subseteq Q_s(R) \subseteq Q_{mr}(R)$. The overrings $Q_s(R)$ and $Q_{mr}(R)$ of $R$ are semiprime rings with the same center $C$, which is a regular self-injective ring. The ring $C$ is called the extended centroid of $R$. Also, $R$ is a prime ring if and only if $C$ is a field. We refer the reader to the book [1] for details.

An additive map $d: R \rightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. For $b \in R$, we let $ad(b)$ denote the map $x \mapsto [b, x] := bx - xb$ for $x \in R$. Clearly, $ad(b)$ is a derivation of $R$, which is called the inner derivation of $R$ induced...
by the element $b$. It is known that any derivation $d$ of $R$ can be uniquely extended to a derivation of $Q_{mr}(R)$. A derivation $d: R \to R$ is called $X$-inner if its extension to $Q_{mr}(R)$ is inner. In this case, it is easy to check that $d = \text{ad}(q)$ for some $q \in Q_1(R)$. An additive map $\delta: R \to R$ is called a generalized derivation if there exists a derivation $d$ of $R$ such that $\delta(xy) = \delta(x)y + xd(y)$ for all $x, y \in R$ (see [2, 14, 18]). The derivation $d$ is uniquely determined by $\delta$, and is called the associated derivation of $\delta$.

Let $\sigma$ be an automorphism of $R$. An additive map $\delta: R \to R$ is called a (right) $\sigma$-derivation if $\delta(xy) = x\delta(y) + \delta(x)\sigma(y)$ for $x, y \in R$. Basic examples of $\sigma$-derivations are derivations and $\sigma - 1$. Given $b \in R$, the map $x \mapsto xb - b\sigma(x)$ for $x \in R$ obviously defines a $\sigma$-derivation, which is called the inner $\sigma$-derivation induced by $b$. It is clear that any $\sigma$-derivation of $R$ can be uniquely extended to a $\sigma$-derivation of $Q_{mr}(R)$. In [21], Lee and Liu gave a common generalization of both generalized derivations and $\sigma$-derivations. An additive map $g: R \to R$ is called a right generalized $\sigma$-derivation if there exists an additive map $\delta: R \to R$ such that $g(xy) = xg(y) + \delta(x)\sigma(y)$ for all $x, y \in R$. It is clear that $\delta$ is uniquely determined by the map $g$. The additive map $\delta$ is called the associated map of $g$. Our present study is motivated by the following results.

Let $d: R \to R$ be a derivation, $\delta: R \to R$ a generalized derivation, $g: R \to R$ a right generalized $\sigma$-derivation, and $n$ a fixed positive integer. Also, the rings $R$ in (4)–(6) are prime.

1. Suppose that $d(x)^n = 0$ for all $x \in R$. Then $d = 0$ (see [10, 12, 13]).

2. Let $\lambda$ be a left ideal of $R$. Suppose that $d(x)^n = 0$ for all $x \in \lambda$. Then $\lambda d(\lambda) = 0$ (see [16, Theorem 6]).

3. Suppose that $\delta(x)^n = 0$ for all $x \in R$. Then $\delta = 0$ (see [18, Theorem 5]).

4. Suppose that $\delta(x)^n = 0$ for all $x \in \rho$, a right ideal of $R$. Then there exist $b, c \in Q_{mr}(R)$ and $\beta \in C$ such that $\delta(x) = bx - xc$ for all $x \in R$ and $(b - \beta)\rho = 0 = (c - \beta)\rho$ (see [18, Theorem 6]).

5. Suppose that $g(x)^n = 0$ for all $x \in R$. Then $g = 0$ (see [21, Theorem 2.7]).

6. Let $a, b, q \in Q_{mr}(R)$. Suppose that $(a\delta(qx) - bx)^n = 0$ for all $x \in R$. Then either $a\delta(q) - b = 0 = aq$ or there exist $a_0, b_0 \in Q_{mr}(R)$ and $\mu \in C$ such that $\delta(x) = a_0x + xb_0$ for $x \in R$ and $aa_0q - b = -b_0aq = \mu aq$.

Let us consider a special case of (5). Suppose that the extension of $\sigma$ to $Q_{ml}(R)$ is inner; that is, there exists a unit $u \in Q_{ml}(R)$ such that $\sigma(x) = uxu^{-1}$ for $x \in R$. Let $\delta$ be the associated map of $g$. Then $g(xy) = xg(y) + d(x)y^{-1}$ for all $x, y \in R$, where $d(x) := \delta(x)u$ for $x \in R$. Notice that $d: R \to Q_{ml}(R)$. See [3, 4] for the Lie ideal case.

In (6), let $d: R \to R$ be the associated derivation of $\delta$; that is, $\delta(xy) = \delta(x)y + xd(y)$ for $x, y \in R$. We let $\tilde{\delta}(x) := a\delta(qx) - bx$ for $x \in R$. Then $\tilde{\delta}(x) = aqd(x) + (a\delta(q) - b)x$ for $x \in R$. A direct computation shows that $\tilde{\delta}(xy) = \tilde{\delta}(x)y + (aq)x$ for all $x, y \in R$. Since $d$ can be uniquely extended to $Q_{mr}(R)$, so can $\tilde{\delta}$. In view of [17, Theorem 3] (or see Fact 1.5 below), $R$ and $Q_{mr}(R)$ satisfy the same differential identities. Thus, $\tilde{\delta}(x)^n = 0$ for all $x \in Q_{mr}(R)$.
Motivated by the results (1)–(6) above, we give the following definition.

**Definition 1.1.** (1) Let \( d : R \to Q_{mr}(R) \) be an additive map and \( b \in Q_{mr}(R) \). An additive map \( \delta : R \to Q_{mr}(R) \) is called a (left) \( b \)-generalized derivation with associated map \( d \) if \( \delta(xy) = \delta(x)y + bx\delta(y) \) for all \( x, y \in R \).

(2) Let \( d : R \to Q_{ml}(R) \) be an additive map and \( b \in Q_{ml}(R) \). An additive map \( \delta : R \to Q_{ml}(R) \) is called a right \( b \)-generalized derivation with associated map \( d \) if \( \delta(xy) = x\delta(y) + d(x)yb \) for all \( x, y \in R \).

Clearly, a generalized derivation is a 1-generalized derivation and a right generalized \( \sigma \)-derivation is a right \( u^{-1} \)-generalized derivation if \( \sigma(x) = uuxu^{-1} \) for \( x \in R \), where \( u \) is a unit in \( Q_{ml}(R) \). For \( a, b, c \in Q_{mr}(R) \), the map \( x \mapsto ax + bxc \) for \( x \in R \) is a left \( b \)-generalized derivation. Analogously, for \( a, b, c \in Q_{ml}(R) \), the map \( x \mapsto xa + bxc \) for \( x \in R \) is a right \( c \)-generalized derivation. We note that left or right \( b \)-generalized derivations appear canonically in [7, Theorems 1.1 and 1.3]. The goal of the paper is to give a complete characterization of \( b \)-generalized derivations having nilpotent values of bounded index. By symmetry, it suffices to deal with one of left and right \( b \)-generalized derivations. For simplicity of notation, a \( b \)-generalized derivation always means a left \( b \)-generalized derivation.

To state the main theorem of the paper, we have to recall some basic properties of idempotents of \( C \). We write \( B \) for the set of all idempotents of \( C \). The set \( B \) forms a Boolean algebra with respect to the operations \( e + h := e + h - 2eh \) and \( e \cdot h := eh \) for all \( e, h \in B \). It is complete with respect to the partial order \( e \leq h \) (defined by \( eh = e \)) in the sense that any subset \( S \) of \( B \) has a supremum \( \bigvee S \) and an infimum \( \bigwedge S \). Given a subset \( S \) of \( Q_{mr}(R) \), we define \( E[S] \) to be the infimum of \( e \in B \) such that \( ex = x \) for all \( x \in S \). If \( S = \{b\} \), we write \( E[b] \) instead of \( E[S] \) for simplicity. Note that, for \( a, b \in Q_{mr}(R), \) \( aRb = 0 \) if and only if \( E[a]E[b] = 0 \). By the characterization, it is easy to see that if a \( b \)-generalized derivation \( \delta \) has associated maps \( d \) and \( d' \), then \( E[b]d(x) = E[b]d'(x) \) for all \( x \in R \). We refer the reader to the book [1] for details.

We are now in a position to state the main theorems of the paper.

**Theorem 1.2.** Let \( R \) be a semiprime ring, \( b \in Q_{mr}(R) \), and let \( \delta : R \to Q_{mr}(R) \) be a \( b \)-generalized derivation with associated map \( d \). Suppose that \( \delta(x)^n = 0 \) for all \( x \in R \), where \( n \) is a positive integer. Then there exists \( q \in Q_{mr}(R) \) such that \( E[b]d(x) = [q, x] \) for \( x \in R, \) \( \delta(x) = -bxq \) for \( x \in R \), and \( qb = 0 \).

By symmetry, we also have the following result whose proof parallels that of Theorem 1.2.

**Theorem 1.3.** Let \( R \) be a semiprime ring, \( b \in Q_{ml}(R) \), and let \( \delta : R \to Q_{ml}(R) \) be a right \( b \)-generalized derivation with associated map \( d \). Suppose that \( \delta(x)^n = 0 \) for all \( x \in R \), where \( n \) is a positive integer. Then there exists \( q \in Q_{ml}(R) \) such that \( E[b]d(x) = [q, x] \) for \( x \in R, \) \( \delta(x) = qxb \) for \( x \in R \), and \( bq = 0 \).

Let \( I \) be an ideal of \( R \). By the semiprimeness of \( R \), the left annihilator of \( I \) in \( R \) coincides with the right annihilator of \( I \) in \( R \). The ideal \( I \) is called dense if
its left annihilator in $R$ is zero. We write $C[X_1, X_2, \ldots]$ for the free algebra over $C$ in noncommutative indeterminates $X_1, X_2, \ldots$ and $Q_{mr}(R) * C [X_1, X_2, \ldots]$ for the free product of the $C$-algebras $Q_{mr}(R)$ and $C[X_1, X_2, \ldots]$. Let $f(X_i) \in Q_{mr}(R) * C C [X_1, X_2, \ldots]$ and $T$ be a subring of $Q_{mr}(R)$. We say that $f$ is a GPI (that is, a generalized polynomial identity) of $T$ if $f(x_i) = 0$ for all $x_i \in T$. By a derivation word $\Delta$, we mean that $\Delta$ is of the form $d_1 d_2 \cdots d_s$, where each $d_i$ is either a derivation of $Q_{mr}(R)$ or the identity map of $Q_{mr}(R)$. By a differential polynomial $f(X_i^{\Delta})$, we mean that all $\Delta_j$ are derivation words and $f(Z_{ij})$ is a generalized polynomial in noncommutative indeterminates $Z_{ij}$. The differential polynomial $f(X_i^{\Delta})$ is called a differential identity of $T$ if $f(x_i^{\Delta}) = 0$ for all $x_i \in T$. We will use the following facts in the proofs below.

**Fact 1.4.** Let $I$ be a dense ideal of $R$. Then $I$ and $Q_{mr}(R)$ satisfy the same GPIs with coefficients in $Q_{mr}(R)$ (see [1, Theorem 6.4.1] for a semiprime ring $R$ and [6, Theorem 2] for a prime ring $R$).

**Fact 1.5.** Let $I$ be a dense ideal of $R$. Then $I$ and $Q_{mr}(R)$ satisfy the same differential identities with coefficients in $Q_{mr}(R)$ (see [17, Theorem 3]).

**Fact 1.6.** Let $\rho$ be a right ideal of $R$ and $a \in Q_{mr}(R)$. Suppose that $(ax)^n = 0$ for all $x \in \rho$. Then $a \rho = 0$ (see Fact 1.4 and [11, Lemma 1.1]).

**Fact 1.7.** Let $\phi: I \rightarrow Q_{mr}(R)$ be a right $R$-module map, where $I$ is a dense ideal of $R$. Then there exists $a \in Q_{mr}(R)$ such that $\phi(x) = ax$ for all $x \in I$ (see [19, Lemma 2.1] with the same proof by replacing ‘a nonzero ideal in a prime ring’ with ‘a dense ideal in a semiprime ring’).

**Fact 1.8.** Let $d: R \rightarrow Q_{mr}(R)$ be a derivation. Then $d$ can be uniquely extended to a derivation from $Q_{mr}(R)$ to itself (see, for instance, [17, Lemma 2]).

2. The prime case

We begin with the following key result.

**Proposition 2.1.** Let $R$ be a prime ring, $a, b, c \in R$, and $n$ a positive integer. Suppose that $(ax + bxc)^n = 0$ for all $x \in R$. Then there exists $\beta \in C$ such that $a = \beta b$ and $(c + \beta)b = 0$.

A prime ring $R$ is called a GPI-ring if it satisfies a nontrivial (that is, nonzero) generalized polynomial with coefficients in $Q_{mr}(R)$. The prime ring $R$ is called centrally closed if $R = RC$. In particular, the prime ring $Q_{mr}(R)$ is centrally closed. The following lemma is a special case of [24, Theorem 1]. Since the proof below is neat and self-contained, we give its proof here for the convenience of the reader. We also remark that Chang proved the following lemma with the extra assumption that $b$ is invertible in $R$ (see [5, Lemma 2.1]).
Lemma 2.2. Let $R$ be a prime ring, $a, b, c \in R$, and $n$ a positive integer. Suppose that $(b(ax + xc))^n = 0$ for all $x \in R$. Then there exists $\beta \in C$ such that $b(a - \beta) = 0$ and $(c + \beta)b = 0$.

Proof. Suppose first that $R$ is not a GPI-ring. This implies that $(bAX + Xc)^n$ is a trivial generalized polynomial. In particular, $ba$ and $b$ are dependent over $C$. That is, $b(a - \beta) = 0$ for some $\beta \in C$. Thus,

$$0 = (c + \beta)(b(ax + xc))^n bx = (c + \beta)(b((a - \beta)x + x(c + \beta)))^n bx = ((c + \beta)bx)^{n+1} \quad (2.1)$$

for all $x \in R$. In view of Fact 1.6, $(c + \beta)b = 0$.

Suppose next that $R$ is a GPI-ring. It follows from Fact 1.4 that

$$(b(ax + xc))^n = 0 \quad (2.2)$$

for all $x \in RC$. Let $F$ denote the algebraic closure of $C$ if $C$ is an infinite field and let $F = C$ if $C$ is a finite field. Then (2.2) holds for all $x \in \bar{R}$ (see [22, Lemma 2.3]), where $\bar{R} := RC \otimes_C F$. In view of [8, Theorem 3.5], $\bar{R}$ is a centrally closed prime $F$-algebra. By [23, Theorem 3], $\bar{R}$ is a primitive ring with a minimal idempotent $e$ such that $eRe = F e$. Hence, there exists a left vector space $V$ over $F$ such that $\bar{R}$ acts densely on $F V$.

Given $v \in V$, we claim that $v(ba)$ and $vb$ are dependent over $F$. Suppose not; then there exists $x \in \bar{R}$ such that $v(ba)x = v$ and $vbx = 0$. Then $0 = v(b(ax + xc))^n = v$, which is a contradiction. This proves the claim.

If $\dim_F Vb \geq 2$, it is routine to prove that there exists $\beta \in C$ such that $ba = \bar{\beta}b$; that is, $b(a - \beta) = 0$. Thus, by (2.1) we have $(c + \bar{\beta})b = 0$. Suppose next that $\dim_F Vb = 1$. Choose $v_0 \in V$ such that $Vb = Fv_0b$. Write $v_0ba = \gamma v_0b$ for some $\gamma \in F$. Let $v \in V$. Then $vb = \overline{\alpha}v_0b$ for some $\overline{\alpha} \in F$. Then $vba = \overline{\alpha}v_0ba = \overline{\alpha}\gamma v_0b = \gamma vb$.

In either case, there exists $\beta \in F$ such that $ba = \bar{\beta}b$. Choose a basis $\mu_0, \mu_1, \ldots$ for $F$ over $C$, where $\mu_0 = 1$, and write $\bar{\beta} = \beta \mu_0 + \beta_1 \mu_1 + \cdots$, where $\beta, \beta_1, \ldots \in C$. Then $ba = \beta a$. That is, $b(a - \beta) = 0$. It follows from (2.1) that $(c + \beta)b = 0$. \qed

Proof of Proposition 2.1. It follows from Fact 1.4 that

$$(ax + bxc)^n = 0 \quad (2.3)$$

for all $x \in Q_{mr}(R)$. We claim that $a \in bQ_{mr}(R)$. Clearly, we may assume $a \neq 0$.

Suppose that $R$ is not a GPI-ring. Then $a$ and $b$ are dependent over $C$. In particular, $a \in bQ_{mr}(R)$, as asserted. Suppose next that $R$ is a GPI-ring. In this case, $Q_{mr}(R)$ is also a prime GPI-ring (see Fact 1.4). Since $Q_{mr}(R)$ is a centrally closed prime ring, it follows from [23, Theorem 3] that $Q_{mr}(R)$ is a primitive ring with nonzero socle. Write $H := \text{soc}(Q_{mr}(R))$, the socle of $Q_{mr}(R)$. Note that $H$ is a regular ring (see [9]); that is, for any $w \in H$, $zw = w$ for some $z \in H$. For $z \in H$, we write $\ell_H(z)$ for the left annihilator of $z$ in $H$; that is, $\ell_H(z) = \{x \in H \mid xz = 0\}$. 

https://www.cambridge.org/core/terms. https://doi.org/10.1017/S1446788713000670
We first consider the case that \(a, b \in H\). Let \(w \in \ell_H(b)\). By (2.3),
\[
0 = w(a(xw) + b(xw)c)\alpha x = (wax)^{n+1}
\]
for all \(x \in Q_{mr}(R)\). In view of Fact 1.6, \(wa = 0\). That is, \(w \in \ell_H(a)\). Up to now, we have proved that \(\ell_H(b) \subseteq \ell_H(a)\)

Since \(a, b \in H\), there exist \(u, v \in H\) such that \(aua = a\) and \(bvb = b\). Set \(f := au\) and \(g := bv\). Then \(f, g\) are idempotents. Then \(\ell_H(g) \subseteq \ell_H(f)\); that is, \(H(1 - g) \subseteq H(1 - f)\).

So \((1 - g)f = 0\). Then \(a = fa = gfa = bvfa \in bH\), as asserted.

For the general case, let \(w \in H\). We see that \((awx + bwxc)^n = 0\) for all \(x \in Q_{mr}(R)\). Since \(aw, bw \in H\), the first case implies that \(aw \in bwH\). Write \(aw = bw \gamma\) for some \(\gamma \in H\), depending on \(w\). Replacing \(x\) by \(wx\) in (2.3),
\[
(bw(tx + xc))^n = (awx + bwxc)c^n = 0
\]
for all \(x \in Q_{mr}(R)\). By Lemma 2.2, there exists \(\beta_w \in C\), depending on \(w\), such that \(bw(t - \beta_w) = 0\). That is, \(aw = \beta_w bw\) for \(w \in H\).

Fix an idempotent \(e_0 \in H\) such that \(ae_0 \neq 0\). Then \(ae_0 = \beta be_0\) for some \(\beta \in C\). Let \(f\) be an idempotent of \(H\). Then \(af = \beta f\) for some \(\beta \in C\). We claim that \(\beta f = \beta f\) if \(af \neq 0\). Indeed, there exists \(h = h^2 \in H\) such that \(e_0 h + fH = h^2 H\) and \(ah = \beta h bh\) for some \(\beta \in C\). Note that \(he_0 = e_0\) and \(hf = f\). Thus,
\[
\beta e_0 = ahe_0 = \beta h b h \beta e_0 = \beta h be_0,
\]

implying that \(\beta h = \beta\). Similarly, \(\beta_h = \beta_f\) and so \(\beta = \beta_f\). Thus, \((a - \beta b)f = 0\) if \(af \neq 0\).

Let \(f = f^2 \in H\) with \(af = 0\). We claim that \(bf = 0\). By Litoff’s theorem [9], there exists an idempotent \(h \in H\) such that \(e_0 f \in hHh\). If \(ah = 0\) then \(ae_0 = ahe_0 = 0\), which is a contradiction. Thus, neither \(ah\) nor \(a(h - f)\) is zero. Note that \(h - f\) is an idempotent. Then
\[
ah = \beta bh \quad \text{and} \quad a(h - f) = \beta b(h - f).
\]

This implies that \(\beta b f = 0\), so \(bf = 0\) follows. Up to now, we have proved that \((a - \beta b)f = 0\) for any idempotent \(f \in H\) with \(af = 0\).

In either case, \((a - \beta b)f = 0\) for any idempotent \(f \in H\). Since \(H\) is a regular ring, \((a - \beta b)H = 0\) and so \(a = \beta b\). Rewrite (2.3) as \((bx(\epsilon + \beta))^n = 0\) for all \(x \in Q_{mr}(R)\). So \((x(c + \beta))^{n+1} = 0\) for all \(x \in Q_{mr}(R)\). By Fact 1.6, \((c + \beta)\gamma = 0\) follows. \(\Box\)

The following characterizes \(b\)-generalized derivations of semiprime rings.

**Theorem 2.3.** Let \(R\) be a semiprime ring, \(b \in Q_{mr}(R),\) and let \(\delta: R \rightarrow Q_{mr}(R)\) be a \(b\)-generalized derivation with associated map \(d\). Then \(E[b]d: R \rightarrow Q_{mr}(R)\) is a derivation and there exists \(\tilde{b} \in Q_{mr}(R)\) such that \(\tilde{b}(x) = bd(x) + bx\) for all \(x \in R\).

**Proof.** Expanding \(\delta((xy)z)\) and \(\delta(x(yz))\) respectively, we see that
\[
bx(d(yz) - yd(z) - d(y)z) = 0
\]
for all $x, y, z \in R$. The semiprimeness of $R$ implies that $E[b]d(yz) = yE[b]d(z) + E[b]d(y)z$ for all $y, z \in R$; that is, $E[b]d : R \to Q_{mr}(R)$ is a derivation. Let $\mu : R \to Q_{mr}(R)$ be the map defined by $\mu(x) = bd(x)$ for $x \in R$. Then

$$
\mu(xy) = bE[b]d(xy) = bE[b]d(x)y + bxE[b]d(y) = bd(x)y + bxd(y) = \mu(x)y + bxd(y)
$$

for all $x, y \in R$. Thus, we have $(\delta - \mu)(xy) = (\delta - \mu)(x)y$ for all $x, y \in R$. In view of Fact 1.7, there exists $b \in Q_{mr}(R)$ such that $\delta(x) = bd(x) + bx$ for all $x \in R$.

**Theorem 2.4.** Let $R$ be a prime ring, $b \in Q_{mr}(R)$, and let $\delta : R \to Q_{mr}(R)$ be a nonzero $b$-generalized derivation with associated map $d$. Suppose that $\delta(x)^n = 0$ for all $x \in R$, where $n$ is a positive integer. Then there exists $q \in Q_{mr}(R)$ such that $d = ad(q)$,

$$
\delta(x) = -bxq\text{ for }x \in R, \text{ and } qb = 0.
$$

**Proof.** In view of Theorem 2.3, there exists $b \in Q_{mr}(R)$ such that $\delta(x) = bd(x) + bx$ for all $x \in R$. By assumption,

$$
(bd(x) + bx)^n = 0 \tag{2.4}
$$

for all $x \in R$. By Fact 1.8, $d$ can be uniquely extended to a derivation from $Q_{mr}(R)$ to itself, also denoted by $d$. In view of Fact 1.5, (2.4) holds for all $x \in Q_{mr}(R)$. Note that $Q_{mr}(R)$ is ring-isomorphic to the direct product $\prod_{y \in \Lambda} Q_{mr}(R)$.

Suppose first that $d$ is X-outer. In view of [15, Theorem 2], $(by + bx)^n = 0$ for all $x, y \in Q_{mr}(R)$. Then $b = 0 = b$ (see Fact 1.6). This implies that $\delta = 0$, which is a contradiction. Thus, $d$ is X-inner. Then there exists $q' \in Q_{mr}(R)$ such that $d(x) = [q', x]$ for $x \in R$. Since $R$ and $Q_{mr}(R)$ satisfy the same GPIs (see Fact 1.4), we rewrite (2.4) as

$$
((bq' + b)x - bxq')^n = 0
$$

for all $x \in Q_{mr}(R)$. In view of Proposition 2.1, there exists $\mu \in C$ such that $bq' + b = \mu b$ and $(q' - \mu)b = 0$. Let $q := q' - \mu$. Then $d = ad(q)$, $bq = -b$ and $qb = 0$. Therefore,

$$
\delta(x) = bd(x) + bx = bq(x - xq) - bqx = -bxq \quad \text{for } x \in R,
$$

as asserted. \hfill \Box

**3. Proof of Theorem 1.2**

Let $R$ be a semiprime ring with extended centroid $C$. We call $\{e_v \mid v \in \Lambda\} \subseteq B$ an orthogonal subset if $e_ve_\mu = 0$ for $v \neq \mu$ and a dense subset of $B$ if $\sum_{v \in \Lambda} e_vC$ is an essential ideal of $C$. The ring $Q_{mr}(R)$ is orthogonally complete in the following sense: Given any dense orthogonal subset $\{e_v \mid v \in \Lambda\}$ of $B$, $Q_{mr}(R)$ is ring-isomorphic to the direct product $\prod_{v \in \Lambda} Q_{mr}(R)e_v$ via the map

$$
x \mapsto \langle xe_v \rangle \in \prod_{v \in \Lambda} Q_{mr}(R)e_v \quad \text{for } x \in Q_{mr}(R).\]
Therefore, given any subset \( \{a_v \in Q_{mr}(R) \mid v \in \Lambda \} \), there exists a unique \( a \in Q_{mr}(R) \) such that \( a \mapsto \langle a_v e_v \rangle \). The element \( a \) is written as \( \sum_{v \in \Lambda} a_v e_v \) and is characterized by the property that \( ae_v = a_v e_v \) for all \( v \in \Lambda \). A subset \( T \) of \( Q_{mr}(R) \) is called orthogonally complete if \( 0 \in T \) and \( \sum_{v \in \Lambda} a_v e_v \in T \) for any dense orthogonal subset \( \{ e_v \mid v \in \Lambda \} \) of \( B \) and any subset \( \{a_v \mid v \in \Lambda \} \subseteq T \). Denote by \( \text{Spec}(B) \) the set of all maximal ideals of the complete Boolean algebra \( B \). Let \( T \) be a subset of \( Q_{mr}(R) \). The intersection of all orthogonally complete subsets of \( Q_{mr}(R) \) containing \( T \) is called the orthogonal completion of \( T \) and is denoted by \( \widehat{T} \). In view of [1, Proposition 3.1.14 and Corollary 3.1.15], \( \widehat{R} \) is a subring of \( Q_{mr}(R) \) and

\[
\widehat{R} = \left\{ \sum_{\alpha \in \Lambda} x_\alpha e_\alpha \mid \{ e_\alpha \mid \alpha \in \Lambda \} \text{ is a dense orthogonal subset of } B \text{ and } x_\alpha \in R \text{ for all } \alpha \in \Lambda \right\}.
\]

Moreover, \( \widehat{R} \cap mQ_{mr}(R) \) is a prime ideal of \( \widehat{R} \) for all \( m \in \text{Spec}(B) \) (see [1, Theorem 3.2.15]).

**Proposition 3.1.** A derivation \( d : Q_{mr}(R) \to Q_{mr}(R) \) is X-inner if and only if \( d : Q_{mr}(R)/mQ_{mr}(R) \to Q_{mr}(R)/mQ_{mr}(R) \) is X-inner for any \( m \in \text{Spec}(B) \).

The proof of Proposition 3.1 is the same as that of [20, Proposition 2.2]. Let \( m \in \text{Spec}(B) \). It is known that \( mQ_{mr}(R) \) is a prime ideal of \( Q_{mr}(R) \). We use the notations: \( \overline{Q_{mr}(R)} = Q_{mr}(R)/mQ_{mr}(R) \), \( \overline{C} = C + mQ_{mr}(R)/mQ_{mr}(R) \), and \( \overline{\widehat{R}} = \overline{R} + mQ_{mr}(R)/mQ_{mr}(R) \). Then both \( \overline{Q_{mr}(R)} \) and \( \overline{\widehat{R}} \) are prime rings having the same extended centroid \( \overline{C} \) (see [1]). Keeping these notations we have the following.

**Lemma 3.2.** Let \( v, x \in Q_{mr}(R) \). Suppose that \( \overline{x} \in \overline{Cv} \) for any \( m \in \text{Spec}(B) \), where \( \overline{x} := z + mQ_{mr}(R) \) for \( z \in Q_{mr}(R) \). Then \( x \in Cv \).

**Proof.** Consider the set \( \Sigma = \{ e \in B \mid ex \in Cv \} \). We see that if \( e \leq f \in \Sigma \) then \( e \in \Sigma \). Also, if \( e, f \in \Sigma \) are orthogonal then clearly \( e + f \in \Sigma \). This means that \( \Sigma \) is an ideal of the complete Boolean algebra \( B \). If \( 1 \in \Sigma \) then \( x \in Cv \), as asserted. Suppose on the contrary that \( 1 \notin \Sigma \). By Zorn’s lemma, there exists \( m \in \text{Spec}(B) \) such that \( \Sigma \subseteq m \). We work in \( Q_{mr}(R)/mQ_{mr}(R) \). Since \( \overline{x} \in \overline{Cv} \), there exists \( a \in Cv \) such that \( \overline{x} = \overline{a} \). Therefore, \( ex = ea \) for some \( e \in B \setminus m \). Note that \( ea \in Cv \), implying \( e \in \Sigma \). This is a contradiction.

The next theorem extends Proposition 2.1 to the semiprime case.

**Theorem 3.3.** Let \( R \) be a semiprime ring, \( a, b, c \in R \), and \( n \) a positive integer. Suppose that \( (ax + bxc)^n = 0 \) for all \( x \in R \). Then there exists \( \beta \in C \) such that \( a = \beta b \) and \( (c + \beta)b = 0 \).

**Proof.** By Fact 1.4, \( (ax + bxc)^n = 0 \) for all \( x \in Q_{mr}(R) \). Let \( m \in \text{Spec}(B) \). Working in \( Q_{mr}(R)/mQ_{mr}(R) \), we see that \( (\overline{a} \overline{x} + \overline{b} \overline{xc})^n = 0 \) for all \( \overline{x} \in Q_{mr}(R)/mQ_{mr}(R) \). In view of Proposition 2.1, \( \overline{a} \in \overline{C} \overline{b} \). Since \( m \in \text{Spec}(B) \) is arbitrary, it follows from Lemma 3.2.
that $a \in Cb$. Write $a = \beta b$ for some $\beta \in C$. Then $(bx(c + \beta))^n = 0$ for all $x \in R$. By Fact 1.6, $(c + \beta)b = 0$ follows, as asserted.

**Lemma 3.4.** Theorem 1.2 holds if $E[b] = 1$.

**Proof.** Since $E[b] = 1$, it follows from Theorem 2.3 that $d' : R \to Q_{mr}(R)$ is a derivation. By Fact 1.8, $d$ can be uniquely extended to a derivation $\hat{d} : Q_{mr}(R) \to Q_{mr}(R)$. Clearly,

$$\hat{d} \left( \sum_{\nu \in \Lambda} x_{\nu}e_{\nu} \right) = \sum_{\nu \in \Lambda} d(x_{\nu})e_{\nu},$$

where $x_{\nu} \in R$. We claim that $\delta$ can be also uniquely extended to a $b$-generalized derivation of $\hat{R}$, say $\hat{\delta}$, with associated map $\hat{d} : \hat{R} \to Q_{mr}(R)$, by defining

$$\hat{\delta} \left( \sum_{\nu \in \Lambda} x_{\nu}e_{\nu} \right) = \sum_{\nu \in \Lambda} \hat{\delta}(x_{\nu})e_{\nu},$$

where $x_{\nu} \in R$. Indeed, let $\sum_{\nu \in \Lambda} x_{\nu}e_{\nu} = 0$, where $x_{\nu} \in R$. Then $x_{\nu}e_{\nu} = 0$ for any $\nu$. Fix an $x_{\nu}$. Choose a dense ideal $I$ of $R$ such that $x_{\nu}I \subseteq R$. Note that $d(ye_{\nu}) = \hat{d}(ye_{\nu}) = \hat{d}(y)e_{\nu} = d(y)e_{\nu}$ for $y \in I$ since $\hat{d}$ is a derivation. Thus,

$$0 = \delta(x_{\nu}(ye_{\nu})) = \delta(x_{\nu})ye_{\nu} + bx_{\nu}d(ye_{\nu}) = \delta(x_{\nu})ye_{\nu},$$

implying that $\delta(x_{\nu})ye_{\nu} = 0$ for all $y \in I$. Fact 1.4 implies that $\delta(x_{\nu})ye_{\nu} = 0$ for all $y \in Q_{mr}(R)$. By Fact 1.6, $\delta(x_{\nu})e_{\nu} = 0$. So $\sum_{\nu \in \Lambda} \delta(x_{\nu})e_{\nu} = 0$. This proves that $\hat{\delta}$ is well defined. It is routine to check that $\hat{\delta}$ is an additive map.

We claim that $\hat{\delta} : \hat{R} \to Q_{mr}(R)$ is a $b$-generalized derivation with associated map $\hat{d}$. Indeed, let $\hat{x}, \hat{y} \in \hat{R}$. Write

$$\hat{x} = \sum_{\nu \in \Lambda} x_{\nu}e_{\nu} \quad \text{and} \quad \hat{y} = \sum_{\nu \in \Lambda} y_{\nu}e_{\nu},$$

where $x_{\nu}, y_{\nu} \in R$. Then $\hat{x}\hat{y} = \sum_{\nu \in \Lambda} (x_{\nu}y_{\nu})e_{\nu}$ and

$$\hat{\delta}(\hat{x}\hat{y}) = \sum_{\nu \in \Lambda} \hat{\delta}(x_{\nu}y_{\nu})e_{\nu} = \sum_{\nu \in \Lambda} \left( \delta(x_{\nu})y_{\nu} + bx_{\nu}d(y_{\nu}) \right)e_{\nu} = \left( \sum_{\nu \in \Lambda} \delta(x_{\nu})e_{\nu} \right) \left( \sum_{\nu \in \Lambda} y_{\nu}e_{\nu} \right) + \left( \sum_{\nu \in \Lambda} x_{\nu}e_{\nu} \right) \left( \sum_{\nu \in \Lambda} d(y_{\nu})e_{\nu} \right) = \hat{\delta}(\hat{x})\hat{y} + b\hat{x}\hat{d}(\hat{y}),$$

as asserted.

Let $m \in \text{Spec}(B)$. Clearly, $\hat{d}(m\hat{R}) \subseteq mQ_{mr}(R)$ since $\hat{d}$ is a derivation. We claim that $\hat{\delta}(m\hat{R}) \subseteq mQ_{mr}(R)$. Let $x \in m\hat{R}$. Then $xe = 0$ for some $e \in B \setminus m$. Applying the same argument as in the first paragraph, we see that $\hat{\delta}(x)e = 0$. Thus $\hat{\delta}(x) \in mQ_{mr}(R)$. This proves our claim.

Thus, $\hat{\delta}$ and $\hat{d}$ canonically induce the maps $\hat{\delta}_m : \hat{R}/m\hat{R} \to Q_{mr}(R)/mQ_{mr}(R)$ and $\hat{d}_m : \hat{R}/m\hat{R} \to Q_{mr}(R)/mQ_{mr}(R)$, where

$$\hat{\delta}_m(\hat{x}) := \hat{\delta}(\hat{x}) \quad \text{and} \quad \hat{d}_m(\hat{x}) := \hat{d}(\hat{x})$$
for $\tilde{x} = \tilde{x} + m\hat{R}$, where $\tilde{x} \in \hat{R}$. Note that $Q_{mr}(R)/mQ_{mr}(R) \subseteq Q_{mr}(\hat{R}/m\hat{R})$. It is clear that $\tilde{d}_m$ is a $\hat{b}$-generalized derivation with associated map $d_m$. Note that $\hat{b} \neq \hat{0}$ since $E[b] = 1$.

We work in the prime ring $\hat{R}/m\hat{R}$ with extended centroid $\hat{C}(= C + m\hat{R}/m\hat{R})$. Let $\tilde{x} = \tilde{x} + m\hat{R} \in \hat{R}/m\hat{R}$, where $\tilde{x} \in \hat{R}$. Write $\tilde{x} = \sum_{v \in A} x_v e_v$, where $x_v \in R$. Then

$$
\delta(\tilde{x}) = \sum_{v \in A} \delta(x_v)e_v
$$

and

$$
\tilde{d}_m(\tilde{x})^n = \tilde{\delta}(\tilde{x})^n = \left(\sum_{v \in A} \delta(x_v)e_v\right)^n = \sum_{v \in A} \delta(x_v)^n e_v = 0.
$$

In view of Theorem 2.4, the derivation $\tilde{d}_m$ is X-inner. It follows from Proposition 3.1 that $\tilde{d}$ is X-inner. Thus, $\tilde{d} = \text{ad}(q')$ for some $q' \in Q_{mr}(R)$. Moreover, in view of Theorem 2.4, for any $m \in \text{Spec}(B)$ we have $q' \hat{b} = q'b \in \hat{C}\hat{b}$. By Lemma 3.2, $q'b = \beta b$ for some $\beta \in C$. Set $q := q' - \beta$. Then $d = \text{ad}(q)$ and $qb = 0$.

Let $x, y \in R$. Then

$$
\delta(xy) = \delta(x)y + bxyq = \delta(x)y + bx(qy - yq),
$$

implying that

$$
\delta(xy) + bxyq = (\delta(x) + bxq)y.
$$

By Fact 1.7, there exists $w \in Q_{mr}(R)$ such that $\delta(x) = -bxq + wx$ for all $x \in R$. Thus, $(wx - bxq)^n = 0$ for all $x \in R$ and hence for all $x \in Q_{mr}(R)$ (see Fact 1.4). In view of Theorem 3.3, there exists $\mu \in C$ such that $w = \mu b$ and $(q - \mu)b = 0$. Thus, by the fact that $qb = 0$, we see that $\mu = 0$ and $w = 0$. That is, $\delta(x) = -bxq$ for all $x \in R$, as asserted.

**Proof of Theorem 1.2.** Let $e := E[b]$, $\delta_1(x) := e\delta(x)$ and $d_1(x) := ed(x)$ for $x \in R$. Then $(1 - e)\delta(xy) = (1 - e)\delta(x)y$ for all $x, y \in R$. By Fact 1.7, there exists $w \in Q_{mr}(R)$ such that $(1 - e)\delta(x) = wx$ for all $x \in R$. But $(wx)^n = 0$ for all $x \in R$. This implies that $w = 0$; that is, $(1 - e)\delta(x) = 0$ for all $x \in R$.

Note that $\delta_1 : R \to Q_{mr}(R)$, $d_1 : R \to Q_{mr}(R)$, and $\delta_1(xy) = \delta_1(x)y + bxd_1(y)$ for all $x, y \in R$. Applying the same argument given in the proof of Lemma 3.4, $d_1$ is a derivation and can be uniquely extended to a derivation $\tilde{d}_1 : \hat{R} \to Q_{mr}(R)$ by defining

$$
\tilde{d}_1\left(\sum_{v \in A} x_v e_v\right) = \sum_{v \in A} (ed(x_v))e_v,
$$

where $x_v \in R$.

On the other hand, $\delta_1$ can be extended to a map $\tilde{\delta}_1 : \hat{R} \to Q_{mr}(R)$ by defining

$$
\tilde{\delta}_1\left(\sum_{v \in A} x_v e_v\right) = \sum_{v \in A} (e\delta(x_v))e_v,
$$

where $x_v \in R$.

Note that $\tilde{d}_1(e\hat{R}) \subseteq eQ_{mr}(R)$ and $\tilde{\delta}_1(e\hat{R}) \subseteq eQ_{mr}(R)$. Working on $eQ_{mr}(R)$,

$$
\tilde{\delta}_1(xy) = \tilde{\delta}_1(x)y + bxd_1(y)
$$

for all $x, y \in e\hat{R}$. Note that $Q_{mr}(e\hat{R}) = eQ_{mr}(R)$ and that $(\tilde{\delta}_1(x))^n = 0$ for all $x \in e\hat{R}$. Since $E[b] = e$ and the extended centroid of $e\hat{R}$ is equal to $eC$, it follows from Lemma 3.4 that
there exists \( q \in eQ_{mr}(R) \) such that \( ed(x) = [q, x] \) for \( x \in e\hat{R}, e\delta(x) = -bxq \) for \( x \in e\hat{R}, \) and \( qb = 0. \)

Choose a dense ideal \( I \) of \( R \) such that \((1 - e)I \subseteq R. \) Let \( x, y, z \in I. \) Then
\[
\delta(x(1 - e)y) = \delta(x)(1 - e)y + bxd((1 - e)y) \\
= bxed((1 - e)y) = bx(ed(y) - ed(e)y - ed(y)) = 0,
\]
since \( \delta(x)(1 - e) = 0 \) and \( ed \) is a derivation on \( Q_{mr}(R). \) So \( \delta((1 - e)I^2) = 0. \) Let \( x \in I^2. \) Then
\[
\delta(x) = e\delta(x) = e\delta(ex + (1 - e)x) = -b(ex)q = -bxq.
\]
Up to now, we have proved that \( \delta(x) = -bxq \) for \( x \in I^2. \) Let \( y \in R \) and \( x \in I^2. \) We notice that \( ed(x) = ed(ex) = e[q, ex] = [q, x], \) Then \( yx \in I^2 \) and
\[ -byxq = \delta(yx) = \delta(y)x + byd(x) = \delta(y)x + byed(x) = \delta(y)x + by[q, x], \]
implying that \( (\delta(y) + byq)x = 0. \) That is, \( (\delta(y) + byq)I^2 = 0 \) and so \( \delta(y) = -byq, \) as asserted. \( \Box \)

References


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